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Research Article

Strong Ramsey Numbers of Graphs

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Abstract

The Ramsey number $R(F, H)$ of two graphs F and H without isolated vertices is the minimum positive integer n such that every red-blue coloring of the complete graph K_n of order n results in a subgraph isomorphic to F all of whose edges are colored red or a subgraph isomorphic to H all of whose edges are colored blue. The (diagonal) Ramsey number $R(F) = R(F, F)$ of F is the minimum positive integer n such that every red-blue coloring of K_n results in a subgraph of K_n isomorphic to F all of whose edges are colored the same (a monochromatic F). The strong Ramsey number $\bar{R}(F, H)$ of two graphs F and H without isolated vertices is the minimum positive integer n such that every red-blue coloring of K_n results in two edge-disjoint monochromatic subgraphs, one isomorphic to F and the other isomorphic to H. Strong Ramsey numbers $\bar{R}(F, H)$ have been determined for several pairs F, H of well-known graphs of small size. For many pairs F, H of graphs, it is shown that $\bar{R}(F, H) = \max\{R(F), R(H)\}$ while an example of two graphs F and H is given for which $\bar{R}(F, H) = 1 + \max\{R(F), R(H)\}$, resulting in an open question.

Keywords: red-blue coloring; Ramsey number; strong Ramsey number.

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1. Introduction

For two graphs F and H without isolated vertices, the *Ramsey number* $R(F, H)$ of F and H is the minimum positive integer n such that for every red-blue coloring of the complete graph K_n of order n there exists a subgraph isomorphic to F all of whose edges are colored red (a *red* F) *or* a subgraph isomorphic to H all of whose edges are colored blue (a *blue* H). Thus, $R(F, H) = R(H, F)$. If $F \cong H$, then the (*diagonal*) *Ramsey number* $R(F, H) = R(F, F) = R(F)$ of F is the minimum positive integer n such that every red-blue coloring of K_n results in a subgraph of K_n isomorphic to F all of whose edges are colored the same (a monochromatic F). The following results give the Ramsey numbers of some well-known graphs, namely cycles C_n and paths P_n of order n and stars $K_{1,n}$ and matchings nK_2 of size n.

Theorem 1.1. [7] *For $n = 3, 4$, $R(C_n) = 6$ and for $n \geq 5$,*
$$R(C_n) = \begin{cases} 2n-1 & \text{if } n \text{ is odd} \\ \frac{3n}{2} - 1 & \text{if } n \text{ is even.} \end{cases}$$

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Theorem 1.2. [8] *For* $n \geq 3$, $R(P_n) = \begin{cases} \dfrac{3(n-1)}{2} & \text{if } n \text{ is odd} \\ \dfrac{3n}{2} - 1 & \text{if } n \text{ is even.} \end{cases}$

Theorem 1.3. [2] *For an integer* $n \geq 2$, $R(K_{1,n}) = \begin{cases} 2n & \text{if } n \text{ is odd} \\ 2n-1 & \text{if } n \text{ is even.} \end{cases}$

Theorem 1.4. [5,6] *For every positive integer* n, $R(nK_2) = 3n - 1$.

The known Ramsey numbers of all pairs of small complete graphs are listed below (see [1,10,11], for example).

$$\begin{array}{ccc} R(K_3,K_3)=6 & R(K_3,K_6)=18 & R(K_3,K_9)=36 \\ R(K_3,K_4)=9 & R(K_3,K_7)=23 & R(K_4,K_4)=18 \\ R(K_3,K_5)=14 & R(K_3,K_8)=28 & R(K_4,K_5)=25. \end{array}$$

While the Ramsey number of two graphs F and H without isolated vertices asks for the smallest order of a complete graph for which every red-blue coloring yields either a monochromatic (red) subgraph isomorphic to F *or* a monochromatic (blue) subgraph isomorphic to H, we now consider the smallest order of a complete graph in which every every red-blue coloring results in two edge-disjoint monochromatic subgraphs either of different colors or the same color, one isomorphic to F *and* the other isomorphic to H.

For two graphs F and H without isolated vertices, the *strong Ramsey number* $\bar{R}(F,H)$ of F and H is the minimum positive integer n such that every red-blue coloring of K_n results in two edge-disjoint monochromatic subgraphs one isomorphic to F and the other isomorphic to H. If $F \cong H$, then $\bar{R}(F,F)$ is the *strong Ramsey number* of F, namely the minimum positive integer n such that every red-blue coloring of K_n results in two edge-disjoint monochromatic copies of F (see [3]). Here, we only consider the strong Ramsey number $\bar{R}(F,H)$ where F and H are non-isomorphic graphs. Since the Ramsey number $R(F,H)$ exists for every pair F,H of graphs by a result of Ramsey [12], the strong Ramsey number $\bar{R}(F,H)$ exists as well. It is not surprising that strong Ramsey numbers of graphs are related to their Ramsey numbers.

Proposition 1.5. *For every two graphs* F *and* H *without isolated vertices,* $\bar{R}(F,H)$ *exists and*

$$\max\{R(F),R(H)\} \leq \bar{R}(F,H) \leq \max\{R(F),R(H)\} + \min\{|V(F)|,|V(H)|\}.$$

Proof. Let $\bar{R}(F,H) = n$. To verify the lower bound, for a given red-blue coloring of K_n, there are edge-disjoint monochromatic copies of F and H. Hence, $R(F) \leq n$ and $R(H) \leq n$, which implies that $\max\{R(F),R(H)\} \leq n = \bar{R}(F,H)$.

To verify the upper bound, let n_F be the order of F and n_H the order of H. We may assume that $n_F \leq n_H$. Let $N = \max\{R(F),R(H)\} + n_F$. For a given red-blue coloring of K_N, since $N \geq R(F)$, there is a monochromatic subgraph F in K_N. Then $K_N - V(F) \cong K_{N-n_F}$. Since $N - n_F \geq R(H)$, there is a monochromatic subgraph H in $K_N - V(F)$ that is edge-disjoint from F. Thus, there are edge-disjoint monochromatic subgraphs F and H in K_N and so $\bar{R}(F,H) \leq N$. □

Listed below are some observations dealing with strong Ramsey numbers.

Observation 1.6. *For every two graphs* F *and* H *without isolated vertices,* $\bar{R}(F,H) = \bar{R}(H,F)$.

Observation 1.7. *If* F *is a graph without isolated vertices and* $F \neq K_2$, *then* $\bar{R}(F,K_2) = R(F)$.

Observation 1.8. *If F, F', H, and H' are graphs without isolated vertices such that $F' \subseteq F$ or $H' \subseteq H$, then $\bar{R}(F', H) \le \bar{R}(F, H)$ and $\bar{R}(F, H') \le \bar{R}(F, H)$.*

We refer to the book [4] for notation and terminology not defined here. More information on Ramsey numbers can be found in [9]. We assume that all graphs under consideration have no isolated vertices.

2. Strong Ramsey Numbers $\bar{R}(F, H)$ for Graphs H of Size 2

By Observation 1.7, $\bar{R}(F, K_2) = R(F)$ for every nontrivial graph F. We now determine $\bar{R}(F, H)$ where $|E(F)| \ge |E(H)| = 2$. There are two graphs of size 2 without isolated vertices, namely P_3 and $2K_2$. By Theorems 1.2 and 1.4, $R(P_3) = 3$ and $R(2K_2) = 5$. We first consider the strong Ramsey numbers $\bar{R}(F, P_3)$ where $F \ne P_3$ is a graph of size at least 2 without isolated vertices. First, we present some preliminary results.

Lemma 2.1. *If F is a graph of order $p \ge 3$, then $R(F) \ge p$ and $R(F) = p$ only when $F = P_3$.*

Proof. Since $R(F) \ge p$ and $R(P_3) = 3$, it remains to show that if F is a graph of order $p \ge 3$ without isolated vertices such that $F \ne P_3$, then $R(F) > p$. If F is a star of order $p \ge 4$, then $R(F) > p$ by Theorem 1.3. Thus, we may assume that F is not a star. Since the red-blue coloring of K_p with red subgraph K_{p-1} and blue subgraph $K_{1,p-1}$ does not contain a monochromatic subgraph F, it follows that $R(F) > p$. □

Proposition 2.2. *If F is a graph of order $p \ge 3$ with $F \ne P_3$, then $\bar{R}(F, K_{1,t}) = R(F)$ for every integer t with $1 \le t \le \lceil p/2 \rceil$.*

Proof. Let $R(F) = n$ and $H = K_{1,t}$ where $1 \le t \le \lceil p/2 \rceil$. Since $\bar{R}(F, H) \ge n$ by Proposition 1.5, it remains to show that $\bar{R}(F, H) \le n$. By Observation 1.8, it suffices to show that $\bar{R}(F, H) \le n$ when $H = K_{1,\lceil p/2 \rceil}$. Let there be given a red-blue coloring of $G = K_n$. Since $R(F) = n$, there is a monochromatic subgraph F in G. Since $F \ne P_3$ and the order of F is at least 3, it follows by Lemma 2.1 that $3 \le p \le n-1$. Thus, there is a vertex in $V(G) - V(F)$ that is joined to at least $\lceil p/2 \rceil$ vertices of F by edges having the same color. Hence, G contains a monochromatic subgraph $H = K_{1,\lceil p/2 \rceil}$ that is edge-disjoint from F. Therefore, $\bar{R}(F, H) \le n$ when $H = K_{1,\lceil p/2 \rceil}$. Consequently, $\bar{R}(F, K_{1,t}) \le n$ for each integer t with $1 \le t \le \lceil p/2 \rceil$ by Observation 1.8. □

Lemma 2.1 and Proposition 2.2 provide the following result.

Theorem 2.3. *If F is a graph of order 3 or more with $F \ne P_3$, then $\bar{R}(F, P_3) = R(F)$.*

Proof. Let F be a graph of order $p \ge 3$ such that $F \ne P_3$. Then $R(F) > p$ by Lemma 2.1. Since $\lceil p/2 \rceil \ge \lceil 3/2 \rceil = 2$ and $P_3 = K_{1,2}$, it follows by Proposition 2.2 that $\bar{R}(F, P_3) = R(F)$. □

Next, we determine $\bar{R}(F, 2K_2)$ for all graphs F of order $p \ge 3$ beginning with $3 \le p \le R(F) - 2$.

Theorem 2.4. *If F is a graph of order p where $3 \le p \le R(F) - 2$, then $\bar{R}(F, 2K_2) = R(F)$.*

Proof. Let $R(F) = n \ge 5$. Since $\bar{R}(F, 2K_2) \ge n$ by Proposition 1.5, it remains to show that $\bar{R}(F, 2K_2) \le n$. Let there be given a red-blue coloring of $G = K_n$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Since $R(F) = n$, there is a monochromatic subgraph F in G. We may assume that $V(F) = \{v_1, v_2, \ldots, v_p\}$ where $3 \le p \le n - 2$. Let $H = K_{2,n-2}$ be a complete bipartite subgraph of G with partite sets $\{v_{n-1}, v_n\}$ and $V(G) - \{v_{n-1}, v_n\}$.

Then H is edge-disjoint from F. If either the red subgraph H_r of H or the blue subgraph H_b contains $2K_2$, then there are edge-disjoint monochromatic subgraphs F and $2K_2$ in G. Thus, we may assume that $H_r = K_{1,n-2}$ is centered at v_{n-1} and $H_b = K_{1,n-2}$ is centered at v_n. We may further assume that $v_{n-1}v_n$ is red. Let $G' = G - \{v_{n-1}, v_n\} \cong K_{n-2}$. Since $R(K_p) \geq p + 3$ for each integer $p \geq 3$, it follows that $E(G') - E(F) \neq \emptyset$. If $E(G') - E(F)$ contains a red edge e, then there are edge-disjoint monochromatic subgraphs F and $2K_2$ in G where $2K_2$ is red with $E(2K_2) = \{e, v_{n-1}v_n\}$. Thus, we may assume that every edge in $E(G') - E(F)$ is blue. Let $f = v_iv_j \in E(G') - E(F)$ where $1 \leq i < j \leq n - 2$. Since $n - 2 \geq p \geq 3$, there is an integer k such that $1 \leq k \leq n - 2$ and $k \notin \{i, j\}$. Then there is a blue $2K_2$ in G with $E(2K_2) = \{f, v_kv_n\}$ that is edge-disjoint from F. Thus, there are edge-disjoint monochromatic subgraphs F and $2K_2$ in G. Hence, $\bar{R}(F, 2K_2) \leq n$ and so $\bar{R}(F, 2K_2) = n = R(F)$. □

By Theorem 2.4, if F is a graph of order p such that $3 \leq p \leq R(F) - 2$, then $\bar{R}(F, 2K_2) = R(F)$. By Lemma 2.1, if F is a graph of order $p \geq 3$ and $F \neq P_3$, then $F(F) \geq p + 1$. We now consider graphs F of order $p \geq 3$ and small size such that $R(F) = p + 1$. For two vertex-disjoint graphs F and H, the graph $F + H$ denotes the *union* of F and H. For integers $a, b \geq 2$, the *double star* $S_{a,b}$ is a tree of diameter 3 whose central vertices have degrees a and b.

Proposition 2.5. *If F is a graph of order $p \geq 3$ and size $m \leq 6$ with $F \neq 2K_2$ such that $R(F) = p + 1$, then $\bar{R}(F, 2K_2) = R(F)$.*

Proof. There are 14 graphs F of order $p \geq 4$ and size $m \leq 6$ without isolated vertices such that $R(F) = p + 1$ (see [1], for example). These 14 graphs are $2K_2$, P_4, P_5, $P_3 + K_2$, $S_{2,3}$ (the double star of order 5 whose central vertices have degree 2 or 3), the graph of order 5 obtained by adding a pendant edge at a vertex of C_4, $2P_3$, $K_{1,3} + K_2$, the tree of order 6 obtained by adding two pendant edges at an end-vertex of P_4, $S_{2,4}$ (the double star of order 6 whose central vertices have degree 2 or 4), $K_{1,3} + P_3$, the graph of order 6 obtained by adding two pendant edge at a vertex of C_4, $K_{1,4} + P_3$, and $2K_{1,3}$. These graphs are shown in Figure 2.1. Next, we show that $\bar{R}(F, 2K_2) = R(F)$ for each of those 13 graphs F with $F \neq 2K_2$. By Proposition 1.5, it suffices to show that $\bar{R}(F, 2K_2) \leq R(F)$.

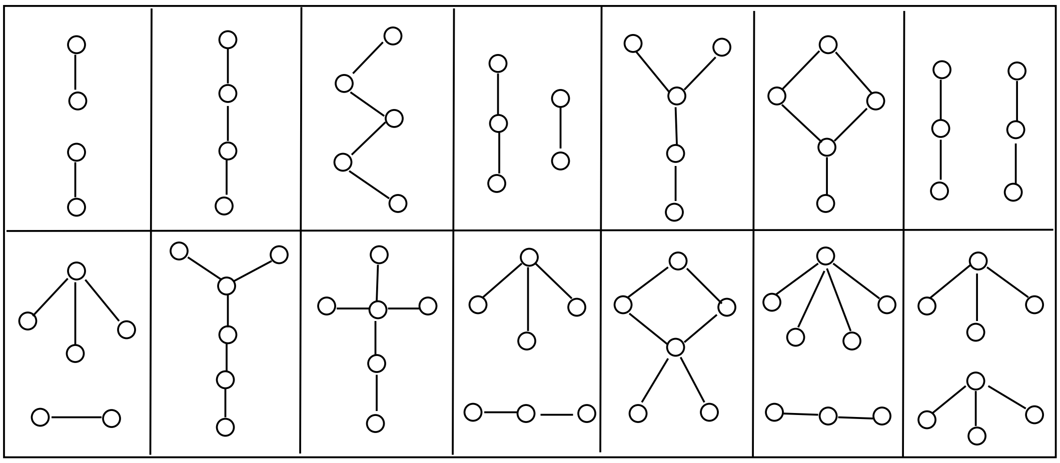

Figure 2.1: The 14 graphs F of order $p \geq 4$ and size $m \leq 6$ with $R(F) = p + 1$.

First, we consider the graph $F = P_4$ of order 4 and Ramsey number 5. Let there be given a red-blue coloring of $G = K_5$ with $V(G) = \{v_1, v_2, \ldots, v_5\}$. Then G contains a monochromatic subgraph P_4. We may assume that $P_4 = (v_1, v_2, v_3, v_4)$. At least two of the four edges v_iv_5 ($1 \le i \le 4$) have the same color, say red. First, suppose that v_1v_5 and v_2v_5 are red. If one of v_1v_3 and v_2v_4 is red, then there is a red $2K_2$ that is edge-disjoint from P_4. If both v_1v_3 and v_2v_4 are blue, then there is a blue $2K_2$ with $E(2K_2) = \{v_1v_3, v_2v_4\}$ that is edge-disjoint from P_4. Next, suppose that v_1v_5 and v_3v_5 are red. We may further assume that v_2v_5 is blue. Then, regardless of the color of v_1v_4, there is a monochromatic $2K_2$ that is edge-disjoint from P_4. Finally, suppose that v_2v_5 and v_3v_5 are red. We may further assume that v_1v_5 is blue. Then, regardless of the color of v_2v_4, there is a monochromatic $2K_2$ that is edge-disjoint from P_4. Therefore, $\bar{R}(P_4, 2K_2) \le 5 = R(P_4)$ and so $\bar{R}(P_4, 2K_2) = R(P_4)$.

In order to show that $\bar{R}(F, 2K_2) \le R(F)$ for the remaining 12 graphs F of order $p \ge 5$, size $m \le 6$, and $R(F) = p + 1$, we first make an observation.

> **Observation:** Let F be a graph with $R(F) = n \ge 6$ and let there be given a red-blue coloring of $G = K_n$. If $G' = K_n - E(F)$ contains three independent edges, then at least two of these three edges have the same color and so G' contains a monochromatic $2K_2$ that is edge-disjoint from F.

For each of the remaining 12 graphs F of order $p \ge 5$ with $R(F) = p+1$, it can be shown that $K_{p+1} - E(F)$ contains three independent edges. It then follows by the observation that for every red-blue coloring of K_{p+1} there are edge-disjoint monochromatic subgraphs of F and $2K_2$. Therefore, $\bar{R}(F, 2K_2) \le R(F)$ and so $\bar{R}(F, 2K_2) = R(F)$. □

The following is a consequence of Theorem 2.4 and Proposition 2.5.

Corollary 2.6. *If F is a graph of order $p \ge 3$ and size $m \le 6$, then $\bar{R}(F, 2K_2) = R(F)$.*

With the aid of known results on complete graphs, paths, cycles, stars, and matchings, the following is a consequence of Theorem 2.4 and Proposition 2.5.

Proposition 2.7. *If F is a complete graph, path, cycle, star, or matching of size at least 3, then $\bar{R}(F, 2K_2) = R(F)$.*

Proof. Let F be a complete graph, path, cycle, star, or matching of order p and size at least 3. If F is neither P_4 nor P_5, then $3 \le p \le R(F) - 2$ and so $\bar{R}(F, 2K_2) = R(F)$ by Theorem 2.4. If $F = P_4$ or $F = P_5$, then $\bar{R}(F, 2K_2) = R(F)$ by Proposition 2.5. □

By Theorem 2.3, if F is a graph of order 3 or more with $F \ne P_3$, then $\bar{R}(F, P_3) = R(F)$. We know of no graph F of order $p \ge 3$ for which $\bar{R}(F, 2K_2) \ne R(F)$. Therefore, we conclude this section with the following conjecture.

Conjecture 2.8. *If F is a graph of order 3 or more such that $F \ne 2K_2$, then $\bar{R}(F, 2K_2) = R(F)$.*

3. Strong Ramsey Numbers $\bar{R}(F, H)$ for Graphs H of Size 3

We now consider $\bar{R}(F, H)$ where $|E(F)| \ge |E(H)| = 3$. If H is a graph of size 3 without isolated vertices, then $H \in \{K_{1,3}, P_4, K_3, P_3 + K_2, 3K_2\}$. Here, $R(P_4) = 5$, $R(K_{1,3}) = R(K_3) = R(P_3 + K_2) = 6$, and $R(3K_2) = 8$. We begin with $H = K_{1,3}$.

Theorem 3.1. *If F is a graph of size 3 or more with $F \neq K_{1,3}$, then*

$$\bar{R}(F, K_{1,3}) = \max\{R(F), R(K_{1,3})\}.$$

Proof. Let $p \geq 3$ be the order of F. First, suppose that $p \geq 5$. Then $R(F) \geq 6$ by Lemma 2.1. Since $R(K_{1,3}) = 6$ and $\bar{R}(F, K_{1,3}) = R(F)$ by Proposition 2.2, it follows that $\bar{R}(F, K_{1,3}) = \max\{R(F), R(K_{1,3})\}$. Next, suppose that $p \in \{3, 4\}$. Let m be the size of F. Then $m \in \{3, 4, 5, 6\}$ and

$$F \in \{P_4, K_3, C_4, K_{1,3} + e, K_4 - e, K_4\}.$$

The Ramsey numbers of these graphs are $R(P_4) = 5$, $R(K_3) = R(C_4) = 6$, $R(K_{1,3}+e) = 7$, $R(K_4-e) = 10$, and $R(K_4) = 18$. By Proposition 1.5, it remains to show that $\bar{R}(F, K_{1,3}) \leq \max\{R(F), R(K_{1,3})\}$.

- ⋆ If $F = P_4$, then $R(F) = 5$ and $\max\{R(P_4), R(K_{1,3})\} = 6$. Let there be given a red-blue coloring of $G = K_6$. Since $R(K_{1,3}) = 6$, there is a monochromatic subgraph $H = K_{1,3}$ in G. Let v be the center of H. Since $G - v \cong K_5$ and $R(P_4) = 5$, there is a monochromatic $F = P_4$ that is edge-disjoint from H. Therefore, $\bar{R}(P_4, K_{1,3}) \leq 6$.

- ⋆ If $F \neq P_4$, then $R(F) = n \geq 6$. Let there be given a red-blue coloring of $G = K_n$. Since $R(F) = n$, there is a monochromatic subgraph F in G. Since the order of F is at most 4, there is a vertex $v \in V(G) - V(F)$. At least three of the $n - 1 \geq 5$ edges incident with v in G have the same color. Hence, there is a monochromatic $K_{1,3}$ in G that is edge-disjoint from F. Therefore, $\bar{R}(F, K_{1,3}) \leq n$.

□

Next, suppose that $H \in \{P_4, K_3, P_3 + K_2, 3K_2\}$. First, we present a lemma dealing with the graphs P_4 and $P_3 + K_2$. For two vertex-disjoint graphs F and H, the graph $F \vee H$ is the *join* of F and H. For a graph G, the graph $\overline{G}$ is the *complement* of G.

Lemma 3.2. *If $H \in \{P_4, P_3 + K_2\}$, then every ed-blue coloring of $K_3 \vee \overline{K}_3$ results in a monochromatic subgraph H.*

Proof. Let $G = K_3 \vee \overline{K}_3$ with $F_1 = K_3$ where $V(F_1) = \{u_1, u_2, u_3\}$ and $F_2 = \overline{K}_3$ where $V(F_2) = \{v_1, v_2, v_3\}$. Since at least two edges in F_1 have the same color, we may assume that u_1u_2 and u_2u_3 are red.

First, suppose that $H = P_4$. If one of u_1v_i $(i = 1, 2, 3)$ is red, say u_1v_1 is red, then (v_1, u_1, u_2, u_3) is a red P_4. Thus, we may assume that each edge u_1v_i $(i = 1, 2, 3)$ is blue. Similarly, each edge u_3v_i $(i = 1, 2, 3)$ is blue. Thus, (v_1, u_1, v_3, u_3) is a blue P_4.

Next, suppose that $H = P_3 + K_2$. At least two of the three edges v_1u_1, v_1u_2, v_1u_3 have the same color. There are two possibilities, depending on the colors of these two edges.

Case 1. *Two of the three edges v_1u_1, v_1u_2, v_1u_3 are blue.* By symmetry, we may assume that either v_1u_1 and v_1u_2 are blue or v_1u_1 and v_1u_3 are blue.

- ⋆ First suppose that v_1u_1 and v_1u_2 are blue. Then v_2u_3 and v_3u_3 are red, for otherwise there is a blue $P_3 + K_2$. Then v_2u_1 is blue, for otherwise $(u_2, u_3, v_3) + (v_2, u_1)$ is a red $P_3 + K_2$. Then u_2v_3 is red, for otherwise $(v_1, u_1, v_2) + (u_2, v_3)$ is a blue $P_3 + K_2$. Then $(u_1, u_2, v_3) + (v_2, u_3)$ is a red $P_3 + K_2$.

- ⋆ Next, suppose that v_1u_1 and v_1u_3 are blue. Then v_2u_2 and v_3u_2 are red, for otherwise there is a blue $P_3 + K_2$. If v_2u_1 or v_3u_3 is red, say the former, then $(v_3, u_2, u_3) + (v_2, u_1)$ is a red $P_3 + K_2$. Thus, we may assume that v_2u_1 and v_3u_2 are blue. Then $(v_1, u_1, v_2) + (v_3, u_3)$ is a blue $P_3 + K_2$.

Case 2. *Two of the three edges v_1u_1, v_1u_2, v_1u_3 are red.* By symmetry, we may assume that either v_1u_1 and v_1u_2 are red or v_1u_1 and v_1u_3 are red.

- ⋆ First, suppose that v_1u_1 and v_1u_2 are red. Then v_2u_3 and v_3u_3 are blue, for otherwise there is a blue $P_3 + K_2$. Assume first that u_1u_3 is red. If v_2u_2 or v_3u_2 is red, say the former, then $(v_1, u_1, u_3) + (v_2, u_2)$ is a red $P_3 + K_2$. Thus, we may assume that v_2u_2 and v_3u_2 are blue. Then v_2u_1 is red, for otherwise $(u_2, v_3, u_3) + (v_2, u_1)$ is a blue $P_3 + K_2$. Thus, $(v_1, u_1, v_2) + (u_2, u_3)$ is a red $P_3 + K_2$. Next, assume that u_1u_3 is blue. If v_2u_2 or v_3u_2 is blue, say the former, then $(v_3, u_3, u_1) + (v_2, u_2)$ is a blue $P_3 + K_2$. Thus, we may assume that v_2u_2 and v_3u_2 are red. Then $(v_2, u_2, v_3) + (v_1, u_1)$ is a red $P_3 + K_2$.

- ⋆ Next, suppose that v_1u_1 and v_1u_3 are red. We may assume that v_1u_2 is blue. Then v_2u_1 and v_3u_1 are blue, for otherwise there is red $P_3 + K_2$. Thus, $(v_2, u_1, v_3) + (v_1, u_2)$ is a blue $P_3 + K_2$.

□

Theorem 3.3. *Let F be a graph of order p with $3 \le p \le R(F) - 3$ and size 3 or more. If $F \ne H$ and $H \in \{P_4, P_3 + K_2\}$, then*

$$\bar{R}(F, H) = \max\{R(F), R(H)\}.$$

Proof. Let $n = \max\{R(F), R(H)\}$ where $F \ne H$ and let $H \in \{P_4, P_3 + K_2\}$. Since $R(P_4) = 5$ and $R(F) \ge 6$ if $F \ne P_4$, it follows that $n \ge 6$. By Proposition 1.5, it remains to show that $\bar{R}(F, H) \le n$. Let there be given a red-blue coloring of $G = K_n$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Since $R(F) \le n$, there is a monochromatic subgraph F of G. We may assume that $V(F) = \{v_1, v_2, \ldots, v_p\}$. Let $G' = G - E(F)$. Since $3 \le p \le n - 3$, it follows that G' contains the complete bipartite subgraph $K_{3,n-3}$ with partite sets $\{v_{n-2}, v_{n-1}, v_n\}$ and $V(G) - \{v_{n-2}, v_{n-1}, v_n\}$. Since $n - 3 \ge p \ge 3$, it follows that $K_3 \vee \overline{K}_3 \subseteq G'$. It then follows by Lemma 3.2 that G' contains a monochromatic subgraph $H \in \{P_4, P_3 + K_2\}$ that is edge-disjoint from F. Therefore, $\bar{R}(F, H) \le n$ and so $\bar{R}(F, H) = n$. □

Theorem 3.4. *If F is a graph of size 3 with $F \ne P_4$, then $\bar{R}(F, P_4) = R(F)$.*

Proof. If F is a graph of size 3 with $F \ne P_4$, then $F \in \{K_3, K_{1,3}, P_3 + K_2, 3K_2\}$. If $F = K_3$, then the order of K_3 is 3 and $R(F) = 6$. Thus, $\bar{R}(K_3, P_4) = R(K_3) = 6$ by Theorem 3.3. If $F = K_{1,3}$, then $\bar{R}(K_{1,3}, P_4) = R(K_{1,3}) = 6$ by Theorem 3.1. Thus, we may assume that $F = P_3 + K_2$ or $F = 3K_2$. By Proposition 1.5, it remains to show that $\bar{R}(F, P_4) \le R(F)$.

- If $F = P_3 + K_2$, then $R(F) = 6$. Let there be given a red-blue coloring of $G = K_6$ with

  $$V(G) = \{v_1, v_2, \ldots, v_6\}.$$

  Since $R(P_3 + K_2) = 6$, there is a monochromatic subgraph $P_3 + K_2$ in G. We may assume that $P_3 + K_2 = (v_1, v_2, v_3) + (v_4, v_5)$. At least two of the three edges v_6v_1, v_6v_3, v_6v_4 have the same color. We may assume that either (i) v_6v_1 and v_6v_3 are red or (ii) v_6v_3 and v_6v_4 are red. If (i) occurs, then every edge in $\{v_1v_4, v_1v_5, v_3v_4, v_3v_5\}$ is blue, for otherwise there is a red P_4 that is edge-disjoint from F. Then (v_4, v_3, v_5, v_1) is a blue P_4 that is edge-disjoint from F. If (ii) occurs, then we may assume that v_6v_1 is blue by (i). Then both v_3v_1 and v_3v_5 are blue, for otherwise there is a red P_4 that is edge-disjoint from F. Then (v_6, v_1, v_3, v_5) is a blue P_4 that is edge-disjoint from F. Therefore, $\bar{R}(P_3 + K_2, P_4) \le 6 = R(P_3 + K_2)$.

- If $F = 3K_2$, then $R(F) = 8$. Let there be given a red-blue coloring of $G = K_8$ with

  $$V(G) = \{v_1, v_2, \ldots, v_8\}.$$

  Since $R(3K_2) = 8$, there is a monochromatic subgraph $F = 3K_2$ in G. We may assume that $E(F) = \{v_1v_2, v_3v_4, v_5v_6\}$. At least two of the three edges v_7v_1, v_7v_3, v_7v_5 have the same color. We may assume that v_7v_1 and v_7v_3 are red. Thus, v_8v_1 and v_8v_3 are blue, for otherwise there is a red P_4 that is edge-disjoint from F. Regardless of the color of v_2v_3, there is a monochromatic P_4 that is edge-disjoint from F. Therefore, $\bar{R}(3K_2, P_4) \leq 8 = R(3K_2)$.

□

Theorem 3.5. *If F is a graph of size 3 with $F \neq 3K_2$, then $\bar{R}(F, 3K_2) = R(3K_2)$.*

Proof. Let F is a graph of size 3 with $F \neq 3K_2$. Then $F \in \{P_4, K_3, K_{1,3}, P_3 + K_2\}$. By Theorems 3.1 and 3.4, we may assume $F = K_3$ or $F = P_3 + K_2$. Since $R(K_3) = R(P_3 + K_2) = 6$ and $R(3K_2) = 8$, it follows by Proposition 1.5 that $\bar{R}(F, 3K_2) \geq 8$. Thus, it remains to show that $\bar{R}(F, 3K_2) \leq 8$. Note that $K_3 \subseteq K_3 + K_2$ and $P_3 + K_2 \subseteq K_3 + K_2$. By Observation 1.8, it suffices to show that $\bar{R}(K_3 + K_2, 3K_2) \leq 8$.

Let there be given a red-blue coloring of $G = K_8$ with $V(G) = \{v_1, v_2, \ldots, v_8\}$. Since $R(K_3 + K_2) = 6$, there is a monochromatic subgraph $F = K_3 + K_2$ in G. We may assume that $F = (v_1, v_2, v_3, v_1) + (v_4, v_5)$. Let $X = G - E(F)$. If three of the four edges $v_1v_5, v_2v_6, v_3v_7, v_4v_8$ in X have the same color, then there is a monochromatic $3K_2$ that is edge-disjoint from F. Thus, we may assume that two of these edges are red and the other two are blue. We may further assume that either (1) v_1v_5, v_2v_6 are red and v_3v_7, v_4v_8 are blue or (2) v_1v_5, v_4v_8 are red and v_2v_6, v_3v_7 are blue. We consider these two cases.

Case 1. *Edges v_1v_5, v_2v_6 are red and v_3v_7, v_4v_8 are blue.* Since v_3v_7 and v_4v_8 are blue, each edge $e \in \{v_1v_6, v_2v_5, v_5v_6\}$ is red, for otherwise there is a blue $3K_2$ with $E(3K_2) = \{e, v_3v_7, v_4v_8\}$. Similarly, every edge in $\{v_3v_4, v_3v_8, v_3v_7, v_4v_7, v_7v_8\}$ is blue. We now consider two subcases, according to the color of v_1v_4.

Subcase 1.1. *v_1v_4 is red.* We proceed with the following steps.

1. Since v_1v_4 and v_2v_5 are red, every edge $e \in \{v_6v_3, v_6v_4, v_6v_7, v_6v_8\}$ is blue, for otherwise there is a red $3K_2$ with $E(3K_2) = \{e, v_1v_4, v_2v_5\}$ in X.

2. The edge v_2v_4 is red, for otherwise there is a blue $3K_2$ with $E(3K_2) = \{v_2v_4, v_3v_7, v_6v_8\}$ in X.

3. The edge v_5v_7 is blue, for otherwise there is a red $3K_2$ with $E(3K_2) = \{v_5v_7, v_1v_4, v_2v_6\}$ in X.

Thus, there is a blue $3K_2$ with $E(3K_2) = \{v_5v_7, v_6v_8, v_3v_4\}$ in X.

Subcase 1.2. *v_1v_4 is blue.* We proceed with the following steps.

1. Since v_1v_4 and v_3v_8 are blue, every edge $e \in \{v_7v_2, v_7v_5, v_7v_6\}$ is red, for otherwise there is a blue $3K_2$ with $E(3K_2) = \{e, v_1v_4, v_3v_8\}$ in X.

2. The ege v_6v_8 is blue, for otherwise there is a red $3K_2$ with $E(3K_2) = \{v_6v_8, v_1v_5, v_2v_7\}$ in X.

Thus, there is a blue $3K_2$ with $E(3K_2) = \{v_1v_4, v_3v_7, v_6v_8\}$ in X.

Case 2. *v_1v_5, v_4v_8 are red and v_2v_6, v_3v_7 are blue.* Since v_1v_5 and v_4v_8 are red, each edge e in $\{v_2v_7, v_3v_6, v_6v_7\}$ is blue, for otherwise there is a red $3K_2$ with $E(3K_2) = \{e, v_1v_5, v_4v_8\}$ in X. Similarly, each edge f in $\{v_1v_4, v_1v_8, v_5v_8\}$ is red, for otherwise there is a blue $3K_2$ with $E(3K_2) = \{f, v_2v_6, v_3v_7\}$ in X. We consider two subcases, depending on the color of v_2v_4.

Subcase 2.1. v_2v_4 *is red.* We proceed with the following steps.

1. Since v_2v_4 and v_1v_8 are red, each edge e in $\{v_2v_5, v_5v_6, v_5v_7\}$ is blue, for otherwise there is a red $3K_2$ with $E(3K_2) = \{e, v_2v_4, v_1v_8\}$ in X.

2. The edge v_7v_8 is red, for otherwise there is a blue $3K_2$ with $E(3K_2) = \{v_7v_8, v_2v_5, v_3v_6\}$ in X.

Thus, there is a red $3K_2$ with $E(3K_2) = \{v_7v_8, v_2v_4, v_1v_5\}$ in X.

Subcase 2.2. v_2v_4 *is blue.* We proceed with the following steps.

1. Since v_2v_4 and v_3v_7 are blue, each edge $e \in \{v_1v_6, v_4v_6, v_5v_6, v_6v_8\}$ is red, for otherwise there is a blue $3K_2$ with $E(3K_2) = \{e, v_2v_4, v_3v_7\}$ in X.

2. Since v_1v_8 and v_4v_6 are red, each edge $f \in \{v_2v_5, v_3v_5, v_5v_7\}$ is blue, for otherwise there is a red $3K_2$ with $E(3K_2) = \{f, v_1v_8, v_4v_6\}$ in X.

Thus, there is a blue $3K_2$ with $E(3K_2) = \{v_2v_4, v_3v_5, v_6v_7\}$ in X.

In either cace, there are edge-disjoint monochromatic subgraphs $K_3 + K_2$ and $3K_2$ in G and so $\bar{R}(K_3 + K_2, 3K_2) \le 8$. Therefore, $\bar{R}(K_3 + K_2, 3K_2) = 8$. Consequently, $\bar{R}(F, 3K_2) = 8$ for $F = K_3$ or $F = P_3 + K_2$. □

Theorem 3.6. *If F is a graph of size 3 and $F \neq P_3 + K_2$, then*

$$\bar{R}(F, P_3 + K_2) = \max\{R(F), R(P_3 + K_2)\}.$$

Proof. Let F be a graph of size 3 and $F \neq P_3 + K_2$. Then $F \in \{P_4, K_3, K_{1,3}, 3K_2\}$. By Theorems 3.1, 3.4 and 3.5, it follows that $\bar{R}(F, P_3 + K_2) = \max\{R(F), R(P_3 + K_2)\}$ if $F \in \{P_4, K_{1,3}, 3K_2\}$. It remains to show that $\bar{R}(K_3, P_3 + K_2) = 6$. By Proposition 1.5, it suffices to show that $\bar{R}(K_3, P_3 + K_2) \le 6$. Let there be given a red-blue coloring of $G = K_6$ with $V(G) = \{v_1, v_2, \ldots, v_6\}$. Since $R(K_3) = 6$, there is a monochromatic subgraph $F = K_3$ in G. We may assume that $F = (v_1, v_2, v_3, v_1)$. Let $G' = G[\{v_4, v_5, v_6\}] = K_3$. At least two of the three edges of G' have the same color, say v_4v_5 and v_5v_6 are red. At least two of the three edges v_4v_1, v_4v_2, v_4v_3 in $G - E(F)$ have the same color, say v_4v_1 and v_4v_2 have the same color. If v_4v_1 and v_4v_2 are red, then $(v_1, v_4, v_2) + (v_5, v_6)$ is a red $P_3 + K_2$ that is edge-disjoint from F. Thus, we may assume that v_4v_1 and v_4v_2 are blue. Then v_3v_5 and v_3v_6 are red, for otherwise, there is a blue $P_3 + K_2$ that is edge-disjoint from F. As with the vertex v_4, at least two of the three edges v_6v_1, v_6v_2, v_6v_3 are blue. Since v_6v_3 is red, it follows that v_6v_1 and v_6v_2 are blue. Regardless of the color of v_2v_5, there is a monochromatic $P_3 + K_2$ that is edge-disjoint from F and so $\bar{R}(K_3, P_3 + K_2) \le 6$. □

The following is a consequence of Theorems 3.1, 3.4 3.5, and 3.6.

Corollary 3.7. *If F and H are graphs of size 3 and $F \neq H$, then*

$$\bar{R}(F, H) = \max\{R(F), R(H)\}.$$

4. Two Strong Ramsey Numbers

For every pair F, H of graphs without isolated vertices that we have considered, it has been shown that $\bar{R}(F, H) = \max\{R(F), R(H)\}$. In particular, this is the case when F and H have size 3 by Corollary 3.7. This is also the case for the two cycles C_3 and C_4 where $R(C_3) = R(C_4) = 6$ as we show next.

Theorem 4.1. $\bar{R}(C_3, C_4) = 6$.

Proof. Since $R(C_3) = R(C_4) = 6$, it follows by Proposition 1.5 that $\bar{R}(C_3, C_4) \geq 6$. Thus, it remains to show that $\bar{R}(C_3, C_4) \leq 6$. Let there be given a red-blue coloring of $G = K_6$ with $V(G) = \{u_1, u_2, u_3, v_1, v_2, v_3\}$. Since $R(K_3) = 6$, there is a monochromatic subgraph $F = K_3$ in G. We may assume that $F = (v_1, v_2, v_3, v_1)$. Let $H = G[\{u_1, u_2, u_3\}] = K_3$. We consider two cases, according to the colors of the edges of H.

Case 1. *H is a monochromatic triangle*. We may assume that H is a red triangle. We consider two subcases, depending on the colors of the edges of the triangle F.

Subcase 1.1. *F is a red triangle.* Thus, F and H are both red triangles here. At least two of the three edges u_1v_i $(i = 1, 2, 3)$ have the same color. If two of these three edges, say u_1v_1 and u_1v_2, are red, then $(u_1, v_1, v_3, v_2, u_1)$ is a red C_4 that is edge-disjoint from H. Suppose that two of the edges u_1v_i $(i = 1, 2, 3)$ are blue. Similarly, at least two of the three edges u_2v_i $(i = 1, 2, 3)$ are blue. If either u_1 or u_2 is joined to all three vertices of F by blue edges, then there is a blue C_4 that is edge-disjoint from H. Thus, we may assume that u_1 and u_2 are joined to two vertices of F by blue edges and one vertex of F by a red edge. Then there is a red C_4 that is edge-disjoint from H.

Subcase 1.2. *F is a blue triangle.* At least two of the three edges u_1v_i $(i = 1, 2, 3)$ have the same color. If two of these three edges are blue, say u_1v_1 and u_1v_2 are blue, then $(u_1, v_1, v_3, v_2, u_1)$ is a blue C_4 that is edge-disjoint from H. Thus, at least two of the edges u_1v_i $(i = 1, 2, 3)$ are red. This is also true for u_2 and u_3. Hence, there is a vertex of F that is adjacent to two vertices of H by red edges. We may assume that u_1v_1 and u_2v_1 are red. Then $(v_1, u_1, u_3, u_2, v_1)$ is a red C_4 that is edge-disjoint from F.

Case 2. *H is not a monochromatic triangle*. Thus, H has two edges of one color and one edge of the other color. We may assume that u_1u_2 and u_2u_3 are red, while u_1u_3 is blue. We consider two subcases, depending on the colors of the edges of the triangle F.

Subcase 2.1. *F is a red triangle.* At least one of v_1u_1 and v_1u_3 is blue, for otherwise $(v_1, u_1, u_2, u_3, v_1)$ is a red C_4 that is edge-disjoint from F. Thus, there are two possibilities, namely (1) both v_1u_1 and v_1u_3 are blue or (2) one of v_1u_1 and v_1u_3 is blue and the other is red. In either case, we may assume that v_1u_3 is blue.

Subcase 2.1.1. *Both v_1u_1 and v_1u_3 are blue.* Then $T = (v_1, u_1, u_3, v_1)$ is a blue triangle.

- ⋆ First, suppose that v_1u_2 is red. Then u_2v_2, u_2v_3, u_3v_2 are blue, for otherwise there is a red C_4 that is edge-disjoint from T. Now consider the edge u_3v_3. If u_3v_3 is red, then $(v_1, u_2, u_3, v_3, v_1)$ is a red C_4 that is edge-disjoint from T. If u_3v_3 is blue, then there is a blue $C_4 = (v_2, u_2, v_3, u_3, v_2)$ that is edge-disjoint from F.

- ⋆ Next, suppose that v_1u_2 is blue. If v_2u_1 and v_2u_3 are both blue, then $(v_1, u_1, v_2, u_3, v_1)$ is a blue C_4 that is edge-disjoint from F. Thus, we may assume that v_2u_1 is red and v_2u_3 is blue. Then v_2u_2 is red for otherwise, $(v_1, u_2, v_2, u_3, v_1)$ is a blue C_4 that is edge-disjoint from F. Furthermore, both u_1v_3 and u_2v_3 are blue, for otherwise there is a red C_4 that is edge-disjoint from T. Now consider the edge u_3v_3. If u_3v_3 is red, then $(v_2, u_2, u_3, v_3, v_2)$ is a red C_4 that is edge-disjoint from T. If u_3v_3 is blue then $(v_1, u_2, v_3, u_3, v_1)$ is a blue C_4 that is edge-disjoint from F.

Subcase 2.1.2. *One of v_1u_1 and v_1u_3 is blue and the other is red, say v_1u_1 is red and v_1u_3 is blue.* By Subcase 2.1.1, we may further assume that for $i = 2, 3$, one of v_iu_1 and v_iu_3 is blue and the other is red. By symmetry, we may assume that either (1) u_1 is joined to all three vertices v_1, v_2, v_3 by red edges or (2) u_1 is joined to exactly two of the three vertices v_1, v_2, v_3 by red edges, say u_1v_1, u_1v_2 are red and u_1v_3 is blue.

- ⋆ First, suppose that u_1 is joined to all three vertices v_1, v_2, v_3 by red edges. Then u_2v_2 is blue, for otherwise the red $C_3 = (v_2, u_1, v_3, v_2)$ and the red $C_4 = (v_1, v_2, u_2, u_1, v_1)$ are edge-disjoint. Now consider the edge u_2v_3. If u_2v_3 is blue, then $(v_2, u_2, v_3, u_3, v_2)$ is a blue C_4 that is edge-disjoint from F. If u_2v_3 is red, then the red $C_3 = (u_1, v_3, u_2, u_1)$ and the red $C_4 = (v_1, u_1, v_2, v_3, v_1)$ are edge-disjoint.

- ⋆ Next, suppose that u_1 is joined to exactly two of the three vertices v_1, v_2, v_3 by red edges, say u_1v_1 and u_1v_2 are red and u_1v_3 is blue. Then u_2v_2 is blue, for otherwise the red $C_3 = (v_1, u_1, v_2, v_1)$ and the red $C_4 = (v_2, u_2, u_3, v_3, v_2)$ are edge-disjoint. Then v_1u_2 is red, for otherwise $(v_1, u_2, v_2, u_3, v_1)$ is a blue C_4 that is edge-disjoint from F. Then the red $C_3 = (v_1, u_1, v_2, v_1)$ and the red

$$C_4 = (v_1, u_2, u_3, v_3, v_1)$$

are edge-disjoint in G.

Subcase 2.2. *F is a blue triangle.* For $i = 1, 2, 3$, at least one of v_iu_1 and v_iu_3 is blue for otherwise, $(v_i, u_1, u_2, u_3, v_i)$ is a red C_4 that is edge-disjoint from $F = C_3$. Thus, there are two possibilities, namely (1) both v_1u_1 and v_1u_3 are blue or (2) one of v_1u_1 and v_1u_3 is blue and the other is red. In either case, we may assume that v_1u_3 is blue.

Subcase 2.2.1. *Both v_1u_1 and v_1u_3 are blue.* Then $T = (v_1, u_1, u_3, v_1)$ is a blue triangle.

- ⋆ First, suppose that v_1u_2 is red. As we mentioned, at least one of v_2u_1 and v_2u_3 is blue. If both v_2u_1 and v_2u_3 are blue, then $(v_1, u_1, v_2, u_3, v_1)$ is a blue C_4 that is edge-disjoint from F. Thus, one of v_2u_1 and v_2u_3 is red and the other is blue. We may assume that v_2u_1 is red and v_2u_3 is blue. Then v_2u_2 is blue, for otherwise, there are a red $C_3 = (v_2, u_2, u_1, v_2)$ and a blue $C_4 = (v_1, u_3, v_2, v_3, v_1)$ that are edge-disjont. Furthermore, u_2v_3 is red, for otherwise, there is a blue $C_4 = (v_1, v_2, u_2, v_3, v_1)$ that is edge-disjoint from T. Now consider the edge v_3u_3. If v_3u_3 is blue, then $(v_2, u_3, v_3, v_1, v_2)$ is a blue C_4 that is edge-disjoint from T. If v_3u_3 is red, there are a red $C_3 = (v_3, u_2, u_3, v_3)$ and a blue $C_4 = (v_1, u_3, v_2, v_3, v_1)$ in G that are edge-disjont.

- ⋆ Next, suppose that v_1u_2 is blue. Again, at least one of v_2u_1 and v_2u_3 is blue. If both v_2u_1 and v_2u_3 are blue, then there is a blue $C_4 = (v_1, u_1, v_2, u_3, v_1)$ that is edge-disjoint from F. Thus, one of v_2u_1 and v_2u_3 is red and the other is blue. We may assume that v_2u_1 is red and v_2u_3 is blue. Consider the color of v_2u_2. If v_2u_2 is blue, then the blue $C_4 = (v_1, u_2, v_2, u_3, v_1)$ is edge-disjoint from F. If v_2u_2 is red, there are a red $C_3 = (v_2, u_1, u_2, v_2)$ and a blue $C_4 = (v_1, v_3, v_2, u_3, v_1)$ in G that are edge-disjont.

Subcase 2.2.2. *One of v_1u_1 and v_1u_3 is blue and the other is red, say v_1u_1 is red and v_1u_3 is blue.* By Subcase 2.2.1, we may further assume that for $i = 2, 3$, one of v_iu_1 and v_iu_3 is blue and the other is red. By symmetry, we may assume that either (1) u_1 is joined to all three vertices v_1, v_2, v_3 by red edges or (2) u_1 is joined to exactly two of the three vertices v_1, v_2, v_3 by red edges, say u_1v_1, u_1v_2 are red and u_1v_3 is blue.

- ⋆ First, suppose that u_1 is joined to all three vertices v_1, v_2, v_3 by red edges. Then u_3 is joined to all three vertices v_1, v_2, v_3 by blue edges. If one of u_2v_2 and u_2v_3 is red, say u_2v_2 is red, then there are a red $C_3 = (v_2, u_1, u_2, v_2)$ and a blue $C_4 = (v_1, u_3, v_2, v_3, v_1)$. Thus, we may assume that u_2v_2 and u_2v_3 are blue. Then the blue $C_4 = (v_2, u_2, v_3, u_3, v_2)$ is edge-disjoint from F.

⋆ Next, suppose that u_1 is joined to exactly two of the three vertices v_1, v_2, v_3 by red edges, say u_1v_1 and u_1v_2 are red and u_1v_3 is blue. Then v_1u_3 and v_2u_3 are blue. If one of v_1u_2 and v_2u_2 is red, say v_1u_2 is red, then there are a red $C_3 = (v_1, u_1, u_2, v_1)$ and a blue $C_4 = (v_1, u_3, v_2, v_3, v_1)$. Hence, we may assume that v_1u_2 and v_2u_2 are blue. Then there is a blue $C_4 = (v_1, u_2, v_2, u_3, v_1)$ that is edge-disjoint from F.

Therefore, for every red-blue coloring of $G = K_6$, there are edge-disjoint monochromatic subgraphs C_3 and C_4. Thus, $\bar{R}(C_3, C_4) \leq 6$ and so $\bar{R}(C_3, C_4) = 6$. □

While $R(C_3) = R(C_4) = 6$ and $\bar{R}(C_3, C_4) = 6$ by Theorem 4.1, it turns out that $R(C_4) = R(P_5) = 6$ as well but $\bar{R}(C_4, P_5) \neq 6$.

Proposition 4.2. $\bar{R}(C_4, P_5) = 7 = 1 + \max\{R(C_4), R(P_5)\}$.

Proof. First, we show that $\bar{R}(C_4, P_5) \geq 7$. Consider the red-blue coloring of K_6 with red subgraph $2K_3$ and blue subgraph $K_{3,3}$. The red subgraph $2K_3$ contains neither P_5 nor C_4. Since $K_{3,3} - E(C_4) \cong S_{3,3}$, the blue subgraph does not contain edge-disjoint subgraphs C_4 and P_5. Therefore, this red-blue coloring of K_6 does not result in edge-disjoint subgraphs C_4 and P_5 and so $\bar{R}(C_4, P_5) \geq 7$.

It remains to show that $\bar{R}(C_4, P_5) \leq 7$. Let there be given a red-blue coloring of $G = K_7$ with $V(G) = \{v_1, v_2, \ldots, v_7\}$. Since $R(C_4) = 6$, there is a monochromatic subgraph $F = C_4$ in G, say

$$F = (v_1, v_2, v_3, v_4, v_1).$$

At least two of the three edges in $G[\{v_5, v_6, v_7\}] \cong K_3$ have the same color. We may assume that (v_5, v_6, v_7) is a red P_3. If every edge joining a vertex in $\{v_5, v_7\}$ and a vertex in $\{v_1, v_2, v_3, v_4\}$ is blue, then there is a blue $P_5 = (v_1, v_5, v_2, v_7, v_3)$ in G that is edge-disjoint from F. Thus, we may assume that there is a red edge joining a vertex in $\{v_5, v_7\}$ and a vertex in $\{v_1, v_2, v_3, v_4\}$, say v_5v_1 is red. Then (v_1, v_5, v_6, v_7) is a red P_4 in G. Hence, we may assume that each of $v_1v_3, v_7v_2, v_7v_3, v_7v_4$ is blue, for otherwise there is a red P_5 in G that is edge-disjoint from F. Then (v_1, v_3, v_7, v_2) is a blue P_4 in G. Hence, we may further assume that each of v_1v_6, v_2v_5, v_2v_6 is red, for otherwise there is a blue P_5 in G that is edge-disjoint from F. Then $(v_7, v_6, v_2, v_5, v_1)$ is a red P_5 in G that is edge-disjoint from F. Therefore, there are edge-disjoint monochromatic subgraphs C_4 and P_5 in G and so $\bar{R}(C_4, P_5) \leq 7$. Therefore, $\bar{R}(C_4, P_5) = 7$. □

5. In Closing

For every two graphs F and H for which $\bar{R}(F, H)$ has been determined, it has been shown that

$$\bar{R}(F, H) = \max\{R(F), R(H)\} \text{ or } \bar{R}(F, H) = 1 + \max\{R(F), R(H)\}.$$

We therefore close with the following question:

Question 1. *Do there exist graphs F and H without isolated vertices for which*

$$\bar{R}(F, H) > 1 + \max\{R(F), R(H)\}?$$

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References

[1] S. A. Burr, Diagonal Ramsey numbers for small graphs, *J. Graph Theory* **7** (1983) 57–67.

[2] S. A. Burr, J. A. Roberts, On Ramsey numbers for stars, *Util. Math.* **4** (1973) 217–220.

[3] G. Chartrand, E. Jent, P. Zhang, Monochromatic subgraphs in graphs, *Contrib. Math.* **9** (2024) 62–68.

[4] G. Chartrand, P. Zhang, *Chromatic Graph Theory*, Second Edition, Chapman & Hall/CRC Press, Boca Raton, 2020.

[5] E. J. Cockayne, P. J. Lorimer, The Ramsey number for stripes, *J. Austral. Math. Soc.* **19** (1975) 252–256.

[6] E. J. Cockayne, P. J. Lorimer, On Ramsey graph numbers for stars and stripes, *Canad. Math. Bull.* **18** (1975) 31–34.

[7] R. J. Faudree, R. H. Schelp, All ramsey numbers for cycles in graphs, *Discrete Math.* **8** (1974) 313–329.

[8] L. Geréncser, A. Gyárfas, On Ramsey-type problems, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **10** (1967) 167–170.

[9] R. L. Graham, B. L. Rothschild, J. H Spencer, *Ramsey Theory*, 2nd Edition, Wiley, New York, 2013.

[10] R. E. Greenwood, A. M. Gleason, Combinatorial relations and chromatic graphs, *Canad. J. Math.* **7** (1955) 1–7.

[11] S. Radzisowski, Small Ramsey numbers, *Electron. J. Combin.* (1994) #DS1, Version 18.

[12] F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.* **30** (1930) 264–286.