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Research Article

Orthogonal Transformations in Three-Dimensional Space: Axis of Rotation, Rodrigues' Formula, and Applications

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Abstract

Euler's theorem on the existence of an axis of rotation for a three-dimensional orthogonal transformation is revisited. The proof requires elementary linear algebra techniques. Then Rodrigues' formula for any orthogonal transformation in $\mathbb{R}^3$ is established. A link between Rodrigues' formula and the symmetric and skew-symmetric parts of an orthogonal transformation is also established, which can be used to easily determine the angle of rotation associated with the transformation. Moreover, Rodrigues' formula for the composition of two orthogonal transformations is extended. In addition, the distributive property of orthogonal transformations with respect to the cross product is established. Finally, the non-commutativity of orthogonal transformations is characterized.

Keywords: orthogonal transformation; axis of rotation; Rodrigues' formula; non-commutativity; composition; cross-product.

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1. Introduction

A remarkable statement in three dimensions says that every rotation of a sphere about its center has an axis. In 1775, Euler [8] stated and proved this result, and provided a geometric construction to obtain the axis. The problem of finding a formula for a rotation in three dimensions in terms of an axis and an angle of rotation was also raised and solved by Euler [8]. The resulting formula is often called the Rodrigues' rotation formula (see [1,4,5,10,11,18,19]), but it would be more appropriate to name it Euler's rotation formula because it is Euler who was the first to derive it [4,5]. Another Euler's theorem states that the composition of two successive rotations is also a rotation, but it is Rodrigues who derived the formula for the axis and angle of rotation of the resulting composition [4,5,7,16,18]. Those problems have been revisited by several researchers. Historical perspectives on the statement of the problem and suggested solutions can be found, for example, in [4,5,12,15].

The goal of this paper is to extend results for proper rotations to improper rotations, and hence to any orthogonal transformations. A unified treatment of Rodrigues' formula and the composition of orthogonal transformations is provided. Orthogonal transformations are useful in numerous applications. They arise in quantum computation [14], computational geometry [2,13], robotics [3,17], and many other fields.

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The organization of the paper is as follows. Section 2 provides a brief introduction to orthogonal transformations in $\mathbb{R}^3$. Mainly based on elementary linear algebra, Section 3 revisits and extends a proof of Euler's theorem on the existence of an axis of rotation for a three-dimensional orthogonal transformation [15]. The proof is divided into two parts: the non-symmetric case is analyzed in Subsection 3.1, while the symmetric case is studied in Subsection 3.2. Two basic examples are discussed in Subsection 3.3. Section 4 is likewise divided into two parts. Subsection 4.1 introduces some preliminary results concerning vectors and operators in $\mathbb{R}^3$, while Subsection 4.2 presents Rodrigues' formula for an orthogonal transformation in $\mathbb{R}^3$. A connection between Rodrigues' formula and the symmetric and skew-symmetric parts of an orthogonal transformation is also established in Subsection 4.2, which can be used to easily determine the angle of rotation associated with the transformation. Examples related to Sections 3 and 4 are provided in Section 5. Section 6 presents various consequences of Rodrigues' formula. In Subsection 6.1, Rodrigues' formula for the composition of two orthogonal transformations is extended. The distributive property of orthogonal transformations with respect to the cross product is established in Subsection 6.2. Finally, the non-commutativity of orthogonal transformations is discussed in Subsection 6.3.

2. Preliminaries on Orthogonal Transformations

For two vectors $\mathbf{u}$ and $\mathbf{v}$ of $\mathbb{R}^3$, the dot and cross products are respectively denoted by $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$. An orthogonal transformation T in $\mathbb{R}^3$ is a linear transformation such that $\mathrm{T}^*\mathrm{T} = \mathrm{I} = \mathrm{T}\mathrm{T}^*$ where T^* is the adjoint operator of T defined by

$$\mathbf{w} \cdot \mathrm{T}^*(\mathbf{v}) = \mathbf{v} \cdot \mathrm{T}(\mathbf{w})$$

for all $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$. It is a dot (inner) product preserving operator, $\mathrm{T}(\mathbf{v}) \cdot \mathrm{T}(\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$, and a norm preserving operator, $\|\mathrm{T}(\mathbf{v})\| = \|\mathbf{v}\|$. Any orthogonal transformation T has

- a symmetric part $\mathrm{S} = \frac{1}{2}(\mathrm{T} + \mathrm{T}^*)$, and
- a skew-symmetric part $\mathrm{A} = \frac{1}{2}(\mathrm{T} - \mathrm{T}^*)$.

So T has a unique representation of the form $\mathrm{T} = \mathrm{S} + \mathrm{A}$.

Lemma 2.1. *Let* T *be an orthogonal transformation, then*

- $\mathrm{TS} = \mathrm{ST}$*;*
- $\mathrm{TA} = \mathrm{AT}$*;*
- $\mathrm{AS} = \mathrm{SA}$.

For any orthonormal basis $\{\mathbf{e}_i \,:\, i = 1, 2, 3\}$ of $\mathbb{R}^3$, we can associate matrices T, S, and A, to the transformations T, S, A, defined by

$$\begin{cases} T &= \left(t_{ji} = \mathbf{e}_j \cdot \mathrm{T}(\mathbf{e}_i)\right)_{j,i=1,2,3}, \\ S &= \left(s_{ji} = \frac{1}{2}(t_{ji} + t_{ij})\right)_{j,i=1,2,3}, \\ A &= \left(a_{ji} = \frac{1}{2}(t_{ji} - t_{ij})\right)_{j,i=1,2,3}. \end{cases}$$

It follows that

$$\begin{cases} t_{ij} &= \quad t_{ji}, \\ a_{ij} &= -a_{ji}, \end{cases} \quad \text{for} \quad i, j = 1, 2, 3.$$

So $s_{jj} = t_{jj}$ and $a_{jj} = 0$ for $j = 1, 2, 3$.

The trace of a linear operator T on $\mathbb{R}^3$, defined by

$$\mathrm{Trace}(\mathrm{T}) = \sum_{i=1}^{3} t_{ii},$$

is independent of the orthonormal basis used to compute it. The determinant associated to T, denoted by $\mathrm{Det}(\mathrm{T})$, can also be computed with any matrix representation T of T. Here, T is named a *proper rotation* for $\mathrm{Det}(\mathrm{T}) = +1$, and an *improper rotation* for $\mathrm{Det}(\mathrm{T}) = -1$. If $\mathrm{T}(\mathbf{v}) = \sigma\mathbf{v}$, we say that $\mathbf{v}$ is an eigenvector associated to the eigenvalue σ of T. The eigenspace associated to σ is

$$\mathcal{E}(\sigma) = \left\{ \mathbf{v} \in \mathbb{R}^3 \ : \ \mathrm{T}(\mathbf{v}) = \sigma\mathbf{v} \right\}.$$

If $\mathrm{Dim}(\mathcal{E}(\sigma))$ is 3 then $\mathrm{T} = \sigma\mathrm{I}$.

3. Euler's Result on the Axis of Rotation

There are several proofs of Euler's theorem concerning the existence of an axis of rotation for an orthogonal transformation in three-dimensional space. In this subsection, we revisit and extend one of the proofs presented in [15], adapting it to the following version of Euler's result.

Theorem 3.1. *For any orthogonal transformation* T *in* $\mathbb{R}^3$*, there exists a value* $\sigma \in \{+1, -1\}$ *such that* $\mathrm{Dim}(\mathcal{E}(\sigma)) = 1$ *or* 3.

The proof is in two parts. In the first part, we prove Theorem 3.1.2 for non-symmetric orthogonal transformations, while in the second part, we prove Theorem 3.2.4 for symmetric orthogonal transformations. In both cases, the proof includes a method to find a direction $\mathbf{v}$ of the axis of rotation, so a unit vector for the axis is $\mathbf{n} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$.

3.1. Non-Symmetric Orthogonal Transformation

Let us start with a basic result which relates the eigenspace $\mathcal{E}(\sigma)$ of T to the kernel of A.

Lemma 3.1.1. *If* $\sigma = \pm 1$*, then* $\mathcal{E}(\sigma) \subseteq \mathrm{Ker}(\mathrm{A})$.

Proof. If $\mathrm{T}(\mathbf{v}) = \sigma\mathbf{v}$, then $\mathrm{T}^*(\mathbf{v}) = \sigma\mathbf{v}$ and $\mathrm{A}(\mathbf{v}) = \mathbf{0}$. □

Theorem 3.1.2. *For any non-symmetric orthogonal transformation* T *in* $\mathbb{R}^3$ *there exists* $\sigma \in \{+1, -1\}$ *such that* $\mathrm{Dim}(\mathcal{E}(\sigma)) = 1$.

Proof. We note that $\mathrm{A} \neq 0$ since $\mathrm{T} \neq \mathrm{T}^*$. So there exists at least one $a_{ji} \neq 0$ for a pair (j, i) such that $j \neq i$. Let us set $\mathbf{v} = a_{23}\mathbf{e}_1 - a_{13}\mathbf{e}_2 + a_{12}\mathbf{e}_3 \neq \mathbf{0}$. Since $\mathrm{A}(\mathbf{v}) = \mathbf{0}$, we have $\mathbf{v} \in \mathrm{Ker}(\mathrm{A})$, and $\mathrm{Dim}(\mathrm{Ker}(\mathrm{A})) \geq 1$. Now, let us suppose $a_{12} \neq 0$, and set

$$\begin{cases} \mathbf{v}_{13} & = & \mathbf{e}_1 & - & \frac{a_{23}}{a_{12}}\mathbf{e}_3 & \neq & \mathbf{0}, \\ \mathbf{v}_{23} & = & \mathbf{e}_2 & + & \frac{a_{13}}{a_{12}}\mathbf{e}_3 & \neq & \mathbf{0}. \end{cases}$$

A similar construction can be done for $a_{13} \neq 0$ or $a_{23} \neq 0$. We observe that $\mathbf{v} \cdot \mathbf{v}_{13} = 0$ and $\mathbf{v} \cdot \mathbf{v}_{23} = 0$. Moreover, the images

$$\begin{cases} \mathrm{A}(\mathbf{v}_{13}) & = & -\frac{1}{a_{12}}\left(a_{13}a_{23}\mathbf{e}_1 + \left(a_{12}^2 + a_{23}^2\right)\mathbf{e}_2 + a_{12}a_{13}\mathbf{e}_3\right) & \neq & \mathbf{0} \\ \mathrm{A}(\mathbf{v}_{23}) & = & \frac{1}{a_{12}}\left(\left(a_{12}^2 + a_{13}^2\right)\mathbf{e}_1 + a_{13}a_{23}\mathbf{e}_2 - a_{12}a_{23}\mathbf{e}_3\right) & \neq & \mathbf{0} \end{cases}$$

are linearly independent, so $\mathrm{Dim}(\mathrm{Range}(\mathrm{A})) \geq 2$. It follows that $\mathrm{Dim}(\mathrm{Ker}(\mathrm{A})) = 1$ and $\mathrm{Dim}(\mathrm{Range}(\mathrm{A})) = 2$. Since $\mathrm{A}(\mathrm{T}(\mathbf{v})) = \mathrm{T}(\mathrm{A}(\mathbf{v})) = \mathbf{0}$, we have $\mathrm{T}(\mathbf{v}) \in \mathrm{Ker}(\mathrm{A})$. Also, we have $\|\mathrm{T}(\mathbf{v})\| = \|\mathbf{v}\|$, and so, $\mathrm{T}(\mathbf{v}) = \sigma\mathbf{v}$ for $\sigma = +1$ or -1. Since $\mathbf{0} \neq \mathbf{v} \in \mathcal{E}(\sigma) \subseteq \mathrm{Ker}(\mathrm{A})$, it follows from Lemma 3.1.1 that $\mathrm{Dim}(\mathcal{E}(\sigma)) = 1$. □

3.2. Symmetric Orthogonal Transformation

Let us define the following two symmetric transformations $\mathrm{T}_+ = \frac{1}{2}(\mathrm{I}+\mathrm{T})$ and $\mathrm{T}_- = \frac{1}{2}(\mathrm{I}-\mathrm{T})$. So $\mathrm{T}_+^* = \mathrm{T}_+$ and $\mathrm{T}_-^* = \mathrm{T}_-$.

Lemma 3.2.1. *For a symmetric orthogonal transformation* T*, the following hold:*

- $\mathrm{I} = \mathrm{T}_+ + \mathrm{T}_-$;
- $\mathrm{T} = \mathrm{T}_+ - \mathrm{T}_-$;
- $\mathrm{T}\mathrm{T}_+ = \mathrm{T}_+ = \mathrm{T}_+\mathrm{T}$;
- $\mathrm{T}\mathrm{T}_- = -\mathrm{T}_- = \mathrm{T}_-\mathrm{T}$;
- $\mathrm{T}_+\mathrm{T}_- = 0 = \mathrm{T}_-\mathrm{T}_+$;
- $\mathrm{T}_+^2 = \mathrm{T}_+$;
- $\mathrm{T}_-^2 = \mathrm{T}_-$.

Let us set

$$\begin{cases} \mathrm{V}^+ &= \{\mathrm{T}_+(\mathbf{w}) \ : \ \mathbf{w} \in \mathbb{R}^3\}\,, \\ \mathrm{V}^- &= \{\mathrm{T}_-(\mathbf{w}) \ : \ \mathbf{w} \in \mathbb{R}^3\}\,. \end{cases}$$

Lemma 3.2.2. *The transformations* T_+ *and* T_- *are the projections onto* V^+ *and* V^-*, and* $\mathbb{R}^3$ *has the orthogonal decomposition* $\mathbb{R}^3 = \mathrm{V}^+ \oplus \mathrm{V}^-$.

Proof. For any $\mathbf{w}_1$ and $\mathbf{w}_2$ in $\mathbb{R}^3$, we have $\mathbf{w}_1^+ \cdot \mathbf{w}_2^- = 0$, where $\mathbf{w}^+ = \mathrm{T}_+(\mathbf{w})$ and $\mathbf{w}^- = \mathrm{T}_-(\mathbf{w})$, and hence, the result follows. ☐

Lemma 3.2.3. *For a symmetric orthogonal transformation* T*,* $\mathcal{E}(1) = \mathrm{V}^+$ *and* $\mathcal{E}(-1) = \mathrm{V}^-$.

Proof. If $\mathbf{v} \in \mathcal{E}(1)$, then $\mathrm{T}_+(\mathbf{v}) = \frac{1}{2}(\mathbf{v} + \mathrm{T}(\mathbf{v})) = \mathbf{v}$ because $\mathrm{T}(\mathbf{v}) = \mathbf{v}$. So $\mathbf{v} \in \mathrm{V}^+$. Conversely, if $\mathbf{v} \in \mathrm{V}^+$, there exists $\mathbf{w} \in \mathbb{R}^3$ such that $\mathbf{v} = \mathrm{T}_+(\mathbf{w})$. So, $\mathrm{T}(\mathbf{v}) = \mathrm{T}\mathrm{T}_+(\mathbf{w}) = \mathrm{T}_+(\mathbf{w}) = \mathbf{v}$, and hence, $\mathbf{v} \in \mathcal{E}(1)$. It follows that $\mathcal{E}(1) = \mathrm{V}^+$. In the same way, we prove that $\mathcal{E}(-1) = \mathrm{V}^-$. ☐

Theorem 3.2.4. *For any symmetric orthogonal transformation* T *in* $\mathbb{R}^3$*, there exists a value* $\sigma \in \{+1, -1\}$ *such that* $\mathrm{Dim}(\mathcal{E}(\sigma)) = 1$ *or* 3.

Proof. From Lemma 3.2.2 and Lemma 3.2.3, we have

$$\begin{cases} \mathcal{E}(+1) &= \mathrm{V}^+ &= \{\mathrm{T}_+(\mathbf{w}) \ : \ \mathbf{w} \in \mathbb{R}^3\} &= \mathrm{Lin}\,\{\mathrm{T}_+(\mathbf{e}_i) \ : \ i = 1,2,3\}\,, \\ \mathcal{E}(-1) &= \mathrm{V}^- &= \{\mathrm{T}_-(\mathbf{w}) \ : \ \mathbf{w} \in \mathbb{R}^3\} &= \mathrm{Lin}\,\{\mathrm{T}_-(\mathbf{e}_i) \ : \ i = 1,2,3\}\,. \end{cases}$$

So

$$\begin{cases} \mathrm{Dim}(\mathcal{E}(1)) &= \mathrm{Dim}(\mathrm{V}^+) &= \mathrm{Dim}(\mathrm{Range}(\mathrm{T}_+)), \\ \mathrm{Dim}(\mathcal{E}(-1)) &= \mathrm{Dim}(\mathrm{V}^-) &= \mathrm{Dim}(\mathrm{Range}(\mathrm{T}_-)), \end{cases}$$

and

$$\mathrm{Dim}(\mathcal{E}(1)) + \mathrm{Dim}(\mathcal{E}(-1)) = \mathrm{Dim}(\mathrm{V}^+) + \mathrm{Dim}(\mathrm{V}^-) = \mathrm{Dim}(\mathbb{R}^3) = 3.$$

Consequently

$$\begin{cases} \mathrm{Dim}(\mathcal{E}(1)) & = \left\{ \begin{array}{l} \text{number of linearly independent vector(s) in} \\ \left\{\mathbf{e}_i^+ = \mathrm{T}_+(\mathbf{e}_i) \ : \ i = 1,2,3\right\}, \end{array} \right. \\ \\ \mathrm{Dim}(\mathcal{E}(-1)) & = \left\{ \begin{array}{l} \text{number of linearly independent vector(s) in} \\ \left\{\mathbf{e}_i^- = \mathrm{T}_-(\mathbf{e}_i) \ : \ i = 1,2,3\right\}. \end{array} \right. \end{cases}$$

The possible cases are:

- $\mathrm{Dim}(\mathrm{V}^+) = \mathrm{Dim}(\mathcal{E}(1)) = 3$, and $\mathrm{Dim}(\mathrm{V}^-) = \mathrm{Dim}(\mathcal{E}(-1)) = 0$, so $\sigma = 1$, $\mathrm{Dim}(\mathcal{E}(\sigma)) = 3$, and $\mathrm{T} = \mathrm{I}$;
- $\mathrm{Dim}(\mathrm{V}^+) = \mathrm{Dim}(\mathcal{E}(1)) = 2$, and $\mathrm{Dim}(\mathrm{V}^-) = \mathrm{Dim}(\mathcal{E}(-1)) = 1$, so $\sigma = -1$ and $\mathrm{Dim}(\mathcal{E}(\sigma)) = 1$;
- $\mathrm{Dim}(\mathrm{V}^+) = \mathrm{Dim}(\mathcal{E}(1)) = 1$, and $\mathrm{Dim}(\mathrm{V}^-) = \mathrm{Dim}(\mathcal{E}(-1)) = 2$, so $\sigma = 1$ and $\mathrm{Dim}(\mathcal{E}(\sigma)) = 1$;
- $\mathrm{Dim}(\mathrm{V}^+) = \mathrm{Dim}(\mathcal{E}(1)) = 0$, and $\mathrm{Dim}(\mathrm{V}^-) = \mathrm{Dim}(\mathcal{E}(-1)) = 3$, so $\sigma = -1$, $\mathrm{Dim}(\mathcal{E}(\sigma)) = 3$, and $\mathrm{T} = -\mathrm{I}$.

This completes the proof. □

3.3. Two Fundamental Examples

In this subsection, we present two basic examples that will be useful in the next section.

Example 3.3.1. *Let* $\sin(\theta) \neq 0$*, and consider the non-symmetric orthogonal transformation*

$$T = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}. \quad \textit{So,} \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2\sin(\theta) \\ 0 & 2\sin(\theta) & 0 \end{pmatrix},$$

and $\mathrm{Det}(T) = \sigma$. *We take* $\mathbf{v} = -2\sin(\theta)\mathbf{e}_1$*, then* $\mathrm{T}(\mathbf{v}) = \sigma\mathbf{v}$*, and* $\mathrm{Dim}(\mathcal{E}(\sigma)) = 1$.

Example 3.3.2. *Let* $\mu = 1$ *or* -1*, and consider*

$$T_\mu = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & \sin(\theta) & -\cos(\theta) \end{pmatrix}.$$

It is a symmetric orthogonal transformation. We have

$$T_{\mu+} = \frac{1}{2}\begin{pmatrix} 1+\mu & 0 & 0 \\ 0 & 1+\cos(\theta) & \sin(\theta) \\ 0 & \sin(\theta) & 1-\cos(\theta) \end{pmatrix} \quad \textit{and} \quad T_{\mu-} = \frac{1}{2}\begin{pmatrix} 1-\mu & 0 & 0 \\ 0 & 1-\cos(\theta) & -\sin(\theta) \\ 0 & -\sin(\theta) & 1+\cos(\theta) \end{pmatrix}.$$

Note that the last two columns of these matrices are linearly dependent. For $\mu = 1$ *or* -1, *we have* $\mathrm{Dim}(\mathcal{E}(\mu)) = 2$ *and* $\mathrm{Dim}(\mathcal{E}(-\mu)) = 1$. *Also,* $\mathrm{Det}(\mathrm{T}_\mu) = -\mu = \sigma$*, and*

$$\mathbf{v}(\mu) = (1-\mu\cos(\theta))\mathbf{e}_2 - \mu\sin(\theta)\mathbf{e}_3.$$

4. Rodrigues' Formula

Using some elementary results about vectors and operators in $\mathbb{R}^3$, we obtain a formula for T in terms of an axis $\mathbf{n}$ and an angle θ of rotation.

4.1. Vectors and Operators

Let $\mathbf{n} \in \mathbb{R}^3$ be a unit vector and let us consider some associated basic linear operators

$$\hat{\mathbf{n}}^k : \mathbb{R}^3 \to \mathbb{R}^3 \quad \text{for} \quad k = 0, 1, 2, \ldots,$$

defined recursively by

$$\begin{cases} \hat{\mathbf{n}}^0(\mathbf{v}) = \mathbf{v} = \mathrm{I}(\mathbf{v}), \\ \hat{\mathbf{n}}^{1+k}(\mathbf{v}) = \mathbf{n} \times \hat{\mathbf{n}}^k(\mathbf{v}) \quad \text{for } k = 0, 1, 2, \ldots. \end{cases}$$

The first three linear operators $\hat{\mathbf{n}}^k$ ($k = 0, 1, 2$) are linearly independent. For $k = 0, 1, 2, \ldots$, we have $\widehat{\lambda\mathbf{n}}^k = \lambda^k \hat{\mathbf{n}}^k$, and

$$\begin{cases} \hat{\mathbf{n}}^{1+2k} &= (-1)^k \hat{\mathbf{n}}^1, \\ \hat{\mathbf{n}}^{2+2k} &= (-1)^k \hat{\mathbf{n}}^2. \end{cases}$$

Their adjoint operators are $\hat{\mathbf{n}}^{0*} = \hat{\mathbf{n}}^0$, and for $k = 0, 1, 2, \ldots$,

$$\begin{cases} \hat{\mathbf{n}}^{(1+2k)*} &= -\hat{\mathbf{n}}^{(1+2k)}, \\ \hat{\mathbf{n}}^{(2+2k)*} &= \hat{\mathbf{n}}^{(2+2k)}. \end{cases}$$

We have $\mathrm{Trace}(\hat{\mathbf{n}}^0) = 3$, $\mathrm{Trace}(\hat{\mathbf{n}}^1) = 0$, and $\mathrm{Trace}(\hat{\mathbf{n}}^2) = -2$.

4.2. Rodrigues' Formula

To obtain a formula for T in terms of an axis $\mathbf{n}$ and an angle (of rotation) θ, we use Theorem 3.1.2 to assume that σ is known and that there exists a unit vector $\mathbf{n} \in \mathcal{E}(\sigma)$ (of dimension 1 or 3), so $\mathrm{T}(\mathbf{n}) = \sigma\mathbf{n}$.

Theorem 4.2.1. *For an orthogonal transformation* T *with* σ *and* $\mathbf{n}$ *given by* Theorem 3.1, *there exists a unique angle* θ *in* $[0, 2\pi)$ *modulo* 2π *such that*

$$\mathrm{T} = \mathrm{T}_{(\sigma,\mathbf{n},\theta)} = \sigma\hat{\mathbf{n}}^0 + \sin(\theta)\hat{\mathbf{n}}^1 + (\sigma - \cos(\theta))\hat{\mathbf{n}}^2.$$

Moreover,

$$\mathrm{T}^*_{(\sigma,\mathbf{n},\theta)} = \sigma\hat{\mathbf{n}}^0 - \sin(\theta)\hat{\mathbf{n}}^1 + (\sigma - \cos(\theta))\hat{\mathbf{n}}^2 = \mathrm{T}_{(\sigma,\mathbf{n},-\theta)}.$$

Proof. We consider any unit vector $\mathbf{n}^\perp$ orthogonal to $\mathbf{n}$, so that $\left\{\mathbf{n}, \mathbf{n}^\perp, \mathbf{n} \times \mathbf{n}^\perp\right\}$ is an orthonormal basis of $\mathbb{R}^3$. The expression of a $\mathbf{v}$ with respect to the orthonormal basis is

$$\mathbf{v} = (\mathbf{n} \cdot \mathbf{v})\mathbf{n} + (\mathbf{n}^\perp \cdot \mathbf{v})\mathbf{n}^\perp + \left((\mathbf{n} \times \mathbf{n}^\perp) \cdot \mathbf{v}\right)\left(\mathbf{n} \times \mathbf{n}^\perp\right).$$

This expression contains two parts :

- the projection of $\mathbf{v}$ onto $\mathbf{n}$:
$$\mathrm{P}_{\mathbf{n}}(\mathbf{v}) = (\mathbf{n} \cdot \mathbf{v})\mathbf{n} = (\hat{\mathbf{n}}^0 + \hat{\mathbf{n}}^2)(\mathbf{v});$$
- the orthogonal part of $\mathbf{v}$ with respect to $\mathbf{n}$:
$$\mathrm{Q}_{\mathbf{n}}(\mathbf{v}) = \begin{cases} \mathbf{v} - \mathrm{P}_{\mathbf{n}}(\mathbf{v}) = -\hat{\mathbf{n}}^2(\mathbf{v}), \\ (\mathbf{n}^\perp \cdot \mathbf{v})\mathbf{n}^\perp + \left((\mathbf{n} \times \mathbf{n}^\perp) \cdot \mathbf{v}\right)\left(\mathbf{n} \times \mathbf{n}^\perp\right). \end{cases}$$

The linearity of T implies that

$$\mathrm{T}(\mathbf{v}) = \mathrm{T}(\mathrm{P}_{\mathbf{n}}(\mathbf{v}) + \mathrm{Q}_{\mathbf{n}}(\mathbf{v})) = \mathrm{T}(\mathrm{P}_{\mathbf{n}}(\mathbf{v})) + \mathrm{T}(\mathrm{Q}_{\mathbf{n}}(\mathbf{v})).$$

For the first part, we have

$$\mathrm{T}(\mathrm{P}_{\mathbf{n}}(\mathbf{v})) = \sigma \mathrm{P}_{\mathbf{n}}(\mathbf{v}).$$

For the second part, the application of T on any vector in the subspace $\mathcal{S} = \mathrm{Lin}\left\{\mathbf{n}^{\perp}, \mathbf{n} \times \mathbf{n}^{\perp}\right\}$ leads to a vector in this subspace. Moreover, a unit vector is transformed into a unit vector. Also, if two vectors are orthogonal in this subspace, their images remain orthogonal. Since $\left\{\mathbf{n}^{\perp}, \mathbf{n} \times \mathbf{n}^{\perp}\right\}$ is an orthonormal basis of this subspace, there exists θ such that

$$\begin{cases} \mathrm{T}(\mathbf{n}^{\perp}) & = & \cos(\theta)\mathbf{n}^{\perp} + \sin(\theta)\left(\mathbf{n} \times \mathbf{n}^{\perp}\right), \\ \mathrm{T}(\mathbf{n} \times \mathbf{n}^{\perp}) & = & \pm\left(-\sin(\theta)\mathbf{n}^{\perp} + \cos(\theta)(\mathbf{n} \times \mathbf{n}^{\perp})\right). \end{cases}$$

The choice of the "$-$" sign leads to the fact that the dimension of $\mathcal{E}(\sigma)$ is 2 as explained in Example 3.3.2, contrary to the initial choice of $\mathbf{n}$. Therefore, in the sequel, we consider only the "$+$" sign. Consequently, $\mathrm{Det}(\mathrm{T}) = \sigma$. Also, we obtain

$$\begin{cases} \mathrm{T}(\mathbf{n}) & = & \sigma\mathbf{n} \\ & = & (\sigma-1)(\hat{\mathbf{n}}^0 + \hat{\mathbf{n}}^2)(\mathbf{n}) + \left(\hat{\mathbf{n}}^0 + \sin(\theta)\hat{\mathbf{n}}^1 + (1-\cos(\theta))\hat{\mathbf{n}}^2\right)(\mathbf{n}), \\ & & \\ \mathrm{T}(\mathbf{n}^{\perp}) & = & \cos(\theta)\mathbf{n}^{\perp} + \sin(\theta)\left(\mathbf{n} \times \mathbf{n}^{\perp}\right) \\ & = & \left(\hat{\mathbf{n}}^0 + \sin(\theta)\hat{\mathbf{n}}^1 + (1-\cos(\theta))\hat{\mathbf{n}}^2\right)(\mathbf{n}^{\perp}), \\ & & \\ \mathrm{T}(\mathbf{n} \times \mathbf{n}^{\perp}) & = & -\sin(\theta)\mathbf{n}^{\perp} + \cos(\theta)(\mathbf{n} \times \mathbf{n}^{\perp}) \\ & = & \left(\hat{\mathbf{n}}^0 + \sin(\theta)\hat{\mathbf{n}}^1 + (1-\cos(\theta))\hat{\mathbf{n}}^2\right)(\mathbf{n} \times \mathbf{n}^{\perp}). \end{cases}$$

Since T is a linear operator, we have

$$\begin{aligned} \mathrm{T}(\mathbf{v}) &= (\mathbf{n}\cdot\mathbf{v})\mathrm{T}(\mathbf{n}) + (\mathbf{n}^{\perp}\cdot\mathbf{v})\mathrm{T}\left(\mathbf{n}^{\perp}\right) + \left((\mathbf{n} \times \mathbf{n}^{\perp})\cdot\mathbf{v}\right)\mathrm{T}\left(\mathbf{n} \times \mathbf{n}^{\perp}\right) \\ &= (\mathbf{n}\cdot\mathbf{v})(\sigma-1)(\hat{\mathbf{n}}^0 + \hat{\mathbf{n}}^2)(\mathbf{n}) + (\mathbf{n}\cdot\mathbf{v})\left(\hat{\mathbf{n}}^0 + \sin(\theta)\hat{\mathbf{n}}^1 + (1-\cos(\theta))\hat{\mathbf{n}}^2\right)(\mathbf{n}) \\ &\quad + (\mathbf{n}^{\perp}\cdot\mathbf{v})\left(\hat{\mathbf{n}}^0 + \sin(\theta)\hat{\mathbf{n}}^1 + (1-\cos(\theta))\hat{\mathbf{n}}^2\right)(\mathbf{n}^{\perp}) \\ &\quad + ((\mathbf{n} \times \mathbf{n}^{\perp})\cdot\mathbf{v})\left(\hat{\mathbf{n}}^0 + \sin(\theta)\hat{\mathbf{n}}^1 + (1-\cos(\theta))\hat{\mathbf{n}}^2\right)(\mathbf{n} \times \mathbf{n}^{\perp}) \\ &= (\sigma-1)(\hat{\mathbf{n}}^0 + \hat{\mathbf{n}}^2)(\mathbf{v}) + \left(\hat{\mathbf{n}}^0 + \sin(\theta)\hat{\mathbf{n}}^1 + (1-\cos(\theta))\hat{\mathbf{n}}^2\right)(\mathbf{v}) \\ &= \left(\sigma\hat{\mathbf{n}}^0 + \sin(\theta)\hat{\mathbf{n}}^1 + (\sigma-\cos(\theta))\hat{\mathbf{n}}^2\right)(\mathbf{v}). \end{aligned}$$

□

We have thus proved that Rodrigues' formula holds not only for proper rotations ($\sigma = 1$), but also for improper rotations ($\sigma = -1$).

Now, we state a number of corollaries.

Corollary 4.2.2. *The following hold:*

- $\mathrm{T}_{(\sigma,\mathbf{n},\theta)} = \mathrm{T}_{(\sigma,-\mathbf{n},-\theta)}$;
- $\mathrm{T}_{(\sigma,\mathbf{n},\theta)} = \mathrm{T}_{(\sigma,\mathbf{n},\theta+2k\pi)}$ *for any* $k \in \mathbb{Z}$;
- $\mathrm{T}^{-1}_{(\sigma,\mathbf{n},\theta)} = \mathrm{T}_{(\sigma,\mathbf{n},-\theta)} = \mathrm{T}_{(\sigma,-\mathbf{n},\theta)}$;
- $\mathrm{T}^{*}_{(\sigma,\mathbf{n},\theta)} = \mathrm{T}^{-1}_{(\sigma,\mathbf{n},\theta)}$.

Corollary 4.2.3 (Orthogonality condition)**.** *For any* $\mathbf{v} \in \mathbb{R}^3$, *it holds that* $\mathbf{n} \cdot \left[\mathrm{T}_{(\sigma,\mathbf{n},\theta)} - \sigma \mathrm{I}\right](\mathbf{v}) = 0$.

Corollary 4.2.4. *It holds that* $\mathrm{Trace}(\mathrm{T}_{(\sigma,\mathbf{n},\theta)}) = \sigma + 2\cos(\theta)$.

As special cases, for $\theta'_\sigma = \frac{1-\sigma}{2}\pi$, we have $\mathrm{T}_{(\sigma,\mathbf{n},\theta'_\sigma)} = \sigma \mathrm{I}$, and for $\theta''_\sigma = \frac{1+\sigma}{2}\pi$, we obtain

$$\mathrm{T}_{(\sigma,\mathbf{n},\theta''_\sigma)} = \sigma\left(\mathrm{I} - 2\mathrm{Q}_{\mathbf{n}}\right) = -\sigma\left(\mathrm{I} - 2\mathrm{P}_{\mathbf{n}}\right).$$

For $\sigma = 1$, the last expression corresponds to a symmetry with respect to the vector $\mathbf{n}$ parallel to the two-dimensional subspace $\mathcal{S}$ orthogonal to $\mathbf{n}$, and for $\sigma = -1$, it corresponds to a mirror reflection along $\mathbf{n}$ with respect to the two-dimensional subspace $\mathcal{S}$ orthogonal to $\mathbf{n}$.

The next result connects the symmetric-antisymmetric representation of an orthogonal transformation and its Rodrigues' formula.

Corollary 4.2.5. *The following hold:*

$$\begin{cases} \mathrm{S}_{(\sigma,\mathbf{n},\theta)} & = & \frac{1}{2}(\mathrm{T}_{(\sigma,\mathbf{n},\theta)} + \mathrm{T}^{*}_{(\sigma,\mathbf{n},\theta)}) & = & \sigma\hat{\mathbf{n}}^0 + (\sigma - \cos(\theta))\hat{\mathbf{n}}^2, \\ \mathrm{A}_{(\sigma,\mathbf{n},\theta)} & = & \frac{1}{2}(\mathrm{T}_{(\sigma,\mathbf{n},\theta)} - \mathrm{T}^{*}_{(\sigma,\mathbf{n},\theta)}) & = & \sin(\theta)\hat{\mathbf{n}}^1, \end{cases}$$

and

$$\mathrm{T}_{(\sigma,\mathbf{n},\theta)} = \sigma\mathrm{I} + \mathrm{A}_{(\sigma,\mathbf{n},\theta)} + (\mathrm{S}_{(\sigma,\mathbf{n},\theta)} - \sigma\mathrm{I}).$$

Remark 4.2.6 (see also [1, 9])**.** *The matrix associated to* $\mathrm{T}_{(\sigma,\mathbf{n},\theta)}$ *with respect to the orthogonal basis* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$*, with* $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$*, is* $\sigma I + \sin(\theta)N + (\sigma - \cos(\theta))M$*, where*

$$N = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} \textit{ and } M = \begin{pmatrix} n_1^2 - 1 & n_1 n_2 & n_1 n_3 \\ n_2 n_1 & n_2^2 - 1 & n_2 n_3 \\ n_3 n_1 & n_3 n_2 & n_3^2 - 1 \end{pmatrix}.$$

Therefore, using $\mathbf{n}$*, we have*

$$\begin{cases} S_{(\sigma,\mathbf{n},\theta)} & = & \sigma I + (\sigma - \cos(\theta))M, \\ A_{(\sigma,\mathbf{n},\theta)} & = & \sin(\theta)N. \end{cases}$$

These relations can be used to easily identify the angle of rotation θ*.*

5. Numerical Examples

In this section, we present non-symmetric and symmetric examples to illustrate the results presented in Sections 3 and 4. We first determine $\mathbf{n}$ and, following Remark 4.2.6, we obtain θ from

$$\begin{cases} \cos(\theta))M & = & \sigma(I + M) - S, \\ \sin(\theta)N & = & A. \end{cases}$$

5.1. Non-Symmetric Orthogonal Transformations

Example 5.1.1. *Consider* $T = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & -1 \\ \sqrt{2} & 0 & \sqrt{2} \\ 1 & -\sqrt{2} & -1 \end{pmatrix}$, *then* $\mathrm{Det}(T) = 1 = \sigma$. *Thus,*

$$A = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & \sqrt{2} \\ 1 & -\sqrt{2} & 0 \end{pmatrix} \text{ and } S = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Also, we obtain $\mathbf{n} = \frac{1}{\sqrt{3}}(\sqrt{2}\,\mathbf{e}_1 + \mathbf{e}_2)$ *in* $\mathcal{E}(1)$*, and hence*

$$N = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -\sqrt{2} \\ -1 & \sqrt{2} & 0 \end{pmatrix} \text{ and } M = \frac{1}{3}\begin{pmatrix} -1 & \sqrt{2} & 0 \\ \sqrt{2} & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

Consequently, we have $\cos(\theta) = -1/2$ *and* $\sin(\theta) = -\sqrt{3}/2$. *Therefore,* $\theta = 4\pi/3$.

Example 5.1.2. *Here, we consider* $T = \frac{1}{3}\begin{pmatrix} 2 & -2 & -1 \\ -1 & -2 & 2 \\ 2 & 1 & 2 \end{pmatrix}$, *then* $\mathrm{Det}(T) = -1 = \sigma$. *So,*

$$A = \frac{1}{6}\begin{pmatrix} 0 & -1 & -3 \\ 1 & 0 & 1 \\ 3 & -1 & 0 \end{pmatrix} \text{ and } S = \frac{1}{6}\begin{pmatrix} 4 & -3 & 1 \\ -3 & -4 & 3 \\ 1 & 3 & 4 \end{pmatrix}.$$

We obtain $\mathbf{n} = \frac{1}{\sqrt{11}}(\mathbf{e}_1 + 3\mathbf{e}_2 - \mathbf{e}_3)$ *in* $\mathcal{E}(-1)$*, and thus*

$$N = \frac{1}{\sqrt{11}}\begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & -1 \\ -3 & 1 & 0 \end{pmatrix} \text{ and } M = \frac{1}{11}\begin{pmatrix} -10 & 3 & -1 \\ 3 & -2 & -1 \\ -1 & -1 & -10 \end{pmatrix}.$$

Consequently, we get $\cos(\theta) = 5/6$ *and* $\sin(\theta) = -\sqrt{11}/6$*, and therefore,* $\theta \approx 1.81\pi$.

5.2. Symmetric Orthogonal Transformations

Example 5.2.1. *We consider* $T = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$, *then* $\mathrm{Det}(T) = 1 = \sigma$. *Also,* $S = T$ *and* $A = 0$.

Thus,

$$T_+ = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \text{ and } T_- = \frac{1}{3}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

Here, we obtain $\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$ *in* $\mathcal{E}(1)$*, and hence*

$$N = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \text{ and } M = \frac{1}{3}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

Consequently, we obtain $\cos(\theta) = -1$ *and* $\sin(\theta) = 0$. *Therefore,* $\theta = \pi$.

Example 5.2.2. *Here, we consider* $T = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} & 0 \\ \sqrt{3} & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, *then* $\operatorname{Det}(T) = -1 = \sigma$. *Thus,* $A = 0$ *and* $S = T$. *So, we have*

$$T_+ = \frac{1}{4}\begin{pmatrix} 3 & \sqrt{3} & 0 \\ \sqrt{3} & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \text{ and } T_- = \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{3} & 0 \\ -\sqrt{3} & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Also, we obtain $\mathbf{n} = \frac{1}{2}(\mathbf{e}_1 - \sqrt{3}\,\mathbf{e}_2)$ *in* $\mathcal{E}(-1)$, *and hence*

$$N = \frac{1}{2}\begin{pmatrix} 0 & 0 & -\sqrt{3} \\ 0 & 0 & -1 \\ \sqrt{3} & 1 & 0 \end{pmatrix} \text{ and } M = \frac{1}{4}\begin{pmatrix} -3 & -\sqrt{3} & 0 \\ -\sqrt{3} & -1 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

Consequently, we obtain $\cos(\theta) = 1$ *and* $\sin(\theta) = 0$. *Therefore,* $\theta = 0$.

6. Miscellaneous Results

There are many applications of Rodrigues' formula. We have already seen that it is useful for determining the angle of rotation of an orthogonal transformation. In this section, we present three results based on Rodrigues' formula.

6.1. Composition of Orthogonal Transformations

The composition $\mathrm{T}_0 = \mathrm{T}_2\mathrm{T}_1$ of two orthogonal transformations T_1 and T_2 is also an orthogonal transformation since $\mathrm{T}_0^* = \mathrm{T}_1^*\mathrm{T}_2^*$. Hence, we aim to determine σ_0, $\mathbf{n}_0$, and θ_0 in the following expression from the known values of σ_i, $\mathbf{n}_i$, and θ_i $(i = 1, 2)$:

$$\mathrm{T}_{(\sigma_2,\mathbf{n}_2,\theta_2)}\mathrm{T}_{(\sigma_1,\mathbf{n}_1,\theta_1)} = \mathrm{T}_{(\sigma_0,\mathbf{n}_0,\theta_0)}.$$

For two orthogonal transformations having the same axis of rotation, it is intuitively obvious. The result can be easily proved using Rodrigues' formula from Theorem 4.2.1.

Theorem 6.1.1. *It holds that* $\mathrm{T}_{(\sigma_2,\mathbf{n},\theta_2)}\mathrm{T}_{(\sigma_1,\mathbf{n},\theta_1)} = \mathrm{T}_{(\sigma_1\sigma_2,\mathbf{n},\theta_1+\theta_2)}$.

To obtain the general result, we first consider a theorem of Euler, which states that the composition of two successive proper rotations is also a proper rotation [4, 5, 7, 16, 18].

Theorem 6.1.2. *Let* $\mathrm{T}_{(1,\mathbf{n}_1,\theta_1)}$ *and* $\mathrm{T}_{(1,\mathbf{n}_2,\theta_2)}$ *be two given proper rotations with arbitrary axes of rotation. The composition of these two proper rotations is a proper rotation such that*

$$\mathrm{T}_{(1,\mathbf{n}_2,\theta_2)}\mathrm{T}_{(1,\mathbf{n}_1,\theta_1)} = \mathrm{T}_{(1,\mathbf{n}_0,\theta_0)},$$

where $\theta_0 \in [0, 2\pi)$ *provided that*

$$\begin{cases} \cos(\theta_0/2) &= \cos(\theta_2/2)\cos(\theta_1/2) - \sin(\theta_2/2)\sin(\theta_1/2)\mathbf{n}_2 \cdot \mathbf{n}_1, \\ \sin(\theta_0/2)\mathbf{n}_0 &= \sin(\theta_2/2)\cos(\theta_1/2)\mathbf{n}_2 + \cos(\theta_2/2)\sin(\theta_1/2)\mathbf{n}_1 + \sin(\theta_2/2)\sin(\theta_1/2)\mathbf{n}_2 \times \mathbf{n}_1. \end{cases}$$

To establish the general result, we first observe that

$$\sigma I = \mathrm{T}_{(\sigma,\mathbf{n},\theta(\sigma))} = \begin{cases} \mathrm{T}_{(1,\mathbf{n},0)} & \text{for} \quad \sigma = 1 \quad \text{and} \quad \theta(\sigma) = 0, \\ \mathrm{T}_{(-1,\mathbf{n},\pi)} & \text{for} \quad \sigma = -1 \quad \text{and} \quad \theta(\sigma) = \pi. \end{cases}$$

Thus, we obtain the following relation:

$$\begin{aligned} \mathrm{T}_{(\sigma_i,\mathbf{n}_i,\theta_i)} &= \mathrm{T}_{(\sigma_i,\mathbf{n}_i,\theta(\sigma_i))}\mathrm{T}_{(\sigma_i,\mathbf{n}_i,\theta(\sigma_i))}\mathrm{T}_{(\sigma_i,\mathbf{n}_i,\theta_i)} \\ &= \sigma_i \mathrm{T}_{(\sigma_i^2,\mathbf{n}_i,\theta_i+\theta(\sigma_i))} \\ &= \sigma_i \mathrm{T}_{(1,\mathbf{n}_i,\theta_i+\theta(\sigma_i))}, \end{aligned}$$

for $i = 1, 2$. Consequently, the next result follows.

Theorem 6.1.3. *Let* $\mathrm{T}_{(\sigma_1,\mathbf{n}_1,\theta_1)}$ *and* $\mathrm{T}_{(\sigma_2,\mathbf{n}_2,\theta_2)}$ *be two given orthogonal transformations with arbitrary axes of rotation. The composition of these two transformations is given by*

$$\begin{aligned} \mathrm{T}_{(\sigma_2,\mathbf{n}_2,\theta_2)}\mathrm{T}_{(\sigma_1,\mathbf{n}_1,\theta_1)} &= \sigma_2\sigma_1 \mathrm{T}_{(1,\mathbf{n}_2,\theta_2+\theta(\sigma_2))}\mathrm{T}_{(1,\mathbf{n}_1,\theta_1+\theta(\sigma_1))} \\ &= \sigma_2\sigma_1 \mathrm{T}_{(1,\mathbf{n}_0',\theta_0')} \\ &= \mathrm{T}_{(\sigma_2\sigma_1,\mathbf{n}_0',\theta(\sigma_2\sigma_1))}\mathrm{T}_{(1,\mathbf{n}_0',\theta_0')} \\ &= \mathrm{T}_{(\sigma_2\sigma_1,\ \mathbf{n}_0',\ \theta_0'+\theta(\sigma_2\sigma_1))}, \end{aligned}$$

where $\mathbf{n}_0'$ *and* θ_0' *can be computed using Theorem 6.1.2.*

Example 6.1.4 (see also [12]). *Suppose that the first rotation is a proper rotation through an angle of* $\pi/3$ *radians about the* x*-axis, and the second rotation is a proper rotation through an angle of* $\pi/6$ *radians about the* z*-axis. We seek* $\mathrm{T}_{(1,\mathbf{n}_0,\theta_0)} = \mathrm{T}_{(1,\mathbf{e}_3,\pi/6)}\mathrm{T}_{(1,\mathbf{e}_1,\pi/3)}$. *For the resulting rotation, we obtain* $\cos(\theta_0/2) = 0.8365$*, and hence* $\theta_0 = 0.369\pi$ *radians. It follows that* $\sin(\theta_0/2) = 0.5480$*, and*

$$\mathbf{n}_0 = (0.8814,\ 0.2362,\ 0.4090).$$

Example 6.1.5. *Suppose that the first rotation is a proper rotation through an angle of* $\pi/3$ *radians about the* x*-axis, and the second rotation is an improper rotation through an angle of* $\pi/6$ *radians about the* z*-axis. We seek* $\mathrm{T}_{(\sigma_0,\mathbf{n}_0,\theta_0)} = \mathrm{T}_{(-1,\mathbf{e}_3,\pi/6)}\mathrm{T}_{(1,\mathbf{e}_1,\pi/3)}$. *We can write*

$$\mathrm{T}_{(\sigma_0,\mathbf{n}_0,\theta_0)} = -\mathrm{T}_{(1,\mathbf{e}_3,\pi/6+\pi)}\mathrm{T}_{(1,\mathbf{e}_1,\pi/3)} = -\mathrm{T}_{(1,\mathbf{n}_0',\theta_0')}.$$

We obtain $\cos(\theta_0'/2) = -0.2241$*, and hence* $\theta_0' = 1.144\pi$ *radians. It follows that* $\sin(\theta_0'/2) = 0.9745$*, and*

$$\mathbf{n}_0' = (-0.1328,\ 0.4956,\ 0.8583).$$

Consequently, $\mathrm{T}_{(\sigma_0,\mathbf{n}_0,\theta_0)} = \mathrm{T}_{(-1,\mathbf{n}_0',\pi)}\mathrm{T}_{(1,\mathbf{n}_0',\theta_0')} = \mathrm{T}_{(-1,\mathbf{n}_0',\theta_0'+\pi)}$. *Therefore, the resulting transformation is an improper rotation (*$\sigma_0 = -1$*), through an angle* $\theta_0 = \theta_0' + \pi = 2.144\pi$ *radians about the axis* $\mathbf{n}_0 = \mathbf{n}_0'$.

6.2. Orthogonal Transformation and Cross Product

An interesting property of orthogonal transformations is that they are distributive with respect to the cross product in $\mathbb{R}^3$, as stated in the next theorem. The proof follows by direct substitution into Rodrigues' formula.

Theorem 6.2.1. $\mathrm{T}_{(\sigma,\mathbf{n},\theta)}(\mathbf{w}_1 \times \mathbf{w}_2) = \sigma \mathrm{T}_{(\sigma,\mathbf{n},\theta)}(\mathbf{w}_1) \times \mathrm{T}_{(\sigma,\mathbf{n},\theta)}(\mathbf{w}_2)$.

6.3. Non-Commutativity of Orthogonal Transformations

From Theorem 6.1.1, we deduce that the composition of two orthogonal transformations about the same axis of rotation is commutative. In general, when the axes of rotation are distinct, the composition is not commutative [6]. The following result characterizes the non-commutativity of two orthogonal transformations. Its proof is again obtained by direct substitution into Rodrigues' formula.

Theorem 6.3.1. *The non-commutativity of two orthogonal transformations* $\mathrm{T}_{(\sigma_1,\mathbf{n}_1,\theta_1)}$ *and* $\mathrm{T}_{(\sigma_2,\mathbf{n}_2,\theta_2)}$ *is characterized by the expression*

$$\mathrm{T}_{(\sigma_2,\mathbf{n}_2,\theta_2)}\mathrm{T}_{(\sigma_1,\mathbf{n}_1,\theta_1)} = \mathrm{T}_{(\sigma_1,\ \sigma_2\mathrm{T}_{(\sigma_2,\mathbf{n}_2,\theta_2)}(\mathbf{n}_1),\ \theta_1)}\mathrm{T}_{(\sigma_2,\mathbf{n}_2,\theta_2)}.$$

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