

Research Article

The A_α -Spectral Radius and Fractional Matchings in GraphsLong Jin[‡]

School of Mathematics, Guangxi University, Nanning, Guangxi, P.R. China

Abstract

A fractional matching of a graph G is a function f that assigns to each edge a value in $[0, 1]$ such that $\sum_{e \in E_G(v)} f(e) \leq 1$ for every vertex $v \in V(G)$, where $E_G(v)$ is the set of edges incident to v . The fractional matching number of G , denoted by $\xi'_*(G)$, is defined as the maximum value of $\sum_{e \in E(G)} f(e)$ taken over all fractional matchings of G . This paper investigates the relationship between the fractional matching number of a graph and the spectral radius of its A_α -matrix. Several sufficient conditions are also established, expressed in terms of the A_α -spectral radii of a graph and its complement, which guarantee the existence of a fractional perfect matching.

Keywords: A_α -spectral radius; fractional matching number; fractional perfect matching.

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1. Introduction

Throughout this paper, all graphs are assumed to be simple and undirected. Let G be a graph with edge set $E(G)$ and vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. The order and size of G are denoted by n and $e(G)$, respectively. That is, $n = |V(G)|$ and $e(G) = |E(G)|$. For $v \in V(G)$, the set of vertices adjacent to v in G is called the neighborhood of v and is denoted by $N_G(v)$. We denote by $d(v) = |N_G(v)|$ the degree of v in G , and by $\delta(G)$ (or δ for short) the minimum degree of G . Given a subset $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S , and by $G - S$ the graph obtained from G by deleting the vertices of S and all their incident edges. The *complement* \bar{G} of G is the graph with the same vertex set $V(G)$ in which two vertices are adjacent if and only if they are nonadjacent in G . For two vertex-disjoint graphs G_1 and G_2 , the *join* $G_1 \vee G_2$ is obtained by adding all possible edges between $V(G_1)$ and $V(G_2)$, while $G_1 \cup G_2$ denotes their disjoint union.

The *adjacency matrix* of G , denoted by $A(G) = [a_{ij}]$, is the $(0, 1)$ -matrix in which

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent in } G, \\ 0, & \text{otherwise.} \end{cases}$$

The *diagonal degree matrix* of G is defined as $D(G) = \text{diag}(d_1, d_2, \dots, d_n)$, where d_i is the degree of v_i in G . The matrix $Q(G) = D(G) + A(G)$ is called the *signless Laplacian matrix* (or the *Q-matrix*) of G . For any real number $\alpha \in [0, 1]$, Nikiforov [9] introduced the A_α -matrix of G as

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G).$$

In particular, $A_0(G) = A(G)$, $A_{1/2}(G) = \frac{1}{2}Q(G)$, and $A_1(G) = D(G)$. For a real symmetric matrix M of order n , let $\rho_1(M)$ denote its *spectral radius*, i.e., the largest eigenvalue of M . We denote by $\lambda_1^0(G) \geq \cdots \geq \lambda_n^0(G)$, $q_1(G) \geq \cdots \geq q_n(G)$ and $\lambda_1^\alpha(G) \geq \cdots \geq \lambda_n^\alpha(G)$ the eigenvalues of $A(G)$, $Q(G)$ and $A_\alpha(G)$, respectively. In particular, $\lambda_1^0(G)$, $q_1(G)$ and $\lambda_1^\alpha(G)$ (or simply λ_1^α) are called the *spectral radius*, *signless Laplacian spectral radius*, and A_α -*spectral radius* of G , respectively.

A set $M \subseteq E(G)$ is called a *matching* if no two edges in M share a common endpoint. A matching of size k is referred to as a k -*matching*, and the maximum cardinality of a matching in G is the *matching number*, denoted by $\xi'(G)$. If every vertex of G is incident with exactly one edge of M , then M is a *perfect matching*. The current literature contains various structural and spectral conditions for the existence of perfect matchings, including those involving toughness [4], neighborhood unions [8], spectral radius [11], and distance spectra [16].

A *fractional matching* of G is a function $f : E(G) \rightarrow [0, 1]$ satisfying

$$\sum_{e \in E_G(v)} f(e) \leq 1 \quad \text{for all } v \in V(G),$$

where $E_G(v)$ is the set of edges incident with v . The *fractional matching number* of G , denoted by $\xi'_*(G)$, is defined as

$$\xi'_*(G) = \max_f \sum_{e \in E(G)} f(e),$$

where the maximum is taken over all fractional matchings of G . A fractional matching is a *fractional perfect matching* if

$$\sum_{e \in E_G(v)} f(e) = 1$$

for all $v \in V(G)$, in which case $\xi'_*(G) = n/2$. Clearly, if f only takes values 0 and 1, it corresponds to an ordinary perfect matching. Moreover, every regular graph admits a fractional perfect matching by assigning each incident edge an appropriate weight. Fractional matchings have also been studied extensively. Lovász and Plummer [7] characterized graphs with fractional perfect matchings, while Liu and Zhang [6] obtained a toughness condition for their existence. Further contributions include lower bounds on $\xi'_*(G)$ and characterizations of extremal graphs [1], sharp comparisons between $\xi'_*(G)$ and $\xi'(G)$ [3], as well as spectral conditions involving the adjacency and signless Laplacian spectral radii [10, 12].

This paper is organized as follows. Section 2 provides several lemmas and definitions to be used in the subsequent sections. Section 3 derives a lower bound on $\xi'_*(G)$ in terms of $\lambda_1^\alpha(G)$ and δ , and characterizes the extremal graphs. Section 4 presents sufficient conditions, via the A_α -spectral radii of G and its complement, for the existence of a fractional perfect matching.

2. Preliminaries

In this section, we present several lemmas and definitions that are used in the rest of the paper.

Lemma 2.1 (see [13]). *For any graph G ,*

$$\xi'_*(G) = \frac{1}{2} \left(n - \max_{S \subseteq V(G)} \{i(G - S) - |S|\} \right),$$

where $i(G - S)$ denotes the number of isolated vertices in $G - S$. The maximum is taken over all subsets $S \subseteq V(G)$.

It follows directly from Lemma 2.1 that $2\xi'_*(G)$ is an integer.

Lemma 2.2 (see [5]). *Let A_1 and A_2 be real nonnegative $n \times n$ matrices such that $A_1 - A_2$ is nonnegative. Then*

$$\rho_1(A_1) \geq \rho_1(A_2),$$

where $\rho_1(A_1)$ and $\rho_1(A_2)$ denote the largest eigenvalues of A_1 and A_2 , respectively.

We next recall several results concerning the A_α -spectral radius of a graph.

Lemma 2.3 (see [9]). *If G is any graph and $\alpha \in [0, 1)$, then*

$$\lambda_1^\alpha(G) \leq \max_{\{u,v\} \in E(G)} \{\alpha d(u) + (1 - \alpha)d(v)\}$$

with equality if and only if G is regular.

Lemma 2.4 (see [9]). *Let K_n be the complete graph of order $n \geq 2$. Then*

$$\lambda_1^\alpha(K_n) = n - 1.$$

Lemma 2.5 (see [9]). *Let G be a connected graph and H a subgraph of G . Then, for $\alpha \in [0, 1)$,*

$$\lambda_1^\alpha(H) \leq \lambda_1^\alpha(G).$$

We now review the notions of equitable matrices and equitable partitions. Given a partition $\pi = (X_1, X_2, \dots, X_k)$ of $\{1, 2, \dots, n\}$ and a matrix M whose rows and columns are indexed by $\{1, 2, \dots, n\}$, we can express M in the block form

$$M = \begin{bmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,k} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ M_{k,1} & M_{k,2} & \cdots & M_{k,k} \end{bmatrix}$$

with respect to π . The *quotient matrix* M_π is the $k \times k$ matrix (m_{ij}) in which m_{ij} is the average of the row sums of $M_{i,j}$. The partition π is said to be *equitable* if each block $M_{i,j}$ has constant row sum m_{ij} . In this case, M_π is called an equitable quotient matrix of M .

Lemma 2.6 (see [2, 5]). *Let M be a real symmetric matrix and let $\rho_1(M)$ denote its largest eigenvalue. If M_π is an equitable quotient matrix of M , then every eigenvalue of M_π is also an eigenvalue of M . Moreover, if M is nonnegative and irreducible, then*

$$\rho_1(M) = \rho_1(M_\pi).$$

Let $\beta_1 \geq \beta_2 \geq \dots \geq \beta_n$ and $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_m$ be two real sequences with $n > m$. The sequence (γ_i) is said to *interlace* (β_i) if $\beta_i \geq \gamma_i \geq \beta_{n-m+i}$, $i = 1, 2, \dots, m$.

Lemma 2.7 (see [15]). *Let M_π be a quotient matrix of a symmetric matrix M corresponding to a given partition. Then the eigenvalues of M_π interlace the eigenvalues of M .*

Following the construction given in [10], we introduce a canonical bipartite family that plays a pivotal role in the characterization of extremal graphs.

Definition 2.8. For integers $\delta \geq 1$ and $0 < k < n$, let $\mathcal{H}(\delta, k)$ denote the family of connected bipartite graphs H with bipartition $V(H) = V_1 \cup V_2$ such that

- (i) every vertex in V_1 has degree δ ;
- (ii) $|V_1| = |V_2| + k$;
- (iii) all vertices in V_2 have the same degree.

The exact values of the fractional matching number and spectral radius for graphs in $\mathcal{H}(\delta, k)$ are given in the next lemma.

Lemma 2.9 (see [10]). *If $H \in \mathcal{H}(\delta, k)$, then*

$$\xi'_*(H) = \frac{|V(H)| - k}{2} \quad \text{and} \quad \lambda_1^0(H) = \delta \sqrt{1 + \frac{2k}{|V(H)| - k}}.$$

We now determine the A_α -spectral radius of graphs in $\mathcal{H}(\delta, k)$.

Lemma 2.10. *Let $H \in \mathcal{H}(\delta, k)$ with $n = |V(H)|$ and $\alpha \in [0, 1)$. Then*

$$\lambda_1^\alpha(H) = \frac{\alpha\delta \cdot \frac{2n}{n-k} + \delta \sqrt{\left(\alpha \cdot \frac{2n}{n-k}\right)^2 - 4(2\alpha - 1) \cdot \frac{n+k}{n-k}}}{2}.$$

Proof. Let $|V_1| = (n + k)/2$ and $|V_2| = (n - k)/2$. The matrix $A_\alpha(H)$ has an equitable partition with respect to (V_1, V_2) , and its quotient is

$$M_\pi = \begin{pmatrix} \alpha\delta & (1 - \alpha)\delta \\ (1 - \alpha)\delta \frac{|V_1|}{|V_2|} & \alpha\delta \frac{|V_1|}{|V_2|} \end{pmatrix} = \begin{pmatrix} \alpha\delta & (1 - \alpha)\delta \\ (1 - \alpha)\delta \frac{n+k}{n-k} & \alpha\delta \frac{n+k}{n-k} \end{pmatrix}.$$

By Lemma 2.6, $\lambda_1^\alpha(H) = \rho(M_\pi)$ equals the larger root of

$$x^2 - \text{tr}(M_\pi)x + \det(M_\pi) = 0,$$

where

$$\text{tr}(M_\pi) = \alpha\delta \left(1 + \frac{n+k}{n-k}\right) = \alpha\delta \cdot \frac{2n}{n-k}$$

and

$$\det(M_\pi) = \delta^2 \left(\alpha^2 - (1 - \alpha)^2\right) \frac{n+k}{n-k} = \delta^2(2\alpha - 1) \frac{n+k}{n-k}.$$

Hence the discriminant is

$$\Delta = \text{tr}(M_\pi)^2 - 4 \det(M_\pi) = \left(\alpha\delta \cdot \frac{2n}{n-k}\right)^2 - 4\delta^2(2\alpha - 1) \frac{n+k}{n-k},$$

and the larger root gives the stated closed form. □

3. A Relationship Between $\lambda_1^\alpha(G)$ and $\xi'_*(G)$

In this section, we examine the relationship between the A_α -spectral radius of a graph with prescribed minimum degree δ and its fractional matching number.

Lemma 3.1. *Let G be a connected graph on n vertices with minimum degree δ , and let $k \in (0, n)$ be a real number. For $\alpha \in [0, 1)$, if*

$$\lambda_1^\alpha(G) < \frac{\alpha\delta \cdot \frac{2n}{n-k} + \delta\sqrt{\left(\alpha \cdot \frac{2n}{n-k}\right)^2 - 4(2\alpha - 1) \cdot \frac{n+k}{n-k}}}{2},$$

then

$$\xi'_*(G) > \frac{n-k}{2}.$$

Proof. Assume to the contrary that

$$\xi'_*(G) \leq \frac{n-k}{2}.$$

By Lemmas 2.1, there exists a vertex set $S \subseteq V(G)$ such that

$$i(G-S) - |S| \geq k > 0,$$

where $i(G-S)$ denotes the number of isolated vertices in $G-S$. Let A be the set of isolated vertices of $G-S$. Then

$$|A| = i(G-S) \geq |S| + \lceil k \rceil \geq 1.$$

Consider the bipartite subgraph H with bipartition $V(H) = A \cup S$, whose edges are those of G joining A and S . Let $r = |E(H)|$. Since every vertex in A has degree at least δ in G and no neighbors in A , we have $r \geq \delta|A|$.

For the partition (A, S) , the quotient matrix of $A_\alpha(H)$ is

$$B_\pi = \begin{pmatrix} \frac{\alpha r}{|A|} & \frac{(1-\alpha)r}{|A|} \\ \frac{(1-\alpha)r}{|S|} & \frac{\alpha r}{|S|} \end{pmatrix}.$$

Its largest eigenvalue is

$$\rho_1(B_\pi) = \frac{\alpha r \left(\frac{1}{|A|} + \frac{1}{|S|} \right) + \sqrt{\left[\alpha r \left(\frac{1}{|A|} + \frac{1}{|S|} \right) \right]^2 - 4(2\alpha - 1) \frac{r^2}{|A||S|}}}{2}.$$

By Lemmas 2.2, 2.6 and 2.7, we have

$$\lambda_1^\alpha(G) \geq \lambda_1^\alpha(H) = \rho_1(B_\pi).$$

Since $r \geq \delta|A|$, $|A| \geq |S| + \lceil k \rceil$, and $|S| \geq \delta$ (because each vertex in A has at least δ neighbors in S), while

$$n \geq |A| + |S| \geq 2|S| + k,$$

we obtain

$$\lambda_1^\alpha(G) \geq \frac{\alpha\delta \cdot \frac{2n}{n-k} + \delta\sqrt{\left(\alpha \cdot \frac{2n}{n-k}\right)^2 - 4(2\alpha - 1) \cdot \frac{n+k}{n-k}}}{2}.$$

This contradicts the assumed upper bound on $\lambda_1^\alpha(G)$. Hence, the assumption is false, and we conclude that $\xi'_*(G) > \frac{n-k}{2}$. \square

Theorem 3.2. *Let G be a connected graph on n vertices with minimum degree δ , and let $\alpha \in [0, 1)$, then*

$$\xi'_*(G) \geq \frac{n\delta(\alpha\lambda_1^\alpha(G) - (2\alpha - 1)\delta)}{(\lambda_1^\alpha(G))^2 - (2\alpha - 1)\delta^2},$$

with equality if and only if either G is regular or $G \in H(\delta, k)$ with

$$k = n \cdot \frac{\left(\frac{\lambda_1^\alpha(G)}{\delta}\right)^2 - 2\alpha\left(\frac{\lambda_1^\alpha(G)}{\delta}\right) + (2\alpha - 1)}{\left(\frac{\lambda_1^\alpha(G)}{\delta}\right)^2 - (2\alpha - 1)}$$

being a positive integer.

Proof. Fix $k \in (0, n)$. By Lemma 3.1, if

$$\lambda_1^\alpha(G) < \frac{\alpha\delta \cdot \frac{2n}{n-k} + \delta\sqrt{\left(\alpha \cdot \frac{2n}{n-k}\right)^2 - 4(2\alpha - 1) \cdot \frac{n+k}{n-k}}}{2},$$

then

$$\xi'_*(G) > \frac{n - k}{2}.$$

Observe that the function

$$F(k) := \frac{\alpha\delta \cdot \frac{2n}{n-k} + \delta\sqrt{\left(\alpha \cdot \frac{2n}{n-k}\right)^2 - 4(2\alpha - 1) \cdot \frac{n+k}{n-k}}}{2}$$

is strictly increasing on the interval $[0, n)$ for fixed $\alpha \in [0, 1)$. Since $\lambda_1^\alpha(G)$ is fixed, there exists a unique $z \in (0, n)$ such that $\lambda_1^\alpha(G) = F(z)$. For every $k > z$, it follows from Lemma 3.1 that $\xi'_*(G) > \frac{n-k}{2}$. Taking the limit as $k \rightarrow z^+$ yields

$$\xi'_*(G) \geq \frac{n - z}{2}.$$

Substituting the expression for z , we find

$$z = n \cdot \frac{\left(\frac{\lambda_1^\alpha(G)}{\delta}\right)^2 - 2\alpha\left(\frac{\lambda_1^\alpha(G)}{\delta}\right) + (2\alpha - 1)}{\left(\frac{\lambda_1^\alpha(G)}{\delta}\right)^2 - (2\alpha - 1)}.$$

Thus,

$$\xi'_*(G) \geq \frac{n\delta(\alpha\lambda_1^\alpha(G) - (2\alpha - 1)\delta)}{(\lambda_1^\alpha(G))^2 - (2\alpha - 1)\delta^2}.$$

For the equality cases, if $k = 0$, then G is regular with $\lambda_1^\alpha(G) = \delta$ and $\xi'_*(G) = \frac{n}{2}$, which matches the formula. If $k \in (0, n)$, then $k = \lceil k \rceil$ must be a positive integer. Moreover, since the inequalities in Lemma 3.1 become equalities in this case, the graph G satisfies

$$r = \delta|A|, \quad |A| = |S| + k, \quad n = 2|S| + k, \quad |S| = \delta,$$

and thus G belongs to the family $\mathcal{H}(\delta, k)$. □

From the definition of the graph matrices, it follows that

$$\lambda_1^{\frac{1}{2}}(G) = \frac{1}{2}q_1(G).$$

Therefore, when $\alpha = \frac{1}{2}$, our result immediately implies a theorem of Pan et al. [12].

Corollary 3.3 (see [12]). *If G is an n -vertex connected graph with minimum degree δ , then*

$$\xi'_*(G) \geq \frac{n\delta}{q_1(G)}$$

with equality if and only if G is a regular graph or an element of $\mathcal{H}(\delta, k)$, where $k = \frac{n(q_1(G) - 2\delta)}{q_1(G)}$ is a positive integer.

Let $g(G)$ denote the girth of G , that is, the length of a shortest cycle in G . Let $\tau(G)$ denote the independence number of G , that is, the size of a largest independent vertex set.

Theorem 3.4. *Let G be a graph with independence number $\tau(G)$. For $\alpha \in [0, 1)$, if $g(G) \geq 5$, then*

$$\lambda_1^\alpha(G) < \begin{cases} 1 + (1 - \alpha)\tau(G), & 0 \leq \alpha \leq \frac{1}{2}, \\ 1 + \alpha\tau(G), & \frac{1}{2} \leq \alpha < 1. \end{cases}$$

Proof. Without loss of generality, assume that G is connected and

$$\alpha d(u_1) + (1 - \alpha)d(v_1) = \max_{\{u,v\} \in E(G)} \{\alpha d(u) + (1 - \alpha)d(v)\}$$

by Lemma 2.3. Set $A = N(u_1) \setminus \{v_1\}$ and $B = N(v_1) \setminus \{u_1\}$. Since $g(G) \geq 5$, there are no cycles of length less than 5, which implies that there are no edges inside A or B , and no edges between A and B . Hence, $A \cup B$ forms an independent set, and thus,

$$|A| + |B| \leq \tau(G).$$

Note that

$$\alpha d(u_1) + (1 - \alpha)d(v_1) = 1 + \alpha|A| + (1 - \alpha)|B| \leq \begin{cases} 1 + (1 - \alpha)\tau(G), & 0 \leq \alpha \leq \frac{1}{2}, \\ 1 + \alpha\tau(G), & \frac{1}{2} \leq \alpha < 1. \end{cases}$$

By Lemma 2.3, it follows that

$$\lambda_1^\alpha(G) \leq \begin{cases} 1 + (1 - \alpha)\tau(G), & 0 \leq \alpha \leq \frac{1}{2}, \\ 1 + \alpha\tau(G), & \frac{1}{2} \leq \alpha < 1. \end{cases}$$

If the equality holds, then $\tau(G) = |A| + |B|$ and G is regular. Without loss of generality, assume that $|A| \geq |B|$ and let $w_1 \in B$. Since G is regular, $d(u_1) = d(w_1) \implies |N(w_1) \setminus \{v_1\}| = |A|$. Observe that $N(w_1) \cup A$ is an independent set. Hence,

$$\tau(G) \geq |N(w_1) \cup A| = 2|A| + 1,$$

which contradicts $\tau(G) = |A| + |B|$. □

Define the function f as

$$f(x) = \frac{n\delta(\alpha x - (2\alpha - 1)\delta)}{x^2 - (2\alpha - 1)\delta^2},$$

where n and δ are positive integers and $\alpha \in [0, 1)$. A direct calculation shows that $f(x)$ is decreasing for $\alpha \in [\frac{1}{2}, 1)$. Combining Theorems 3.2 and 3.4, we obtain the following lower bound for the fractional matching number in terms of the independence number and minimum degree.

Corollary 3.5. *Let G be an n -vertex connected graph with independence number $\tau(G)$ and minimum degree $\delta(G)$. If $g(G) \geq 5$ and $\alpha \in [\frac{1}{2}, 1)$, then*

$$\xi'_*(G) > \frac{n\delta(\alpha^2\tau(G) + \alpha(1 - 2\delta) + \delta)}{\alpha^2\tau(G)^2 + 2\alpha\tau(G) + 1 - (2\alpha - 1)\delta^2}.$$

4. The A_α -Spectral Radius and Fractional Perfect Matching

Some sufficient conditions for the existence of fractional perfect matchings in graphs in terms of the spectral radius were established in [14]. In this section, we present several sufficient conditions for a graph to admit a fractional perfect matching from the perspective of the A_α -spectral radius.

Theorem 4.1. *Let G be a connected graph on n vertices with minimum degree δ . For $\alpha \in [0, 1)$, if*

$$\lambda_1^\alpha(G) < \frac{\delta}{n-1} \left(\alpha n + \sqrt{\alpha^2 n^2 - (2\alpha - 1)(n^2 - 1)} \right),$$

then G admits a fractional perfect matching.

Proof. Suppose that

$$\lambda_1^\alpha(G) < \frac{\alpha\delta \cdot \frac{2n}{n-1} + \delta \sqrt{\left(\alpha \cdot \frac{2n}{n-1}\right)^2 - 4(2\alpha - 1) \cdot \frac{n+1}{n-1}}}{2} = \frac{\delta}{n-1} \left(\alpha n + \sqrt{\alpha^2 n^2 - (2\alpha - 1)(n^2 - 1)} \right).$$

It follows from the argument provided in the proof of Lemma 3.1 that the fractional matching number satisfies the following inequality:

$$\xi'_*(G) > \frac{n-1}{2}.$$

Since $2\xi'_*(G)$ is an integer, we conclude that $\xi'_*(G) = \frac{n}{2}$, which implies that G possesses a fractional perfect matching. □

We now provide a sufficient condition for the existence of a fractional perfect matching in terms of the A_α -spectral radius of the complement graph.

Theorem 4.2. *Let G be an n -vertex connected graph with minimum degree δ . Let \bar{G} be the complement of G . If $\lambda_1^\alpha(\bar{G}) < \delta$, then G admits a fractional perfect matching.*

Proof. Suppose, for the sake of contradiction, that $\xi'_*(G) < \frac{n}{2}$. By Lemma 2.1, there exists a vertex subset $S \subseteq V(G)$ such that $i(G - S) - |S| > 0$, where $i(G - S)$ denotes the number of isolated vertices in the induced subgraph $G - S$. Let A be the set of isolated vertices in $G - S$. Since each isolated vertex in $G - S$ is adjacent only to vertices in S , it follows that $|S| \geq \delta$, and consequently, $|A| \geq |S| + 1 \geq \delta + 1$.

Note that $\bar{G}[A]$, the subgraph of \bar{G} induced by A , forms a clique. By Lemmas 2.4 and 2.5, we have

$$\lambda_1^\alpha(\bar{G}) \geq \lambda_1^\alpha(\bar{G}[A]) = |A| - 1 \geq \delta,$$

which contradicts the assumption. Hence, G must have a fractional perfect matching. □

Theorem 4.3. *Let G be an n -vertex connected graph with minimum degree δ , and let \bar{G} be its complement. If*

$$\lambda_1^\alpha(\bar{G}) < \delta + 1,$$

then G admits a fractional perfect matching unless $G \cong (\delta + 1)K_1 \vee H_1$, where H_1 is a graph of order δ .

Proof. Assume that $\xi'_*(G) < \frac{n}{2}$. By Lemma 2.1, there exists a vertex subset $S \subseteq V(G)$ such that $i(G - S) - |S| > 0$. Let A be the set of isolated vertices in $G - S$. Then $|A| \geq |S| + 1 \geq \delta + 1$. If $|A| \geq \delta + 2$, then \bar{G} contains a clique of order $\delta + 2$, which yields

$$\lambda_1^\alpha(\bar{G}) \geq \delta + 1,$$

a contradiction. Hence, we must have

$$|A| = |S| + 1 = \delta + 1.$$

If $V(G) \neq A \cup S$, then \bar{G} contains a clique of order $\delta + 2$, which leads to a contradiction. Therefore, $V(G) = A \cup S$, and consequently, $G \cong (\delta + 1)K_1 \vee H_1$, where H_1 is a graph of order δ . □

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