Electronic Journal of Mathematics
www.shahindp.com/locate/ejm

Electron. J. Math. **11** (2026) 38–44
DOI: 10.47443/ejm.2025.063

Research Article

A Note on an Analogue of the Sălăgean Class

Myroslav M. Sheremeta‡

Department of Mechanics and Mathematics, Ivan Franko National University of Lviv, 29000 Lviv, Ukraine

Abstract

Let $F(s) = e^s + \sum_{k=1}^{\infty} f_k e^{s\lambda_k}$ be a Dirichlet series that is absolutely convergent in the half-plane $\Pi_0 = \{s: \operatorname{Re} s < 0\}$. This paper provides the conditions under which the inequality $\operatorname{Re} \frac{F^{(j+1)}(s)}{F^{(j)}(s)} > \alpha$ holds true for $\alpha \in [0, 1)$, $s \in \Pi_0$ and $j \geq 0$, where (λ_k) is a sequence of positive numbers with $\lambda_1 > 1$ increasing to $+\infty$. The obtained result is applied to the study of the properties of a solution of the differential equation $\frac{d^2 w}{ds^2} + (\gamma_0 e^{2s} + \gamma_1 e^s + \gamma_2)w = 0$.

Keywords: Dirichlet series; Sălăgean class; differential equation.

2020 Mathematics Subject Classification: 30B50, 30C45.

1. Introduction

For the function

$$f(z) = z + \sum_{k=2}^{\infty} f_k z^k \tag{1}$$

analytic in the disk $\mathbb{D} = \{z: |z| < 1\}$, Sălăgean [5] defined

$$D^0 f(z) = f(z),\ D^1 f(z) = z f'(z),\ D^j f(z) = D^1(D^{j-1} f(z))$$

for $j \in \mathbb{N}$ and introduced a class $S_j(\alpha)$ of functions (1) satisfying

$$\operatorname{Re} \frac{D^{j+1} f(z)}{D^j f(z)} > \alpha \in [0,\ 1)$$

for every $z \in \mathbb{D}$. Sălăgean derivatives have been used in many works (see, for example, [1,2,4,7]). If we use the substitution $z = e^s$ in the series (1), we obtain the Dirichlet series

$$F(s) = e^s + \sum_{k=2}^{\infty} f_k \exp\{ks\}.$$

It is clear that such a Dirichlet series converges absolutely in the half-plane $\Pi_0 = \{s: \operatorname{Re} s < 0\}$ and

$$F'(s) = e^s f'(e^s) = z f'(z) = D^1 f(z),$$

and hence, $F^{(j)}(s) = D^j f(z)$.

Received: 7 October 2025. Received in revised form: 25 December 2025. Accepted: 30 December 2025. Published online: 14 January 2026.
‡E-mail address: m.m.sheremeta@gmail.com

Therefore, the condition

$$\mathrm{Re}\,\frac{D^{j+1}f(z)}{D^j f(z)} > \alpha$$

takes the form

$$\mathrm{Re}\,\frac{F^{(j+1)}(s)}{F^{(j)}(s)} > \alpha \quad (s \in \Pi_0). \tag{2}$$

A generalization of the above-mentioned series is the following Dirichlet series:

$$F(s) = e^s + \sum_{k=1}^{\infty} f_k \exp\{s\lambda_k\} \quad (s = \sigma + it), \tag{3}$$

where $1 < \lambda_k \uparrow +\infty$.

As in [8], we denote by D the class of series (3) that are absolutely convergent in the half-plane Π_0. We say that the series (3) belongs to the class $D_j(\alpha)$ if it satisfies condition (2); it belongs to the class $D_j^+(\alpha)$ if

$$\frac{F^{(j+1)}(\sigma)}{F^{(j)}(\sigma)} > \alpha \quad \text{for all } \sigma < 0;$$

and, finally, it belongs to the class $K_j(\alpha)$ if

$$\sum_{k=1}^{\infty} \lambda_k^j(\lambda_k - \alpha)|f_k| \le 1 - \alpha. \tag{4}$$

Note that if we take $\alpha = j = 0$ in (2), then we obtain the definition of a pseudostarlike function (see [3] and [6, page 147]). The following theorem was proved in [8].

Theorem 1.1. *If $f_k \le 0$ for every $k \ge 1$, then $D_j(\alpha) \subset D_j^+(\alpha) = K_j(\alpha)$ for every $\alpha \in [0, 1)$ and $j \ge 0$.*

Theorem 1.1 was applied to the study of the properties of solutions to the differential equation

$$\frac{d^2w}{ds^2} + (\gamma_0 e^{2s} + \gamma_1 e^s + \gamma_2)w = 0, \tag{5}$$

and the following result was obtained [8].

Theorem 1.2. *Equation (5) with real parameters $\gamma_0, \gamma_1, \gamma_2$ has the only solution $F(s) = e^s$ that belongs to $D_j(\alpha)$ for every $\alpha \in [0, 1)$ and $j \ge 0$, and at the same time $\gamma_0 = \gamma_1 = 0$, $\gamma_2 = -1$. For any $\alpha \in [0, 1)$ and $j \ge 0$, Equation (5) has no other solutions (3) with coefficients $f_k \le 0$ for all $k \ge 1$ that belong to the class $D_j(\alpha)$.*

The condition $f_k \le 0$ for all $k \ge 1$ in Theorem 1.2 is important. If the coefficients are allowed to be complex, one obtains a result that differs significantly from Theorem 1.2. The present note is devoted to the investigation of this problem.

2. Main Result

We first prove an analogue of Theorem 1.1 in the case where the coefficients f_k are allowed to be complex.

Theorem 2.1. *Let $F \in D$ and suppose that either $\alpha \in [1/2, \lambda_1/2)$ or $\alpha \in (0, 1/2)$. Then condition (4) is sufficient to ensure that $F \in D_j(\alpha)$ for every $j \ge 0$.*

Proof. If $m \ge 0$ and $\alpha > 0$ then $(\mathrm{Re}\, z > \alpha) \iff (|z + m\alpha| > |z - (m+2)\alpha|)$ for $z \in \mathbb{C}$. Therefore, condition (2) holds if and only if

$$|F^{(j+1)}(s) + m\alpha F^{(j)}(s)| > |F^{(j+1)}(s) - (m+2)\alpha F^{(j)}(s)|, \quad (s = \sigma + it), \tag{6}$$

that is, if

$$\left|e^s+\sum_{k=1}^{\infty}\lambda_k^{j+1}f_k\exp\{s\lambda_k\}-(m+2)\alpha e^s-(m+2)\alpha\sum_{k=1}^{\infty}\lambda_k^{j}f_k\exp\{s\lambda_k\}\right|$$

$$<\left|e^s+\sum_{k=1}^{\infty}\lambda_k^{j+1}f_k\exp\{s\lambda_k\}+m\alpha e^s+m\alpha\sum_{k=1}^{\infty}\lambda_k^{j}f_k\exp\{s\lambda_k\}\right|,$$

and thus, if

$$\left|1-(m+2)\alpha+\sum_{k=1}^{\infty}\lambda_k^{j}(\lambda_k-(m+2)\alpha)f_k\exp\{s(\lambda_k-1)\}\right|$$

$$<\left|1+m\alpha+\sum_{k=1}^{\infty}\lambda_k^{j}(\lambda_k+m\alpha)f_k\exp\{s(\lambda_k-1)\}\right|. \tag{7}$$

On the other hand, for $s=\sigma+it\in\Pi_0$, we have $\sigma<0$. Hence,

$$\begin{aligned}\left|1+m\alpha+\sum_{k=1}^{\infty}\lambda_k^{j}(\lambda_k+m\alpha)f_k\exp\{s(\lambda_k-1)\}\right| &\geq 1+m\alpha-\left|\sum_{k=1}^{\infty}\lambda_k^{j}(\lambda_k+m\alpha)f_k\exp\{s(\lambda_k-1)\}\right|\\ &\geq 1+m\alpha-\sum_{k=1}^{\infty}\lambda_k^{j}(\lambda_k+m\alpha)|f_k|\exp\{\sigma(\lambda_k-1)\}\\ &\geq 1+m\alpha-\sum_{k=1}^{\infty}\lambda_k^{j}(\lambda_k+m\alpha)|f_k|\end{aligned}$$

and

$$\begin{aligned}&\left|1-(m+2)\alpha+\sum_{k=1}^{\infty}\lambda_k^{j}(\lambda_k-(m+2)\alpha)f_k\exp\{s(\lambda_k-1)\}\right|\\ &\leq|1-(m+2)\alpha|+\sum_{k=1}^{\infty}\lambda_k^{j}|\lambda_k-(m+2)\alpha||f_k|\exp\{\sigma(\lambda_k-1)\}\\ &\leq|1-(m+2)\alpha|+\sum_{k=1}^{\infty}\lambda_k^{j}|\lambda_k-(m+2)\alpha||f_k|.\end{aligned}$$

Thus, (7) holds if

$$|1-(m+2)\alpha|+\sum_{k=1}^{\infty}\lambda_k^{j}|\lambda_k-(m+2)\alpha||f_k|\leq 1+m\alpha-\sum_{k=1}^{\infty}\lambda_k^{j}(\lambda_k+m\alpha)|f_k|,$$

that is, if

$$\sum_{k=1}^{\infty}\lambda_k^{j}(|\lambda_k-(m+2)\alpha|+\lambda_k+m\alpha)|f_k|\leq 1+m\alpha-|1-(m+2)\alpha|. \tag{8}$$

For $m=0$, condition (8) has the following form:

$$\sum_{k=1}^{\infty}\lambda_k^{j}(|\lambda_k-2\alpha|+\lambda_k)\,|f_k|\leq 1-|1-2\alpha|. \tag{9}$$

If $\alpha \in [1/2,\ \lambda_1/2)$, then

$$|\lambda_k - 2\alpha| + \lambda_k = 2(\lambda_k - \alpha)$$

and

$$1 - |1 - 2\alpha| = 1 - |2\alpha - 1| = 2(1 - \alpha).$$

Hence, (9) is equivalent to (4). Therefore, (4) implies (6) and Theorem 2.1 is proved for $\alpha \in [1/2,\ \lambda_1/2)$.

Now, suppose that $0 < \alpha < 1/2$ and choose $m = 1/\alpha - 2$. Substituting this value $m > 0$ into (8), we obtain

$$\sum_{k=1}^{\infty} \lambda_k^j 2(\lambda_k - \alpha)|f_k| \le 2 - 2\alpha.$$

This condition is equivalent to (4). □

Theorems 1.1 and 2.1 imply the following statement.

Corollary 2.2. *Let $F \in D$ and suppose that $\lambda_1 \ge 2$. Then, in order for F to belong to $D_j(\alpha)$ for all $\alpha \in (0,1)$ and $j \ge 0$, it is sufficient that condition (4) holds; moreover, in the case where all $f_k \le 0$, this condition is also necessary.*

3. The Solution to Equation (5) Belonging to the Class $F \in D_j(\alpha)$

The following theorem differs significantly from Theorem 1.2.

Theorem 3.1. *Let $F \in D$, $\alpha \in (0,1)$, and $j \ge 0$. Then, Equation (5) admits a solution of the form (3) which, under the condition*

$$2^{j+3}|\gamma_1| + \left(2^j|\gamma_1| + 3^{j+1}\right)\left(|\gamma_1|^2 + 3|\gamma_0|\right) \le 24(1-\alpha)\left(1 - 2^{j-2}(|\gamma_0| + |\gamma_1|)\right), \qquad (10)$$

belongs to the class $D_j(\alpha)$.

Proof. In [8], conditions on λ_k and f_k were established under which the function

$$F(s) = e^s - \sum_{k=1}^{\infty} f_k e^{s\lambda_k}$$

is a solution of Equation (5). Using this result, it is easy to show that the series (3) is a solution of Equation (5) if and only if

$$\gamma_2 = 1,\ \lambda_k = k+1,\ f_1 = -\frac{\gamma_1}{3},\ f_2 = \frac{\gamma_1 f_1 - \gamma_0}{8},\ f_k = \frac{\gamma_1 f_{k-1} + \gamma_0 f_{k-2}}{(k+1)^2 - 1}\ (k \ge 3).$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{\infty} \lambda_k^j(\lambda_k - \alpha)|f_k| &= \lambda_1^j(\lambda_1 - \alpha)|f_1| + \lambda_2^j(\lambda_2 - \alpha)|f_2| + \sum_{k=3}^{\infty} \lambda_k^j(\lambda_k - \alpha)|f_k| \\ &\le \lambda_1^j(\lambda_1 - \alpha)|f_1| + \lambda_2^j(\lambda_2 - \alpha)|f_2| + \sum_{k=3}^{\infty} \lambda_k^j(\lambda_k - \alpha)\frac{|\gamma_1||f_{k-1}|}{\lambda_k^2 - 1} \\ &\quad + \sum_{k=3}^{\infty} \lambda_k^j(\lambda_k - \alpha)\frac{|\gamma_0||f_{k-2}|}{\lambda_k^2 - 1}\end{aligned}$$

$$
\begin{aligned}
&= \lambda_1^j(\lambda_1-\alpha)|f_1| + \lambda_2^j(\lambda_2-\alpha)|f_2| + \sum_{k=2}^{\infty}\lambda_{k+1}^j(\lambda_{k+1}-\alpha)\frac{|\gamma_1||f_k|}{\lambda_{k+1}^2-1}\\
&\quad + \sum_{k=1}^{\infty}\lambda_{k+2}^j(\lambda_{k+2}-\alpha)\frac{|\gamma_0||f_k|}{\lambda_{k+2}^2-1}\\
&= \lambda_1^j(\lambda_1-\alpha)|f_1| + \lambda_2^j(\lambda_2-\alpha)|f_2| + \lambda_3^j(\lambda_3-\alpha)\frac{|\gamma_1||f_2|}{\lambda_3^2-1}\\
&\quad + \sum_{k=3}^{\infty}\lambda_{k+1}^j(\lambda_{k+1}-\alpha)\frac{|\gamma_1||f_k|}{\lambda_{k+1}^2-1} + \lambda_3^j(\lambda_3-\alpha)\frac{|\gamma_0||f_1|}{\lambda_3^2-1}\\
&\quad + \lambda_4^j(\lambda_4-\alpha)\frac{|\gamma_0||f_2|}{\lambda_4^2-1} + \sum_{k=3}^{\infty}\lambda_{k+2}^j(\lambda_{k+2}-\alpha)\frac{|\gamma_0||f_k|}{\lambda_{k+2}^2-1}\\
&= \sum_{k=3}^{\infty}\left(\frac{\lambda_{k+1}^j(\lambda_{k+1}-\alpha)|\gamma_1|}{\lambda_{k+1}^2-1} + \frac{\lambda_{k+2}^j(\lambda_{k+2}-\alpha)|\gamma_0|}{\lambda_{k+2}^2-1}\right)|f_k|\\
&\quad + \lambda_1^j(\lambda_1-\alpha)|f_1| + \lambda_2^j(\lambda_2-\alpha)|f_2| + \lambda_3^j(\lambda_3-\alpha)\frac{|\gamma_1||f_2|}{\lambda_3^2-1}\\
&\quad + \lambda_3^j(\lambda_3-\alpha)\frac{|\gamma_0||f_1|}{\lambda_3^2-1} + \lambda_4^j(\lambda_4-\alpha)\frac{|\gamma_0||f_2|}{\lambda_4^2-1}\\
&= \sum_{k=1}^{\infty}\left(\frac{\lambda_{k+1}^j(\lambda_{k+1}-\alpha)|\gamma_1|}{\lambda_{k+1}^2-1} + \frac{\lambda_{k+2}^j(\lambda_{k+2}-\alpha)|\gamma_0|}{\lambda_{k+2}^2-1}\right)|f_k|\\
&\quad - \left(\frac{\lambda_2^j(\lambda_2-\alpha)|\gamma_1|}{\lambda_2^2-1} + \frac{\lambda_3^j(\lambda_3-\alpha)|\gamma_0|}{\lambda_3^2-1}\right)|f_1| - \left(\frac{\lambda_3^j(\lambda_2-\alpha)|\gamma_1|}{\lambda_3^2-1} + \frac{\lambda_4^j(\lambda_4-\alpha)|\gamma_0|}{\lambda_4^2-1}\right)|f_2|\\
&\quad + \lambda_1^j(\lambda_1-\alpha)|f_1| + \lambda_2^j(\lambda_2-\alpha)|f_2| + \lambda_3^j(\lambda_3-\alpha)\frac{|\gamma_1||f_2|}{\lambda_3^2-1}\\
&\quad + \lambda_3^j(\lambda_3-\alpha)\frac{|\gamma_0||f_1|}{\lambda_3^2-1} + \lambda_4^j(\lambda_4-\alpha)\frac{|\gamma_0||f_2|}{\lambda_4^2-1}\\
&= \sum_{k=1}^{\infty}\left(\frac{\lambda_{k+1}^j(\lambda_{k+1}-\alpha)|\gamma_1|}{\lambda_{k+1}^2-1} + \frac{\lambda_{k+2}^j(\lambda_{k+2}-\alpha)|\gamma_0|}{\lambda_{k+2}^2-1}\right)|f_k|\\
&\quad + \left(\lambda_1^j(\lambda_1-\alpha) - \frac{\lambda_2^j(\lambda_2-\alpha)|\gamma_1|}{\lambda_2^2-1}\right)|f_1| + \big(\lambda_1^j(\lambda_1-\alpha)|f_1| + \lambda_2^j(\lambda_2-\alpha)\big)|f_2|. \qquad (11)
\end{aligned}
$$

Since

$$\frac{\lambda_{k+1}-\alpha}{\lambda_k-\alpha} = \frac{k+2-\alpha}{k+1-\alpha} \le \frac{3-\alpha}{2-\alpha} \le 2,$$

$$\frac{\lambda_{k+2}-\alpha}{\lambda_k-\alpha} \le 3,$$

$$\frac{\lambda_{k+1}^j}{\lambda_k^j(\lambda_{k+1}^2-1)}=\left(\frac{k+2}{k+1}\right)^j\frac{1}{(k+2)^2-1}\le\left(\frac{3}{2}\right)^j\frac{1}{8},$$

and

$$\frac{\lambda_{k+2}^j}{\lambda_k^j(\lambda_{k+2}^2-1)}\le 2^j\frac{1}{15},$$

we obtain

$$\frac{\lambda_{k+1}^j}{\lambda_k^j(\lambda_{k+1}^2-1)}\frac{\lambda_{k+1}-\alpha}{\lambda_k-\alpha}|\gamma_1|+\frac{\lambda_{k+2}^j}{\lambda_k^j(\lambda_{k+2}^2-1)}\frac{\lambda_{k+2}-\alpha}{\lambda_k-\alpha}|\gamma_0|$$

$$\le\frac{1}{4}\left(\frac{3}{2}\right)^j|\gamma_1|+\frac{1}{5}2^j|\gamma_0|\le 2^{j-2}(|\gamma_0|+|\gamma_1|).$$

Thus,

$$\sum_{k=1}^{\infty}\left(\frac{\lambda_{k+1}^j(\lambda_{k+1}-\alpha)|\gamma_1|}{\lambda_{k+1}^2-1}+\frac{\lambda_{k+2}^j(\lambda_{k+2}-\alpha)|\gamma_0|}{\lambda_{k+2}^2-1}\right)|f_k|\le$$

$$\le 2^{j-2}(|\gamma_0|+|\gamma_1|)\sum_{k=1}^{\infty}\lambda_k^j(\lambda_k-\alpha)|f_k|. \tag{12}$$

Also, we have

$$\left(\lambda_1^j(\lambda_1-\alpha)-\frac{\lambda_2^j(\lambda_2-\alpha)|\gamma_1|}{\lambda_2^2-1}\right)|f_1|\le\lambda_1^{j+1}|f_1|$$

$$=\frac{2^{j+1}|\gamma_1|}{3}$$

$$\le 2^j|\gamma_1|$$

and

$$\left(\lambda_1^j(\lambda_1-\alpha)|f_1|+\lambda_2^j(\lambda_2-\alpha)\right)|f_2|\le(\lambda_2-\alpha)\left(\frac{\lambda_1^j|\gamma_1|}{3}+\lambda_2^j\right)\frac{1}{8}\left(\frac{|\gamma_1|^2}{3}+|\gamma_0|\right)$$

$$\le\frac{(2^j|\gamma_1|+3^{j+1})(|\gamma_1|^2+3|\gamma_0|)}{24}.$$

Therefore, from (11) and (12), it follows that

$$\sum_{k=1}^{\infty}\lambda_k^j(\lambda_k-\alpha)|f_k|\le 2^{j-2}(|\gamma_0|+|\gamma_1|)\sum_{k=1}^{\infty}\lambda_k^j(\lambda_k-\alpha)|f_k|++\frac{2^{j+3}|\gamma_1|+(2^j|\gamma_1|+3^{j+1})(|\gamma_1|^2+3|\gamma_0|)}{24}.$$

From condition (10), it follows that $2^{j-2}(|\gamma_0|+|\gamma_1|)<1$, and thus,

$$(1-2^{j-2}(|\gamma_0|+|\gamma_1|))\sum_{k=1}^{\infty}\lambda_k^j(\lambda_k-\alpha)|f_k|\le\frac{2^{j+3}|\gamma_1|+(2^j|\gamma_1|+3^{j+1})(|\gamma_1|^2+3|\gamma_0|)}{24}.$$

Consequently, in view of (10), we obtain

$$\sum_{k=1}^{\infty}\lambda_k^j(\lambda_k-\alpha)|f_k|\le\frac{2^{j+3}|\gamma_1|+(2^j|\gamma_1|+3^{j+1})(|\gamma_1|^2+3|\gamma_0|)}{24(1-2^{j-2}(|\gamma_0|+|\gamma_1|))}\le 1-\alpha,$$

that is, (4) holds, and by Corollary 2.2, $F\in D_j(\alpha)$. □

Acknowledgment

The author thanks the reviewers for their comments.

References

[1] F. Al-Oboudi, On univalent functions defined by generalized Sălăgean operator, *Int. J. Math. Math. Sci.* **27** (2004) 1429–1436.

[2] M. Çağlar, E. Deniz, Initial coefficients for subclasses of bi-univalent functions defined by Sălăgean differential operator, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **66** (2017) 85–91.

[3] O. M. Holovata, O. M. Mulyava, M. M. Sheremeta, Pseudostarlike, pseudoconvex and close-to-pseudoconvex Dirichlet series satisfying differential equations with exponential coefficients, *Mat. Metody Fiz. Mekh. Polya* **61** (2018) 57–70 (In Ukrainian).

[4] S. B. Joshi, N. D. Sangle, New subclasses of univalent functions defined by using the generalized Sălăgean operator, *J. Indones. Math. Soc.* **15** (2009) 79–89.

[5] G. S. Sălăgean, Subclasses of univalent functions, *Lecture Notes Math.* **1013** (1983) 362–372.

[6] M. M. Sheremeta, *Geometric Properties of Analytic Solutions of Differential Equations*, I. E. Chyzhykov, Lviv, 2019.

[7] M. M. Sheremeta, Hadamard compositions of Gelfond–Leont'ev–Sălăgean and Gelfond–Leont'ev–Ruscheweyh derivatives of functions analytic in the unit disc, *Mat. Stud.* **54** (2020) 115–134.

[8] M. Sheremeta, O. Mulyava, M. Medvedev, On the analog of the Sălăgean class for Dirichlet series and the solutions of one differential equation with exponential coefficients, *Ukrainian Math. J.* **76** (2025) 1591–1598.