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Research Article

Hardy–Hilbert-Type Integral Inequalities With Asymmetric and Inhomogeneous Kernel Functions

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Abstract

This paper presents two new Hardy–Hilbert-type integral inequalities, each associated with a distinct kernel function involving the maximum of two variables. The first kernel function is primarily asymmetric, homogeneous, and governed by three parameters. In particular, its formulation includes a classical symmetric case well known in the literature. The sharpness of the constant factor is established, and several related integral inequalities are also derived. In contrast, the second kernel function is symmetric and depends on a single parameter, but is inhomogeneous, thereby extending the range of applicable inequality forms. Several additional integral inequalities are also obtained.

Keywords: integral inequalities; kernel functions; Hardy–Hilbert integral inequalities; optimality; integral formulas; Hölder integral inequality.

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1. Introduction

Historically, the development of integral inequalities has been strongly influenced by the pioneering work of D. Hilbert and G. H. Hardy. In particular, Hilbert's work in 1908 marked the beginning of a series of important advances in the field, which were subsequently extended and generalized by Hardy in 1925. Since then, such inequalities have become fundamental tools in mathematical analysis, with wide-ranging applications in operator theory, harmonic analysis, and the theory of function spaces; see [4, 5].

The basis of this paper is the so-called Hardy–Hilbert integral inequality, formally stated below. Let $p > 1$, and let $q = \frac{p}{p-1}$ denote the Hölder conjugate of p, i.e., $1/p + 1/q = 1$. Let $f, g : (0, \infty) \to (0, \infty)$ be two functions satisfying the integrability conditions

$$\int_0^\infty f^p(x)\, dx < \infty \quad \text{and} \quad \int_0^\infty g^q(y)\, dy < \infty.$$

Then the following inequality holds:

$$\int_0^\infty \int_0^\infty \frac{1}{x+y} f(x)g(y)dxdy \le \frac{\pi}{\sin(\pi/p)} \left[\int_0^\infty f^p(x)dx\right]^{1/p} \left[\int_0^\infty g^q(y)dy\right]^{1/q}. \tag{1}$$

The double integral in (1) can be written as

$$\int_0^\infty \int_0^\infty k(x, y)f(x)g(y)dxdy,$$

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where k denotes the kernel function $k(x,y) = 1/(x+y)$. Crucial features of this function are its symmetry and homogeneity of degree -1. More precisely, for any $x, y \in (0,\infty)$, we have $k(x,y) = k(y,x)$, and for any $\lambda > 0$ and $x, y \in (0,\infty)$, $k(\lambda x, \lambda y) = \lambda^{-1} k(x,y)$. When $p = 2$, the Hardy–Hilbert integral inequality reduces to the original Hilbert integral inequality, i.e.,

$$\int_0^\infty \int_0^\infty \frac{1}{x+y} f(x)g(y)dxdy \leq \pi \sqrt{\int_0^\infty f^2(x)dx} \sqrt{\int_0^\infty g^2(y)dy}.$$

Refinements and modifications of this inequality have been proposed over the years, often through the introduction of alternative kernel functions or extensions of the underlying functional framework. A comprehensive account of related developments can be found in the excellent survey [2].

In particular, certain inequalities are formulated using kernel functions involving the maximum of the integration variables, namely $\max(x,y)$, thereby giving rise to the class of maximum Hardy–Hilbert-type integral inequalities. One of the most representative examples of such inequalities was established in [5]. In the setting of the Hardy–Hilbert integral inequality, the following inequality holds:

$$\int_0^\infty \int_0^\infty \frac{1}{\max(x,y)} f(x)g(y)dxdy \leq pq \left[\int_0^\infty f^p(x)dx\right]^{1/p} \left[\int_0^\infty g^q(y)dy\right]^{1/q}.$$

The associated kernel function is therefore defined as $k(x,y) = 1/\max(x,y)$. This result has inspired numerous Hardy–Hilbert-type integral inequalities. In particular, a one-parameter extension was introduced in [12]. In [6], a Hardy–Hilbert-type integral inequality with the kernel function

$$k(x,y) = \frac{1}{x+y+\max(x,y)}$$

was studied. Several original one-parameter extensions and functional generalizations of this inequality were subsequently investigated in [9]. Further modifications involving combinations of the maximum and minimum of the integration variables can be found in [1, 3, 5–7, 10, 11].

In this paper, we continue this line of research by investigating two new Hardy–Hilbert-type integral inequalities. The first inequality is associated with the kernel function

$$k(x,y) = \frac{1}{|(\alpha+\beta)y - \alpha x - \beta \max(x,y)|^{1-\gamma}},$$

where α, β, and γ are adjustable parameters. The novelty of this kernel function stems from the power of the absolute value of a linear expression involving both variables, combined with the non-linear contribution of the maximum of the variables. If $\beta \neq 0$, then this kernel function is generally asymmetric because there exist $x_\star, y_\star \in (0,\infty)$ such that $k(x_\star, y_\star) \neq k(y_\star, x_\star)$. This feature distinguishes it from many known maximum-type kernel functions, which are typically symmetric. Such asymmetry enables the derivation of more general integral inequalities that are sensitive to the relative ordering of the variables. Moreover, in the special case $\beta = 0$ and $\alpha = 1$, the kernel function under consideration reduces to the classical form $k(x,y) = |x-y|^{\gamma-1}$. In this case, we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{|x-y|^{1-\gamma}} dxdy \leq 2B\left(\frac{1}{2}(1-\gamma), \gamma\right) \left[\int_0^\infty x^{(1+\gamma)p/2-1} f^p(x)dx\right]^{1/p} \left[\int_0^\infty y^{(1+\gamma)q/2-1} g^q(y)dy\right]^{1/q},$$

where $\gamma \in (0,1)$ and $B(a,b)$ denotes the standard beta function, defined by

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt, \tag{2}$$

with $a, b > 0$; for instance, see [8, Corollary 3]. In this sense, the proposed kernel function may be viewed as a meaningful three-parameter asymmetric generalization of the classical kernel. The same interpretation applies to the corresponding Hardy–Hilbert-type integral inequality, developed in the next section.

Our second Hardy–Hilbert-type integral inequality is associated with the following kernel function:

$$k(x,y) = \frac{1}{1 + 2\alpha\sqrt{x} + 2\alpha\sqrt{y} + \alpha^2 \max(x,y)},$$

where α is an adjustable parameter. This kernel function is clearly symmetric. More importantly, unlike the first main kernel function, it is inhomogeneous. That is, there exists no $\varepsilon \in \mathbb{R}$ such that, for every $\lambda > 0$ and all $x, y \in (0, \infty)$, $k(\lambda x, \lambda y) = \lambda^{\varepsilon} k(x,y)$.

Consequently, it does not possess a simple scaling property, thereby broadening the class of functional inequalities to which it may be applied. Moreover, the structural complexity of this kernel function gives rise to special weighted integral norms of the underlying functions in the corresponding Hardy–Hilbert-type integral inequality, as will be seen later.

Corresponding to each of the above two kernel functions, the main theoretical results are developed in detail, together with supporting propositions. These include new integral inequalities involving only one function. Complete proofs are provided to ensure clarity and mathematical rigor.

The remainder of this paper is organized as follows. Sections 2 and 3 are devoted to the first and second Hardy–Hilbert-type integral inequalities, respectively. Finally, Section 4 concludes the paper.

2. First Hardy–Hilbert-Type Integral Inequality

Before presenting the main Hardy–Hilbert-type integral inequality of this section, we first establish some propositions concerning integral formulas, given in the next subsection.

2.1. Preliminary Propositions

We begin with the following result.

Proposition 2.1.1. *For any $\alpha > 0$, $\beta > -\alpha$, $\gamma, \sigma > 0$ such that $\gamma + \sigma < 1$, and $x \in (0, \infty)$, it holds that*

$$\int_0^{\infty} \frac{y^{-\gamma-\sigma}}{|(\alpha+\beta)y - \alpha x - \beta \max(x,y)|^{1-\gamma}} dy = x^{-\sigma} \left[\frac{1}{(\alpha+\beta)^{1-\gamma}} B(1-\gamma-\sigma, \gamma) + \frac{1}{\alpha^{1-\gamma}} B(\sigma, \gamma) \right],$$

where $B(a,b)$ is the standard beta function defined in Equation (2).

Proof. Using the Chasles integral theorem and standard manipulations on $\max(x,y)$, we obtain

$$\begin{aligned}
&\int_0^{\infty} \frac{y^{-\gamma-\sigma}}{|(\alpha+\beta)y - \alpha x - \beta \max(x,y)|^{1-\gamma}} dy \\
&= \int_0^{x} \frac{y^{-\gamma-\sigma}}{|(\alpha+\beta)y - \alpha x - \beta \max(x,y)|^{1-\gamma}} dy + \int_x^{\infty} \frac{y^{-\gamma-\sigma}}{|(\alpha+\beta)y - \alpha x - \beta \max(x,y)|^{1-\gamma}} dy \\
&= \int_0^{x} \frac{y^{-\gamma-\sigma}}{|(\alpha+\beta)y - \alpha x - \beta x|^{1-\gamma}} dy + \int_x^{\infty} \frac{y^{-\gamma-\sigma}}{|(\alpha+\beta)y - \alpha x - \beta y|^{1-\gamma}} dy \\
&= \frac{1}{(\alpha+\beta)^{1-\gamma}} \int_0^{x} \frac{y^{-\gamma-\sigma}}{(x-y)^{1-\gamma}} dy + \frac{1}{\alpha^{1-\gamma}} \int_x^{\infty} \frac{y^{-\gamma-\sigma}}{(y-x)^{1-\gamma}} dy. \qquad (3)
\end{aligned}$$

Making the change of variables $y = ux$ in the first integral and $y = x/v$ in the second, and recognizing the beta function at particular values of the parameters, we obtain

$$\begin{aligned}
&\frac{1}{(\alpha+\beta)^{1-\gamma}}\int_0^x \frac{y^{-\gamma-\sigma}}{(x-y)^{1-\gamma}}dy+\frac{1}{\alpha^{1-\gamma}}\int_x^\infty \frac{y^{-\gamma-\sigma}}{(y-x)^{1-\gamma}}dy\\
&=\frac{1}{(\alpha+\beta)^{1-\gamma}}\int_0^1 \frac{(ux)^{-\gamma-\sigma}}{(x-ux)^{1-\gamma}}xdu+\frac{1}{\alpha^{1-\gamma}}\int_1^0 \frac{(x/v)^{-\gamma-\sigma}}{(x/v-x)^{1-\gamma}}\left(-\frac{x}{v^2}\right)dv\\
&=\frac{1}{(\alpha+\beta)^{1-\gamma}}x^{-\sigma}\int_0^1 \frac{u^{-\gamma-\sigma}}{(1-u)^{1-\gamma}}du+\frac{1}{\alpha^{1-\gamma}}x^{-\sigma}\int_0^1 \frac{v^{\sigma-1}}{(1-v)^{1-\gamma}}dv\\
&=\frac{1}{(\alpha+\beta)^{1-\gamma}}x^{-\sigma}\int_0^1 u^{(1-\gamma-\sigma)-1}(1-u)^{\gamma-1}du+\frac{1}{\alpha^{1-\gamma}}x^{-\sigma}\int_0^1 v^{\sigma-1}(1-v)^{\gamma-1}dv\\
&=x^{-\sigma}\left[\frac{1}{(\alpha+\beta)^{1-\gamma}}B(1-\gamma-\sigma,\gamma)+\frac{1}{\alpha^{1-\gamma}}B(\sigma,\gamma)\right]. \qquad (4)
\end{aligned}$$

From (3) and (4), the required conclusion follows. □

The following result is analogous to Proposition 2.1.1, but involves integration with respect to y. Due to the asymmetry of the integrand, significant changes have to be taken into account.

Proposition 2.1.2. *For any $\alpha>0$, $\beta>-\alpha$, $\gamma,\sigma>0$ such that $\gamma+\sigma<1$, and $y\in(0,\infty)$, it holds that*

$$\int_0^\infty \frac{x^{-\gamma-\sigma}}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}}dx=y^{-\sigma}\left[\frac{1}{\alpha^{1-\gamma}}B(1-\gamma-\sigma,\gamma)+\frac{1}{(\alpha+\beta)^{1-\gamma}}B(\sigma,\gamma)\right].$$

Proof. We argue in a manner analogous to the proof of Proposition 2.1.1. Using the Chasles integral theorem and standard manipulations on $\max(x,y)$, we obtain

$$\begin{aligned}
&\int_0^\infty \frac{x^{-\gamma-\sigma}}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}}dx\\
&=\int_0^y \frac{x^{-\gamma-\sigma}}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}}dx+\int_y^\infty \frac{x^{-\gamma-\sigma}}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}}dx\\
&=\int_0^y \frac{x^{-\gamma-\sigma}}{|(\alpha+\beta)y-\alpha x-\beta y|^{1-\gamma}}dx+\int_y^\infty \frac{x^{-\gamma-\sigma}}{|(\alpha+\beta)y-\alpha x-\beta x|^{1-\gamma}}dx\\
&=\frac{1}{\alpha^{1-\gamma}}\int_0^y \frac{x^{-\gamma-\sigma}}{(y-x)^{1-\gamma}}dx+\frac{1}{(\alpha+\beta)^{1-\gamma}}\int_y^\infty \frac{x^{-\gamma-\sigma}}{(x-y)^{1-\gamma}}dx.
\end{aligned}$$

Making the change of variables $x=uy$ in the first integral and $x=y/v$ in the second, and recognizing the beta function at particular values of the parameters, we have

$$\begin{aligned}
&\frac{1}{\alpha^{1-\gamma}}\int_0^y \frac{x^{-\gamma-\sigma}}{(y-x)^{1-\gamma}}dx+\frac{1}{(\alpha+\beta)^{1-\gamma}}\int_y^\infty \frac{x^{-\gamma-\sigma}}{(x-y)^{1-\gamma}}dx\\
&=\frac{1}{\alpha^{1-\gamma}}\int_0^1 \frac{(uy)^{-\gamma-\sigma}}{(y-uy)^{1-\gamma}}ydu+\frac{1}{(\alpha+\beta)^{1-\gamma}}\int_1^0 \frac{(y/v)^{-\gamma-\sigma}}{(y/v-y)^{1-\gamma}}\left(-\frac{y}{v^2}\right)dv\\
&=\frac{1}{\alpha^{1-\gamma}}y^{-\sigma}\int_0^1 \frac{u^{-\gamma-\sigma}}{(1-u)^{1-\gamma}}du+\frac{1}{(\alpha+\beta)^{1-\gamma}}y^{-\sigma}\int_0^1 \frac{v^{\sigma-1}}{(1-v)^{1-\gamma}}dv\\
&=\frac{1}{\alpha^{1-\gamma}}y^{-\sigma}\int_0^1 v^{(1-\gamma-\sigma)-1}(1-v)^{\gamma-1}dv+\frac{1}{(\alpha+\beta)^{1-\gamma}}y^{-\sigma}\int_0^1 v^{\sigma-1}(1-v)^{\gamma-1}dv\\
&=y^{-\sigma}\left[\frac{1}{\alpha^{1-\gamma}}B(1-\gamma-\sigma,\gamma)+\frac{1}{(\alpha+\beta)^{1-\gamma}}B(\sigma,\gamma)\right].
\end{aligned}$$

Therefore, we have

$$\int_0^{\infty} \frac{x^{-\gamma-\sigma}}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}}dx = y^{-\sigma}\left[\frac{1}{\alpha^{1-\gamma}}B(1-\gamma-\sigma,\gamma)+\frac{1}{(\alpha+\beta)^{1-\gamma}}B(\sigma,\gamma)\right],$$

which completes the proof. □

The proposition below will be crucial in determining a useful value of σ for the main integral inequality.

Proposition 2.1.3. *For any $\alpha > 0$, $\beta > -\alpha$, $\gamma \in (0,1)$, if $\sigma = (1-\gamma)/2$ then, for any $x, y \in (0,\infty)$, the following identity holds:*

$$x^{\sigma}\int_0^{\infty} \frac{y^{-\gamma-\sigma}}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}}dy = y^{\sigma}\int_0^{\infty} \frac{x^{-\gamma-\sigma}}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}}dx.$$

Proof. We note that $1-\gamma-\sigma = 1-\gamma-\frac{1}{2}(1-\gamma) = \frac{1}{2}(1-\gamma) = \sigma$, and hence,

$$B(1-\gamma-\sigma,\gamma) = B(\sigma,\gamma) = B\left(\frac{1}{2}(1-\gamma),\gamma\right).$$

It follows from Propositions 2.1.1 and 2.1.2 that

$$\begin{aligned} x^{\sigma}\int_0^{\infty} \frac{y^{-\gamma-\sigma}}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}}dy &= \frac{1}{(\alpha+\beta)^{1-\gamma}}B(1-\gamma-\sigma,\gamma)+\frac{1}{\alpha^{1-\gamma}}B(\sigma,\gamma) \\ &= \left[\frac{1}{\alpha^{1-\gamma}}+\frac{1}{(\alpha+\beta)^{1-\gamma}}\right]B\left(\frac{1}{2}(1-\gamma),\gamma\right) \\ &= \frac{1}{\alpha^{1-\gamma}}B(1-\gamma-\sigma,\gamma)+\frac{1}{(\alpha+\beta)^{1-\gamma}}B(\sigma,\gamma) \\ &= y^{\sigma}\int_0^{\infty} \frac{x^{-\gamma-\sigma}}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}}dx. \end{aligned}$$

This completes the proof of Proposition 2.1.3. □

Proposition 2.1.3 thus identifies a value of σ that balances two integral terms, one depending on x and the other on y. This will play a key role in the development of the main integral inequality.

2.2. Main Result of Section 2

The theorem below presents the main Hardy–Hilbert-type integral inequality of Section 2. We recall that this inequality deals with the asymmetric kernel function $k(x,y) = |(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{\gamma-1}$, where α, β, and γ are adjustable parameters.

Theorem 2.2.1. *Let $p > 1$, $q = p/(p-1)$, $\alpha > 0$, $\beta > -\alpha$, $\gamma \in (0,1)$, and $f, g : (0,\infty) \to (0,\infty)$ such that*

$$\int_0^{\infty} x^{(1+\gamma)p/2-1}f^p(x)dx < \infty, \quad \int_0^{\infty} y^{(1+\gamma)q/2-1}g^q(y)dy < \infty.$$

Then the following inequality holds:

$$\begin{aligned} &\int_0^{\infty}\int_0^{\infty} \frac{1}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}}f(x)g(y)dxdy \\ &\leq K_{\alpha,\beta,\gamma}\left[\int_0^{\infty} x^{(1+\gamma)p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{\infty} y^{(1+\gamma)q/2-1}g^q(y)dy\right]^{1/q}, \end{aligned}$$

where

$$K_{\alpha,\beta,\gamma} = \left[\frac{1}{\alpha^{1-\gamma}}+\frac{1}{(\alpha+\beta)^{1-\gamma}}\right]B\left(\frac{1}{2}(1-\gamma),\gamma\right). \tag{5}$$

Proof. By appropriately decomposing the integrand, taking into account the identity $1/p + 1/q = 1$, using Proposition 2.1.3 (whose role will be explained in greater detail later), and applying Hölder's integral inequality, we obtain

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{1}{|(\alpha+\beta)y - \alpha x - \beta \max(x,y)|^{1-\gamma}} f(x)g(y)dxdy \\
&= \int_0^\infty \int_0^\infty x^{(1+\gamma)/(2q)} y^{-(1+\gamma)/(2p)} \left[\frac{1}{|(\alpha+\beta)y - \alpha x - \beta \max(x,y)|^{1-\gamma}}\right]^{1/p} f(x) \\
&\times x^{-(1+\gamma)/(2q)} y^{(1+\gamma)/(2p)} \left[\frac{1}{|(\alpha+\beta)y - \alpha x - \beta \max(x,y)|^{1-\gamma}}\right]^{1/q} g(y)dxdy \\
&\leq \mathfrak{A}^{1/p}\mathfrak{B}^{1/q}, \qquad (6)
\end{aligned}$$

where $\mathfrak{A}$ and $\mathfrak{B}$ are given by

$$\mathfrak{A} = \int_0^\infty \int_0^\infty x^{(1+\gamma)p/(2q)} \frac{y^{-(1+\gamma)/2}}{|(\alpha+\beta)y - \alpha x - \beta \max(x,y)|^{1-\gamma}} f^p(x)dxdy$$

and

$$\mathfrak{B} = \int_0^\infty \int_0^\infty y^{(1+\gamma)q/(2p)} \frac{x^{-(1+\gamma)/2}}{|(\alpha+\beta)y - \alpha x - \beta \max(x,y)|^{1-\gamma}} g^q(y)dxdy.$$

Proposition 2.1.3 was established a priori with the powers $y^{-1-\sigma}$ and $x^{-1-\sigma}$ in the integrand, where $\sigma = (1-\gamma)/2$, in order to achieve a balance between the integral terms arising in subsequent developments. We now examine the expressions of these terms, beginning with $\mathfrak{A}$. Applying the Fubini-Tonelli integral theorem to exchange the order of integration, applying Proposition 2.1.1 with $\sigma = (1-\gamma)/2$, recognizing the constant $K_{\alpha,\beta,\gamma}$, and noting that $p/q = p-1$, we have

$$\begin{aligned}
\mathfrak{A} &= \int_0^\infty x^{(1+\gamma)p/(2q)} f^p(x) \left[\int_0^\infty \frac{y^{-(1+\gamma)/2}}{|(\alpha+\beta)y - \alpha x - \beta \max(x,y)|^{1-\gamma}} dy\right] dx \\
&= \int_0^\infty x^{(1+\gamma)p/(2q)} f^p(x) \left[x^{-(1-\gamma)/2} K_{\alpha,\beta,\gamma}\right] dx \\
&= K_{\alpha,\beta,\gamma} \int_0^\infty x^{(1+\gamma)(p-1)/2-(1-\gamma)/2} f^p(x)dx \\
&= K_{\alpha,\beta,\gamma} \int_0^\infty x^{(1+\gamma)p/2-1} f^p(x)dx. \qquad (7)
\end{aligned}$$

For the term $\mathfrak{B}$, we proceed in a similar way, but applying Proposition 2.1.2. We thus obtain

$$\begin{aligned}
\mathfrak{B} &= \int_0^\infty y^{(1+\gamma)q/(2p)} g^q(y) \left[\int_0^\infty \frac{x^{-(1+\gamma)/2}}{|(\alpha+\beta)y - \alpha x - \beta \max(x,y)|^{1-\gamma}} dx\right] dy \\
&= \int_0^\infty y^{(1+\gamma)q/(2p)} g^q(y) \left[y^{-(1-\gamma)/2} K_{\alpha,\beta,\gamma}\right] dy \\
&= K_{\alpha,\beta,\gamma} \int_0^\infty y^{(1+\gamma)(q-1)/2-(1-\gamma)/2} g^q(y)dy \\
&= K_{\alpha,\beta,\gamma} \int_0^\infty y^{(1+\gamma)q/2-1} g^q(y)dy. \qquad (8)
\end{aligned}$$

It follows from (6), (7), and (8), together with the identity $1/p + 1/q = 1$, that

$$\int_0^{\infty}\int_0^{\infty} \frac{1}{|(\alpha+\beta)y - \alpha x - \beta\max(x,y)|^{1-\gamma}} f(x)g(y)dxdy$$
$$\leq \left[K_{\alpha,\beta,\gamma}\int_0^{\infty} x^{(1+\gamma)p/2-1} f^p(x)dx\right]^{1/p}\left[K_{\alpha,\beta,\gamma}\int_0^{\infty} y^{(1+\gamma)q/2-1} g^q(y)dy\right]^{1/q}$$
$$= K_{\alpha,\beta,\gamma}\left[\int_0^{\infty} x^{(1+\gamma)p/2-1} f^p(x)dx\right]^{1/p}\left[\int_0^{\infty} y^{(1+\gamma)q/2-1} g^q(y)dy\right]^{1/q}.$$

This completes the proof of Theorem 2.2.1. □

If we take $\alpha = 1$ and $\beta = 0$, Theorem 2.2.1 yields

$$\int_0^{\infty}\int_0^{\infty} \frac{1}{|x-y|^{1-\gamma}} f(x)g(y)dxdy \leq K_{\alpha,\beta,\gamma}\left[\int_0^{\infty} x^{(1+\gamma)p/2-1} f^p(x)dx\right]^{1/p}\left[\int_0^{\infty} y^{(1+\gamma)q/2-1} g^q(y)dy\right]^{1/q}.$$

where

$$K_{\alpha,\beta,\gamma} = 2B\left(\frac{1}{2}(1-\gamma),\gamma\right).$$

As outlined in the introduction, this corresponds to a standard Hardy–Hilbert-type integral inequality (see [8, Corollary 3]). In the remaining cases, the maximum of the variables in the integrand becomes active, leading to new integral inequalities. In the next subsection, some additional results based on Theorem 2.2.1 are established.

2.3. Additional Results Based on Theorem 2.2.1

The proposition below formalizes the optimality of the constant factor in Theorem 2.2.1.

Proposition 2.3.1. *In the context of Theorem 2.2.1, the constant $K_{\alpha,\beta,\gamma}$ defined by Equation (5) is the optimal one, i.e., it is the smallest possible constant for which the inequality holds for all the functions f and g.*

Proof. Under the assumption that the constant $K_{\alpha,\beta,\gamma}$ in Equation (5) is not optimal, there exists a constant $\delta \in (0, K_{\alpha,\beta,\gamma})$, such that, for any $f, g : [0,\infty) \to [0,\infty)$,

$$\int_0^{\infty}\int_0^{\infty} \frac{1}{|(\alpha+\beta)y - \alpha x - \beta\max(x,y)|^{1-\gamma}} f(x)g(y)dxdy$$
$$\leq \delta\left[\int_0^{\infty} x^{(1+\gamma)p/2-1} f^p(x)dx\right]^{1/p}\left[\int_0^{\infty} y^{(1+\gamma)q/2-1} g^q(y)dy\right]^{1/q}. \quad (9)$$

To derive a contradiction, we consider two special functions. For any $\epsilon \in (0,\infty)$, let $f_\epsilon, g_\epsilon : [0,\infty) \to [0,\infty)$ be defined by

$$f_\epsilon(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ x^{-(1+\gamma)/2-\epsilon/p} & \text{if } x \in [1,\infty), \end{cases}$$

$$g_\epsilon(y) = \begin{cases} 0 & \text{if } y \in [0,1), \\ y^{-(1+\gamma)/2-\epsilon/q} & \text{if } y \in [1,\infty). \end{cases}$$

We now determine the weighted integral norms of the upper bounds associated with these functions, as follows:

$$\int_0^{\infty} x^{(1+\gamma)p/2-1} f_\epsilon^p(x)dx = \int_1^{\infty} x^{(1+\gamma)p/2-1}(x^{-(1+\gamma)/2-\epsilon/p})^p dx = \int_1^{\infty} x^{-\epsilon-1}dx = \left[-\frac{1}{\epsilon}x^{-\epsilon}\right]_{x=1}^{x\to\infty} = \frac{1}{\epsilon}$$

and

$$\int_0^{\infty} y^{(1+\gamma)q/2-1} g_\epsilon^q(y)dy = \int_1^{\infty} y^{(1+\gamma)q/2-1}(y^{-(1+\gamma)/2-\epsilon/q})^q dy = \int_1^{\infty} y^{-\epsilon-1}dy = \left[-\frac{1}{\epsilon}x^{-\epsilon}\right]_{x=1}^{x\to\infty} = \frac{1}{\epsilon}.$$

This, together with the identity $1/p+1/q=1$ and Equation (9), yields

$$\begin{aligned}\delta &= \delta\epsilon \times \frac{1}{\epsilon^{1/p}} \times \frac{1}{\epsilon^{1/q}} = \epsilon\left\{\delta\left[\int_0^{\infty} x^{(1+\gamma)p/2-1} f_\epsilon^p(x)dx\right]^{1/p}\left[\int_0^{\infty} y^{(1+\gamma)q/2-1} g_\epsilon^q(y)dy\right]^{1/q}\right\} \\ &\geq \epsilon\int_0^{\infty}\int_0^{\infty} \frac{1}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}} f_\epsilon(x)g_\epsilon(y)dxdy. \end{aligned} \tag{10}$$

Using the expressions of f_ϵ and g_ϵ, making the change of variables $x=uy$, applying the Fubini-Tonelli integral theorem and using the identity $1/p+1/q=1$, we obtain

$$\begin{aligned} &\int_0^{\infty}\int_0^{\infty} \frac{1}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}} f_\epsilon(x)g_\epsilon(y)dxdy \\ &= \int_1^{\infty}\int_1^{\infty} \frac{1}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}} x^{-(1+\gamma)/2-\epsilon/p} y^{-(1+\gamma)/2-\epsilon/q}dxdy \\ &= \int_1^{\infty}\left[\int_1^{\infty} \frac{1}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}} x^{-(1+\gamma)/2-\epsilon/p}dx\right] y^{-(1+\gamma)/2-\epsilon/q}dy \\ &= \int_1^{\infty}\left[\int_{1/y}^{\infty} \frac{1}{|(\alpha+\beta)y-\alpha uy-\beta\max(uy,y)|^{1-\gamma}} (uy)^{-(1+\gamma)/2-\epsilon/p}(ydu)\right] y^{-(1+\gamma)/2-\epsilon/q}dy \\ &= \int_1^{\infty}\left[\int_{1/y}^{\infty} \frac{1}{|\alpha+\beta-\alpha u-\beta\max(u,1)|^{1-\gamma}} u^{-(1+\gamma)/2-\epsilon/p}du\right] y^{-(1+\epsilon)}dy. \end{aligned} \tag{11}$$

Using the Chasles integral theorem, the Fubini-Tonelli integral theorem, simple integral calculus and the identity $1/p+1/q=1$, we have

$$\begin{aligned} &\int_1^{\infty}\left[\int_{1/y}^{\infty} \frac{1}{|\alpha+\beta-\alpha u-\beta\max(u,1)|^{1-\gamma}} u^{-(1+\gamma)/2-\epsilon/p}du\right] y^{-(1+\epsilon)}dy \\ &= \int_1^{\infty}\left[\int_{1/y}^{1} \frac{1}{|\alpha+\beta-\alpha u-\beta\max(u,1)|^{1-\gamma}} u^{-(1+\gamma)/2-\epsilon/p}du\right] y^{-(1+\epsilon)}dy \\ &+ \int_1^{\infty}\left[\int_{1}^{\infty} \frac{1}{|\alpha+\beta-\alpha u-\beta\max(u,1)|^{1-\gamma}} u^{-(1+\gamma)/2-\epsilon/p}du\right] y^{-(1+\epsilon)}dy \\ &= \int_0^{1}\left[\int_{1/u}^{\infty} y^{-(1+\epsilon)}dy\right] \frac{1}{|\alpha+\beta-\alpha u-\beta\max(u,1)|^{1-\gamma}} u^{-(1+\gamma)/2-\epsilon/p}du \\ &+ \left[\int_{1}^{\infty} \frac{1}{|\alpha+\beta-\alpha u-\beta\max(u,1)|^{1-\gamma}} u^{-(1+\gamma)/2-\epsilon/p}du\right]\left[\int_1^{\infty} y^{-(1+\epsilon)}dy\right] \\ &= \int_0^{1}\left(\frac{1}{\epsilon}u^{\epsilon}\right) \frac{1}{|\alpha+\beta-\alpha u-\beta\max(u,1)|^{1-\gamma}} u^{-(1+\gamma)/2-\epsilon/p}du \\ &+ \frac{1}{\epsilon}\times\left[\int_{1}^{\infty} \frac{1}{|\alpha+\beta-\alpha u-\beta\max(u,1)|^{1-\gamma}} u^{-(1+\gamma)/2-\epsilon/p}du\right] \end{aligned}$$

$$= \frac{1}{\epsilon} \times \left[\int_0^1 \frac{1}{|\alpha+\beta-\alpha u-\beta \max(u,1)|^{1-\gamma}} u^{-(1+\gamma)/2+\epsilon/q} du \right.$$
$$\left. + \int_1^\infty \frac{1}{|\alpha+\beta-\alpha u-\beta \max(u,1)|^{1-\gamma}} u^{-(1+\gamma)/2-\epsilon/p} du \right]. \tag{12}$$

Using Equations (10), (11), and (12), we obtain

$$\delta \geq \int_0^1 \frac{1}{|\alpha+\beta-\alpha u-\beta \max(u,1)|^{1-\gamma}} u^{-(1+\gamma)/2+\epsilon/q} du + \int_1^\infty \frac{1}{|\alpha+\beta-\alpha u-\beta \max(u,1)|^{1-\gamma}} u^{-(1+\gamma)/2-\epsilon/p} du.$$

Since this inequality is valid for every $\epsilon > 0$, we may consider $\epsilon \to 0^+$ in the limit inferior sense. Using Fatou's integral theorem, together with the facts that

$$\liminf_{\epsilon \to 0^+} u^{\epsilon/q} = 1 \quad \text{for } u \in (0,1),$$

and

$$\liminf_{\epsilon \to 0^+} u^{-\epsilon/p} = 1 \quad \text{for } u \in [1,\infty),$$

as well as the Chasles integral theorem and Proposition 2.1.1 with $x = 1$ and $\sigma = (1-\gamma)/2$, we obtain

$$\delta \geq \liminf_{\epsilon \to 0^+} \int_0^1 \frac{1}{|\alpha+\beta-\alpha u-\beta \max(u,1)|^{1-\gamma}} u^{-(1+\gamma)/2+\epsilon/q} du$$
$$+ \liminf_{\epsilon \to 0^+} \int_1^\infty \frac{1}{|\alpha+\beta-\alpha u-\beta \max(u,1)|^{1-\gamma}} u^{-(1+\gamma)/2-\epsilon/p} du$$
$$\geq \int_0^1 \frac{1}{|\alpha+\beta-\alpha u-\beta \max(u,1)|^{1-\gamma}} u^{-(1+\gamma)/2} \left[\liminf_{\epsilon \to 0^+} u^{\epsilon/q} \right] du$$
$$+ \int_1^\infty \frac{1}{|\alpha+\beta-\alpha u-\beta \max(u,1)|^{1-\gamma}} u^{-(1+\gamma)/2} \left[\liminf_{\epsilon \to 0^+} u^{-\epsilon/p} \right] du$$
$$= \int_0^1 \frac{u^{-(1+\gamma)/2}}{|\alpha+\beta-\alpha u-\beta \max(u,1)|^{1-\gamma}} du$$
$$+ \int_1^\infty \frac{u^{-(1+\gamma)/2}}{|\alpha+\beta-\alpha u-\beta \max(u,1)|^{1-\gamma}} du = \int_0^\infty \frac{u^{-(1+\gamma)/2}}{|\alpha+\beta-\alpha u-\beta \max(u,1)|^{1-\gamma}} du$$
$$= \left[\frac{1}{\alpha^{1-\gamma}} + \frac{1}{(\alpha+\beta)^{1-\gamma}} \right] B\left(\frac{1}{2}(1-\gamma), \gamma \right)$$
$$= K_{\alpha,\beta,\gamma},$$

which implies that $\delta \notin (0, K_{\alpha,\beta,\gamma})$, thereby yielding a contradiction. Therefore, $K_{\alpha,\beta,\gamma}$ is optimal. □

The result below is about a one-function integral inequality derived from Theorem 2.2.1.

Proposition 2.3.2. *Let $p > 1$, $q = p/(p-1)$, $\alpha > 0$, $\beta > -\alpha$, $\gamma \in (0,1)$, and $f : (0,\infty) \to (0,\infty)$ such that*

$$\int_0^\infty x^{(1+\gamma)p/2-1} f^p(x) dx < \infty.$$

Then the following inequality holds:

$$\int_0^\infty y^{-[(1+\gamma)q/2-1](p-1)} \left[\int_0^\infty \frac{1}{|(\alpha+\beta)y-\alpha x-\beta \max(x,y)|^{1-\gamma}} f(x) dx \right]^p dy \leq K^p_{\alpha,\beta,\gamma} \int_0^\infty x^{(1+\gamma)p/2-1} f^p(x) dx,$$

where $K_{\alpha,\beta,\gamma}$ is given by Equation (5).

Proof. By standard integral manipulations, we can write

$$\int_0^{\infty} y^{-[(1+\gamma)q/2-1](p-1)} \left[\int_0^{\infty} \frac{1}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}} f(x)dx\right]^p dy$$

$$= \int_0^{\infty} y^{-[(1+\gamma)q/2-1](p-1)} \left[\int_0^{\infty} \frac{1}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}} f(x)dx\right]^{p-1} \times$$

$$\left[\int_0^{\infty} \frac{1}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}} f(x)dx\right] dy$$

$$= \int_0^{\infty} \left[y^{-(1+\gamma)q/2+1} \int_0^{\infty} \frac{1}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}} f(x)dx\right]^{p-1} \times$$

$$\left[\int_0^{\infty} \frac{1}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}} f(x)dx\right] dy$$

$$= \int_0^{\infty}\int_0^{\infty} \frac{1}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}} f(x)g_{\dagger}(y)dxdy, \tag{13}$$

where

$$g_{\dagger}(y) = \left[y^{-(1+\gamma)q/2+1} \int_0^{\infty} \frac{1}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}} f(x)dx\right]^{p-1}.$$

It follows from Theorem 2.2.1 that

$$\int_0^{\infty}\int_0^{\infty} \frac{1}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}} f(x)g_{\dagger}(y)dxdy$$

$$\leq K_{\alpha,\beta,\gamma}\left[\int_0^{\infty} x^{(1+\gamma)p/2-1} f^p(x)dx\right]^{1/p}\left[\int_0^{\infty} y^{(1+\gamma)q/2-1} g_{\dagger}^q(y)dy\right]^{1/q}. \tag{14}$$

Using $q(p-1)=p$, we have

$$\int_0^{\infty} y^{(1+\gamma)q/2-1} g_{\dagger}^q(y)dy$$

$$= \int_0^{\infty} y^{(1+\gamma)q/2-1}\left[y^{-(1+\gamma)q/2+1} \int_0^{\infty} \frac{1}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}} f(x)dx\right]^{q(p-1)} dy$$

$$= \int_0^{\infty} y^{(1+\gamma)q/2-1}\left[y^{-(1+\gamma)q/2+1} \int_0^{\infty} \frac{1}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}} f(x)dx\right]^{p} dy$$

$$= \int_0^{\infty} y^{-[(1+\gamma)q/2-1](p-1)} \left[\int_0^{\infty} \frac{1}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}} f(x)dx\right]^p dy. \tag{15}$$

Let us now set

$$\mathfrak{C} = \int_0^{\infty} y^{-[(1+\gamma)q/2-1](p-1)} \left[\int_0^{\infty} \frac{1}{|(\alpha+\beta)y-\alpha x-\beta\max(x,y)|^{1-\gamma}} f(x)dx\right]^p dy.$$

Using this notation and combining Equations (13), (14), and (15), we obtain

$$\mathfrak{C} \leq K_{\alpha,\beta,\gamma}\left[\int_0^{\infty} x^{(1+\gamma)p/2-1} f^p(x)dx\right]^{1/p} \mathfrak{C}^{1/q}.$$

From this inequality, together with the identity $1/p+1/q=1$, we derive

$$\mathfrak{C}^{1/p} \leq K_{\alpha,\beta,\gamma}\left[\int_0^{\infty} x^{(1+\gamma)p/2-1} f^p(x)dx\right]^{1/p}.$$

Hence,

$$\mathfrak{C} \leq K^p_{\alpha,\beta,\gamma} \int_0^\infty x^{(1+\gamma)p/2-1} f^p(x)dx,$$

which yields

$$\int_0^\infty y^{-[(1+\gamma)q/2-1](p-1)} \left[\int_0^\infty \frac{1}{|(\alpha+\beta)y - \alpha x - \beta\max(x,y)|^{1-\gamma}} f(x)dx\right]^p dy \leq K^p_{\alpha,\beta,\gamma} \int_0^\infty x^{(1+\gamma)p/2-1} f^p(x)dx.$$

This completes the proof of Proposition 2.3.2. □

Let us introduce the following integral operator:

$$T(f)(y) = \int_0^\infty \frac{1}{|(\alpha+\beta)y - \alpha x - \beta\max(x,y)|^{1-\gamma}} f(x)dx.$$

Then, Proposition 2.3.2 gives

$$\int_0^\infty y^{-[(1+\gamma)q/2-1](p-1)} \left[T(f)(y)\right]^p dy \leq K^p_{\alpha,\beta,\gamma} \int_0^\infty x^{(1+\gamma)p/2-1} f^p(x)dx,$$

so that T defines a bounded (equivalently, continuous) operator between suitably weighted integral spaces. This provides one possible application of Section 2's results among many others, including estimates for singular integral operators, the study of weighted norm inequalities, and the analysis of maximal-type functions arising in harmonic analysis.

As a side remark, throughout this section, the term $(\alpha+\beta)^{1-\gamma}$ may be replaced by $|\alpha+\beta|^{1-\gamma}$ under the assumption that $\beta \neq -\alpha$, and $\alpha^{1-\gamma}$ may be replaced by $|\alpha|^{1-\gamma}$ under the assumption that $\alpha \neq 0$.

3. Second Hardy–Hilbert-Type Integral Inequality

Before presenting the second Hardy–Hilbert-type integral inequality, we first establish a particular proposition concerning an integral formula, stated in the next subsection.

3.1. A Preliminary Proposition

Proposition 3.1.1. *For any $\alpha > 0$, it holds that*

$$\int_0^\infty \frac{y^{-1/2}}{1 + 2\alpha\sqrt{x} + 2\alpha\sqrt{y} + \alpha^2\max(x,y)} dy = \frac{1}{\alpha}\log\left\{1 + \frac{2\alpha\sqrt{x}}{[\alpha\sqrt{x}+1]^2}\right\} + \frac{\sqrt{2}}{\alpha\sqrt{\alpha}x^{1/4}}\arctan\left[\frac{\sqrt{2\alpha}x^{1/4}}{\alpha\sqrt{x}+1}\right].$$

Proof. Using the Chasles integral theorem, standard manipulations of $\max(x,y)$, classical rules for primitives and the formula $\arctan(x) + \arctan(1/x) = \pi/2$ for any $x > 0$, we have

$$\begin{aligned}
&\int_0^\infty \frac{y^{-1/2}}{1 + 2\alpha\sqrt{x} + 2\alpha\sqrt{y} + \alpha^2\max(x,y)} dy \\
&= \int_0^x \frac{y^{-1/2}}{1 + 2\alpha\sqrt{x} + 2\alpha\sqrt{y} + \alpha^2\max(x,y)} dy + \int_x^\infty \frac{y^{-1/2}}{1 + 2\alpha\sqrt{x} + 2\alpha\sqrt{y} + \alpha^2\max(x,y)} dy \\
&= \int_0^x \frac{y^{-1/2}}{1 + 2\alpha\sqrt{x} + 2\alpha\sqrt{y} + \alpha^2 x} dy + \int_x^\infty \frac{y^{-1/2}}{1 + 2\alpha\sqrt{x} + 2\alpha\sqrt{y} + \alpha^2 y} dy \\
&= \int_0^x \frac{y^{-1/2}}{[\alpha\sqrt{x}+1]^2 + 2\alpha\sqrt{y}} dy + \int_x^\infty \frac{y^{-1/2}}{2\alpha\sqrt{x} + [\alpha\sqrt{y}+1]^2} dy
\end{aligned}$$

$$= \left[\frac{1}{\alpha}\log\left\{[\alpha\sqrt{x}+1]^2+2\alpha\sqrt{y}\right\}\right]_{y=0}^{y=x} + \left[\frac{\sqrt{2}}{\alpha\sqrt{\alpha}x^{1/4}}\arctan\left[\frac{\alpha\sqrt{y}+1}{\sqrt{2\alpha}x^{1/4}}\right]\right]_{y=x}^{y\to\infty}$$

$$= \frac{1}{\alpha}\log\left\{1+\frac{2\alpha\sqrt{x}}{[\alpha\sqrt{x}+1]^2}\right\} + \frac{\sqrt{2}}{\alpha\sqrt{\alpha}x^{1/4}}\left\{\frac{\pi}{2}-\arctan\left[\frac{\alpha\sqrt{x}+1}{\sqrt{2\alpha}x^{1/4}}\right]\right\}$$

$$= \frac{1}{\alpha}\log\left\{1+\frac{2\alpha\sqrt{x}}{[\alpha\sqrt{x}+1]^2}\right\} + \frac{\sqrt{2}}{\alpha\sqrt{\alpha}x^{1/4}}\arctan\left[\frac{\sqrt{2\alpha}x^{1/4}}{\alpha\sqrt{x}+1}\right].$$

This completes the proof of Proposition 3.1.1. □

If we take $\alpha = 1$ in Proposition 3.1.1, then we obtain

$$\int_0^\infty \frac{y^{-1/2}}{1+2\sqrt{x}+2\sqrt{y}+\max(x,y)}dy = \log\left\{1+\frac{2\sqrt{x}}{[\sqrt{x}+1]^2}\right\} + \frac{\sqrt{2}}{x^{1/4}}\arctan\left[\frac{\sqrt{2}x^{1/4}}{\sqrt{x}+1}\right].$$

In full generality, the result involves two different types of terms: a logarithmic term and an arctangent term. This sophisticated functional structure will play a central role in the formulation of the main result of Section 3.

3.2. Main Result of Section 3

The theorem below presents our second main Hardy–Hilbert-type integral inequality. We recall that it is associated with the kernel function $k(x,y) = (1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y))^{-1}$, where α is an adjustable parameter. We recall that this kernel function is inhomogeneous, a property that is relatively rare among kernel functions of this type.

Theorem 3.2.1. *Let $p > 1$, $q = p/(p-1)$, $\alpha > 0$, and $f, g : (0,\infty) \to (0,\infty)$ such that*

$$\int_0^\infty x^{(p-1)/2}h(x;\alpha)f^p(x)dx < \infty, \quad \int_0^\infty y^{(q-1)/2}h(y;\alpha)g^q(y)dy < \infty, \text{ where}$$

$$h(x;\alpha) = \frac{1}{\alpha}\log\left\{1+\frac{2\alpha\sqrt{x}}{[\alpha\sqrt{x}+1]^2}\right\} + \frac{\sqrt{2}}{\alpha\sqrt{\alpha}x^{1/4}}\arctan\left[\frac{\sqrt{2\alpha}x^{1/4}}{\alpha\sqrt{x}+1}\right].$$

Then the following inequality holds:

$$\int_0^\infty\int_0^\infty \frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)}f(x)g(y)dxdy$$
$$\leq \left[\int_0^\infty x^{(p-1)/2}h(x;\alpha)f^p(x)dx\right]^{1/p}\left[\int_0^\infty y^{(q-1)/2}h(y;\alpha)g^q(y)dy\right]^{1/q}.$$

Proof. Decomposing appropriately the integrand, taking into account the identity $1/p + 1/q = 1$, and using the Hölder integral inequality, we obtain

$$\int_0^\infty\int_0^\infty \frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)}f(x)g(y)dxdy$$
$$= \int_0^\infty\int_0^\infty x^{1/(2q)}y^{-1/(2p)}\left[\frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)}\right]^{1/p}f(x)$$
$$\times\, x^{-1/(2q)}y^{1/(2p)}\left[\frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)}\right]^{1/q}g(y)dxdy \leq \mathfrak{D}^{1/p}\mathfrak{E}^{1/q}, \tag{16}$$

where $\mathfrak{D}$ and $\mathfrak{E}$ are given by

$$\mathfrak{D} = \int_0^{\infty}\int_0^{\infty} x^{p/(2q)} \frac{y^{-1/2}}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)} f^p(x)dxdy$$

and

$$\mathfrak{E} = \int_0^{\infty}\int_0^{\infty} y^{q/(2p)} \frac{x^{-1/2}}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)} g^q(y)dxdy.$$

Let us now examine the expressions for these terms, starting with $\mathfrak{D}$.

Applying the Fubini–Tonelli integral theorem to exchange the order of integration, applying Proposition 3.1.1, and noting that $p/q = p-1$, we obtain

$$\begin{aligned}\mathfrak{D} &= \int_0^{\infty} x^{p/(2q)} f^p(x)\left[\int_0^{\infty} \frac{y^{-1/2}}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)}dy\right]dx \\ &= \int_0^{\infty} x^{(p-1)/2} f^p(x) h(x;\alpha)dx. \end{aligned} \tag{17}$$

For the term $\mathfrak{E}$, we proceed analogously, again using the function h. We obtain

$$\begin{aligned}\mathfrak{E} &= \int_0^{\infty} y^{q/(2p)} g^q(y)\left[\int_0^{\infty} \frac{x^{-1/2}}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)}dx\right]dy \\ &= \int_0^{\infty} y^{(q-1)/2} g^q(y) h(y;\alpha)dy. \end{aligned} \tag{18}$$

Using Equations (16), (17), and (18), we have

$$\begin{aligned}&\int_0^{\infty}\int_0^{\infty} \frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)} f(x)g(y)dxdy \\ &\leq \left[\int_0^{\infty} x^{(p-1)/2} f^p(x)h(x;\alpha)dx\right]^{1/p}\left[\int_0^{\infty} y^{(q-1)/2} g^q(y)h(y;\alpha)dy\right]^{1/q}.\end{aligned}$$

This completes the proof of Theorem 3.2.1. □

Given the relatively small number of intermediate inequalities involved in the proof of Theorem 3.2.1, the resulting inequality may be regarded as effective. The function h therefore plays a crucial role in determining the weights of the integral norms appearing in the upper bound. The complexity of this upper bound stems from the inhomogeneity of the corresponding kernel function. Nevertheless, some tractable bounds for this function can be obtained, as developed in the next subsection.

3.3. Additional Results Related to Theorem 3.2.1

The following proposition considers the setting of Theorem 3.2.1, but with a simpler weight function for the weighted integral norms of the underlying functions.

Proposition 3.3.1. *Let $p > 1$, $q = p/(p-1)$, $\alpha > 0$, and $f, g : (0,\infty) \to (0,\infty)$ such that*

$$\int_0^{\infty} x^{(p-1)/2} k(x;\alpha) f^p(x)dx < \infty, \quad \int_0^{\infty} y^{(q-1)/2} k(y;\alpha) g^q(y)dy < \infty,$$

where

$$k(x;\alpha) = \frac{2}{\alpha}\left\{1 - \frac{\alpha^2 x}{[\alpha\sqrt{x}+1]^2}\right\}.$$

Then the following inequality holds:

$$\int_0^\infty \int_0^\infty \frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)} f(x)g(y)dxdy$$
$$\leq \left[\int_0^\infty x^{(p-1)/2}k(x;\alpha)f^p(x)dx\right]^{1/p}\left[\int_0^\infty y^{(q-1)/2}k(y;\alpha)g^q(y)dy\right]^{1/q}.$$

Proof. It follows from Theorem 3.2.1 that

$$\int_0^\infty \int_0^\infty \frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)} f(x)g(y)dxdy$$
$$\leq \left[\int_0^\infty x^{(p-1)/2}h(x;\alpha)f^p(x)dx\right]^{1/p}\left[\int_0^\infty y^{(q-1)/2}h(y;\alpha)g^q(y)dy\right]^{1/q}, \tag{19}$$

where

$$h(x;\alpha)=\frac{1}{\alpha}\log\left\{1+\frac{2\alpha\sqrt{x}}{[\alpha\sqrt{x}+1]^2}\right\}+\frac{\sqrt{2}}{\alpha\sqrt{\alpha}x^{1/4}}\arctan\left[\frac{\sqrt{2\alpha}x^{1/4}}{\alpha\sqrt{x}+1}\right].$$

Using the following well-known inequalities: $\log(1+x)\leq x$ and $\arctan(x)\leq x$ for $x\geq 0$, and after some mathematical manipulations, we obtain

$$\begin{aligned}
h(x;\alpha) &\leq \frac{1}{\alpha}\times\frac{2\alpha\sqrt{x}}{[\alpha\sqrt{x}+1]^2}+\frac{\sqrt{2}}{\alpha\sqrt{\alpha}x^{1/4}}\times\frac{\sqrt{2\alpha}x^{1/4}}{\alpha\sqrt{x}+1}\\
&= \frac{1}{\alpha}\times\frac{2\alpha\sqrt{x}}{[\alpha\sqrt{x}+1]^2}+\frac{2}{\alpha}\times\frac{1}{\alpha\sqrt{x}+1}\\
&= \frac{2[2\alpha\sqrt{x}+1]}{\alpha[\alpha\sqrt{x}+1]^2}=\frac{2\{[\alpha\sqrt{x}+1]^2-\alpha^2x\}}{\alpha[\alpha\sqrt{x}+1]^2}=\frac{2}{\alpha}\left\{1-\frac{\alpha^2x}{[\alpha\sqrt{x}+1]^2}\right\}\\
&= k(x;\alpha).
\end{aligned} \tag{20}$$

Using Equations (19) and (20), we have

$$\int_0^\infty \int_0^\infty \frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)} f(x)g(y)dxdy$$
$$\leq \left[\int_0^\infty x^{(p-1)/2}h(x;\alpha)f^p(x)dx\right]^{1/p}\left[\int_0^\infty y^{(q-1)/2}h(y;\alpha)g^q(y)dy\right]^{1/q}$$
$$\leq \left[\int_0^\infty x^{(p-1)/2}k(x;\alpha)f^p(x)dx\right]^{1/p}\left[\int_0^\infty y^{(q-1)/2}k(y;\alpha)g^q(y)dy\right]^{1/q}.$$

This concludes the proof of Proposition 3.3.1. □

The significance of Proposition 3.3.1 lies in its use of more tractable weight functions for the weighted integral norms of the underlying functions. We may proceed one step further in this direction, as explained in the following corollary.

Corollary 3.3.2. *Let* $p>1$, $q=p/(p-1)$, $\alpha>0$, *and* $f,g:(0,\infty)\to(0,\infty)$ *such that*

$$\int_0^\infty x^{(p-1)/2}f^p(x)dx<\infty,\qquad \int_0^\infty y^{(q-1)/2}g^q(y)dy<\infty.$$

Then the following inequality holds:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)}dxdy\leq\frac{2}{\alpha}\left[\int_0^\infty x^{(p-1)/2}f^p(x)dx\right]^{1/p}\left[\int_0^\infty y^{(q-1)/2}g^q(y)dy\right]^{1/q}.$$

Proof. It follows from Proposition 3.3.1 that

$$\int_0^\infty \int_0^\infty \frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)} f(x)g(y)dxdy$$
$$\leq \left[\int_0^\infty x^{(p-1)/2}k(x;\alpha)f^p(x)dx\right]^{1/p}\left[\int_0^\infty y^{(q-1)/2}k(y;\alpha)g^q(y)dy\right]^{1/q}, \tag{21}$$

where

$$k(x;\alpha)=\frac{2}{\alpha}\left\{1-\frac{\alpha^2 x}{[\alpha\sqrt{x}+1]^2}\right\}.$$

Since $\alpha^2 x \leq 1+2\alpha\sqrt{x}+\alpha^2 x=[\alpha\sqrt{x}+1]^2$, we have

$$k(x;\alpha)\leq \frac{2}{\alpha}. \tag{22}$$

Using Equations (21) and (22), together with $1/p+1/q=1$, we obtain

$$\int_0^\infty \int_0^\infty \frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)} f(x)g(y)dxdy$$
$$\leq \left[\int_0^\infty x^{(p-1)/2}f^p(x)\frac{2}{\alpha}dx\right]^{1/p}\left[\int_0^\infty y^{(q-1)/2}g^q(y)\frac{2}{\alpha}dy\right]^{1/q}$$
$$=\frac{2}{\alpha}\left[\int_0^\infty x^{(p-1)/2}f^p(x)dx\right]^{1/p}\left[\int_0^\infty y^{(q-1)/2}g^q(y)dy\right]^{1/q},$$

This concludes the proof. □

The bound given in Corollary 3.3.2 is obtained in a rather crude manner, and no claim of sharpness is made.

The following result presents a one-function integral inequality associated with Theorem 3.2.1.

Proposition 3.3.3. *Let $p>1$, $q=p/(p-1)$, $\alpha>0$, and $f:(0,\infty)\to(0,\infty)$ such that*

$$\int_0^\infty x^{(p-1)/2}h(x;\alpha)f^p(x)dx<\infty,$$

where

$$h(x;\alpha)=\frac{1}{\alpha}\log\left\{1+\frac{2\alpha\sqrt{x}}{[\alpha\sqrt{x}+1]^2}\right\}+\frac{\sqrt{2}}{\alpha\sqrt{\alpha}x^{1/4}}\arctan\left[\frac{\sqrt{2\alpha}x^{1/4}}{\alpha\sqrt{x}+1}\right].$$

Then the following inequality holds:

$$\int_0^\infty [y^{(q-1)/2}h(y;\alpha)]^{-(p-1)}\left[\int_0^\infty \frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)}f(x)dx\right]^p dy$$
$$\leq \int_0^\infty x^{(p-1)/2}h(x;\alpha)f^p(x)dx.$$

Proof. An integral decomposition gives

$$\int_0^\infty [y^{(q-1)/2}h(y;\alpha)]^{-(p-1)}\left[\int_0^\infty \frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)}f(x)dx\right]^p dy$$
$$= \int_0^\infty [y^{(q-1)/2}h(y;\alpha)]^{-(p-1)}\left[\int_0^\infty \frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)}f(x)dx\right]^{p-1} \times$$
$$\left[\int_0^\infty \frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)}f(x)dx\right] dy$$
$$= \int_0^\infty \left\{[y^{(q-1)/2}h(y;\alpha)]^{-1}\int_0^\infty \frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)}f(x)dx\right\}^{p-1} \times$$
$$\left[\int_0^\infty \frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)}f(x)dx\right] dy$$
$$= \int_0^\infty\int_0^\infty \frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)}f(x)g_\diamond(y)dxdy, \tag{23}$$

where

$$g_\diamond(y) = \left\{[y^{(q-1)/2}h(y;\alpha)]^{-1}\int_0^\infty \frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)}f(x)dx\right\}^{p-1}.$$

It follows from Theorem 3.2.1 that

$$\int_0^\infty\int_0^\infty \frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)}f(x)g_\diamond(y)dxdy$$
$$\leq \left[\int_0^\infty x^{(p-1)/2}h(x;\alpha)f^p(x)dx\right]^{1/p}\left[\int_0^\infty y^{(q-1)/2}h(y;\alpha)g_\diamond^q(y)dy\right]^{1/q}. \tag{24}$$

Using the identity $q(p-1)=p$, we obtain

$$\int_0^\infty y^{(q-1)/2}h(y;\alpha)g_\diamond^q(y)dy$$
$$= \int_0^\infty y^{(q-1)/2}h(y;\alpha)\left\{[y^{(q-1)/2}h(y;\alpha)]^{-1}\int_0^\infty \frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)}f(x)dx\right\}^{q(p-1)} dy$$
$$= \int_0^\infty y^{(q-1)/2}h(y;\alpha)\left\{[y^{(q-1)/2}h(y;\alpha)]^{-1}\int_0^\infty \frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)}f(x)dx\right\}^{p} dy$$
$$= \int_0^\infty [y^{(q-1)/2}h(y;\alpha)]^{-(p-1)}\left\{\int_0^\infty \frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)}f(x)dx\right\}^{p} dy. \tag{25}$$

Let us now set

$$\mathfrak{F} = \int_0^\infty [y^{(q-1)/2}h(y;\alpha)]^{-(p-1)}\left\{\int_0^\infty \frac{1}{1+2\alpha\sqrt{x}+2\alpha\sqrt{y}+\alpha^2\max(x,y)}f(x)dx\right\}^{p} dy.$$

Using this notation and combining Equations (23), (24), and (25), we obtain

$$\mathfrak{F} \leq \left[\int_0^\infty x^{(p-1)/2}h(x;\alpha)f^p(x)dx\right]^{1/p}\mathfrak{F}^{1/q}. \tag{26}$$

Using $1 - 1/q = 1/p$ in (26), we have

$$\mathfrak{F}^{1/p} \le \left[\int_0^\infty x^{(p-1)/2} h(x;\alpha) f^p(x) dx\right]^{1/p},$$

which implies

$$\mathfrak{F} \le \int_0^\infty x^{(p-1)/2} h(x;\alpha) f^p(x) dx.$$

Therefore,

$$\int_0^\infty [y^{(q-1)/2} h(y;\alpha)]^{-(p-1)} \left[\int_0^\infty \frac{1}{1 + 2\alpha\sqrt{x} + 2\alpha\sqrt{y} + \alpha^2 \max(x,y)} f(x) dx\right]^p dy \le \int_0^\infty x^{(p-1)/2} h(x;\alpha) f^p(x) dx,$$

which completes the proof. □

We conclude this section by remarking that Proposition 3.3.3 is useful in the study of the following integral operator:

$$U(f)(y) = \int_0^\infty \frac{1}{1 + 2\alpha\sqrt{x} + 2\alpha\sqrt{y} + \alpha^2 \max(x,y)} f(x) dx.$$

4. Conclusion

In this paper, we have established two new Hardy–Hilbert-type integral inequalities, each associated with a particular kernel function involving the maximum of two variables. The first inequality, governed by a three-parameter asymmetric and homogeneous kernel function, provides a flexible framework that includes a known symmetric inequality as a special case. The second inequality, involving a symmetric yet inhomogeneous single-parameter kernel function, yields further generalizations that broaden the applicability of the Hardy–Hilbert-type framework. Several new integral inequalities have been derived. In addition, a variety of related inequalities were established, further enhancing the robustness and versatility of the proposed formulations.

Several promising directions for future research naturally arise from this work. One possible extension is the study of analogous inequalities in higher dimensions or on different domains, such as bounded intervals. Another direction is the investigation of discrete counterparts of the proposed inequalities, which may lead to new developments in the theory of series. Furthermore, potential applications in harmonic analysis, fractional calculus, and the study of integral operators on weighted function spaces deserve further investigation.

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