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Research Article

Enriched Categories of Coalgebras

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Abstract

Let $\mathcal{V}$ be a monoidal category with underlying ordinary category $\mathcal{V}_0$. Denote by $\mathcal{A}$ an enriched category over $\mathcal{V}$. For any $\mathcal{V}$-endofunctor $T : \mathcal{A} \to \mathcal{A}$, a T-coalgebra is defined as a pair (A, τ_A) consisting of an object A in $ob\mathcal{A}$ together with a $\mathcal{V}_0$-morphism $\tau_A : I \to \mathcal{A}(A, TA)$, where I is the unit. The object A is called the underlying object of the T-coalgebra (A, τ_A) and τ_A is called its coalgebra structure. Assuming that $\mathcal{V}$ is symmetric and admits equalizers, it is shown that T-coalgebras form a $\mathcal{V}$-category, denoted by $\mathcal{A}_T$. Consequently, the correspondence $U_T : ob\mathcal{A}_T \to ob\mathcal{A}$ that assigns each T-coalgebra (A, τ_A) to its underlying object A is a $\mathcal{V}$-functor. Moreover, if $\mathcal{V}$ is closed and complete, and $\mathcal{A}$ is a small $\mathcal{V}$-category, then the underlying $\mathcal{V}$-functor $U_T : \mathcal{A}_T \to \mathcal{A}$ is shown to create weighted colimits and weighted limits that are preserved by T.

Keywords: symmetric monoidal closed category; enriched category; coalgebra structure.

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1. Introduction

The notion of an enriched category is a generalization of the notion of an ordinary category. Very often, instead of merely having a set of morphisms from one object to another, a category may have an abelian group of morphisms, a topological space of morphisms, or other similar structures. An enriched category generalizes the idea of a category by replacing hom-sets with hom-objects, which are objects of a fixed monoidal category $\mathcal{V}$. The enriching category $\mathcal{V}$ must be monoidal in order to define a suitable composition law, which reduces to the usual composition of morphisms in the case of ordinary categories. In particular, the underlying ordinary category $\mathcal{V}_0$ of a monoidal category $\mathcal{V}$ is equipped with a bifunctor $\otimes\colon \mathcal{V}_0 \times \mathcal{V}_0 \longrightarrow \mathcal{V}_0$ that is associative up to isomorphism. The monoidal category $\mathcal{V}$ is said to be symmetric if the monoidal product $\otimes$ is commutative up to isomorphism, and closed if, for each object B, the functor $- \otimes B\colon \mathcal{V}_0 \longrightarrow \mathcal{V}_0$ admits a right adjoint $[B, -]$. The categories Ab of abelian groups and group homomorphisms, and Top of topological spaces and continuous maps, are standard examples of monoidal categories (see [7]). An enriched category whose hom-objects lie in a monoidal category $\mathcal{V}$ is called an enriched category over $\mathcal{V}$, or simply a $\mathcal{V}$-category.

Enriched functors play the role of ordinary functors in enriched category theory. Like ordinary functors, they assign objects to objects; however, instead of mapping hom-sets to hom-sets, they assign morphisms in the enriching category between the corresponding hom-objects. These assignments are required to be compatible with composition and units in a natural and coherent way. An enriched functor between enriched categories over a monoidal category $\mathcal{V}$ is called a $\mathcal{V}$-functor.

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Let $T\colon \mathcal{A} \to \mathcal{A}$ be an endofunctor on an ordinary category $\mathcal{A}$, that is, a category enriched over Set. A T-coalgebra, or coalgebra of type T, is a pair (A, a) consisting of an object A of $\mathcal{A}$ together with an $\mathcal{A}$-morphism $a\colon A \to TA$. We call A the underlying object of the T-coalgebra (A, a) and a its coalgebra structure.

A homomorphism between T-coalgebras (A, a) and (B, b) is a morphism $f\colon A \to B$ in $\mathcal{A}$ that respects the coalgebra structures, that is,

$$T(f) \circ a = b \circ f.$$

Homomorphisms of T-coalgebras are stable under composition. Hence, T-coalgebras and their homomorphisms form a category, denoted by $\mathcal{A}_T$. The forgetful functor $U_T\colon \mathcal{A}_T \to \mathcal{A}$ creates colimits; that is, a colimit of a diagram $D\colon \mathcal{D} \to \mathcal{A}_T$ is obtained from a colimit $L = colim(U_T \circ D)$ in $\mathcal{A}$ (assuming it exists) by equipping L with the unique coalgebra structure that turns the colimit cocone into a colimit cocone of homomorphisms (see [2, Theorem 1.1]). In general, limits in coalgebras are not created by U_T (see [9, 10]). However, if $\mathcal{A}$ has limits of a given type that are preserved by T, then U_T creates limits of that type (see [1, Remark 4.4(a)]). Recall that every $\mathcal{A}$-morphism $a : A \to TA$ corresponds under bijection with a mapping

$$\tau_A : \{\star\} \to \mathcal{A}(A, TA)$$

such that $\tau_A(\star) = a$. So, we can consider a T-coalgebra as a pair (A, τ_A) consisting of an object A in $\mathcal{A}$ together with a mapping

$$\tau_A : \{\star\} \to \mathcal{A}(A, TA).$$

This prompts us to define, for each pair of T-coalgebras $\underline{A} = (A, a)$ and $\underline{B} = (B, b)$, the hom-set $\mathcal{A}_T(\underline{A}, \underline{B})$ as the subobject of $\mathcal{A}(A, B)$ represented by the equalizer of the pair of composite morphisms:

$$\begin{array}{c} \mathcal{A}(A,B) \\ \downarrow {\scriptstyle r^{-1}_{\mathcal{A}(A,B)}} \\ \mathcal{A}(A,B) \times \{\star\} \\ \downarrow {\scriptstyle 1\times\tau_B} \\ \mathcal{A}(A,B) \times \mathcal{A}(B,TB) \\ \downarrow {\scriptstyle c_{ABTB}} \\ \mathcal{A}(A,TB) \end{array}$$

and

$$\begin{array}{c} \mathcal{A}(A,B) \\ \downarrow {\scriptstyle l^{-1}_{\mathcal{A}(A,B)}} \\ \{\star\} \times \mathcal{A}(A,B) \\ \downarrow {\scriptstyle 1\times T_{AB}} \\ \{\star\} \times \mathcal{A}(TA,TB) \\ \downarrow {\scriptstyle \tau_A\times 1} \\ \mathcal{A}(A,TA) \times \mathcal{A}(TA,TB) \\ \downarrow {\scriptstyle c_{ATATB}} \\ \mathcal{A}(A,TB), \end{array}$$

where $\tau_A(\star) = a$ and $\tau_B(\star) = b$. In this way, every $\mathcal{A}$-morphism $f : A \to B$ that belongs to $\mathcal{A}_T(\underline{A}, \underline{B})$ is a homomorphism between T-coalgebras $\underline{A} = (A, a)$ and $\underline{B} = (B, b)$. Hence, we say that $\mathcal{A}_T(\underline{A}, \underline{B})$ satisfies the homomorphism condition.

In this paper, $\mathcal{A}$ denotes an enriched category over a monoidal category $\mathcal{V}$. For any $\mathcal{V}$-endofunctor $T : \mathcal{A} \to \mathcal{A}$, a T-coalgebra is defined as a pair (A, τ_A) consisting of an object A in $ob\mathcal{A}$ together with a $\mathcal{V}_0$-morphism $\tau_A : I \to \mathcal{A}(A, TA)$, where I is the unit. The object A is called the underlying object of the T-coalgebra (A, τ_A) and τ_A is called its coalgebra structure. Assuming that $\mathcal{V}$ is symmetric and admits equalizers, it is shown that T-coalgebras form a $\mathcal{V}$-category, denoted by $\mathcal{A}_T$. Consequently, the correspondence $U_T : ob\mathcal{A}_T \to ob\mathcal{A}$ that assigns each T-coalgebra (A, τ_A) to its underlying object A is a $\mathcal{V}$-functor. Moreover, if $\mathcal{V}$ is closed and complete, and $\mathcal{A}$ is a small $\mathcal{V}$-category, then the underlying $\mathcal{V}$-functor $U_T : \mathcal{A}_T \to \mathcal{A}$ is shown to create weighted colimits and weighted limits that are preserved by T.

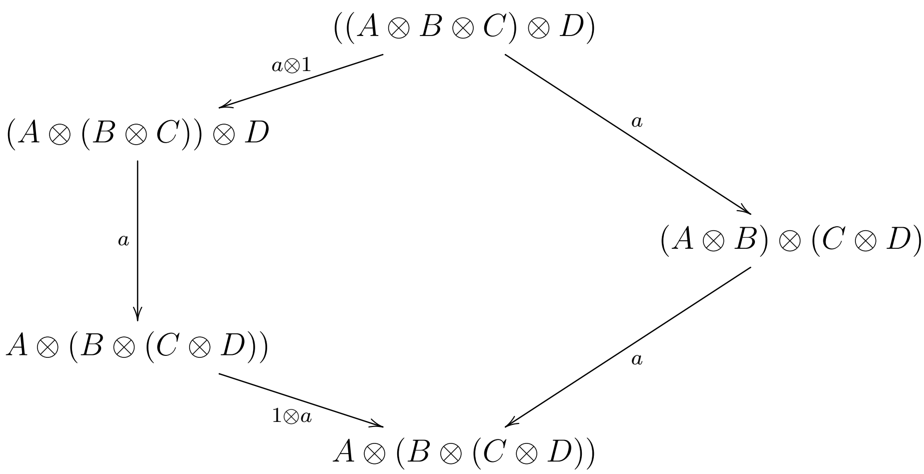
2. Elementary Notions

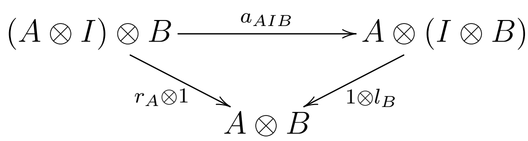
2.1. Monoidal Categories

A *monoidal category* $\mathcal{V} = (\mathcal{V}_0, \otimes, I, a, l, r)$ consists of a category $\mathcal{V}_0$, a bifunctor $\otimes : \mathcal{V}_0 \times \mathcal{V}_0 \to \mathcal{V}_0$, an object I of $\mathcal{V}_0$ (called the unit) and natural isomorphisms

$$a_{ABC} : (A \otimes B) \otimes C \to A \otimes (B \otimes C), \quad l_A : I \otimes A \to A, \quad r_A : A \otimes I \to A,$$

subject to the coherence axioms, expressing the commutativity of the following diagrams:





A special case of a monoidal category, called a *cartesian monoidal category*, is obtained by taking $\mathcal{V}_0$ to be any category with finite products, taking $\otimes$ to be the categorical product $\times$, the unit object I to be the terminal object 1, and the coherence isomorphisms a, l, and r to be the canonical ones. Particular cases of cartesian monoidal categories include the category Set of sets and mappings, the category Top of topological spaces and continuous mappings, and the category SL of sup-complete lattices and order-preserving mappings. Further cartesian examples arise by taking $\mathcal{V}_0$ to be an ordered set with finite intersections, such as the two-element ordinal $2 = \{0, 1\}$.

The category Ab of abelian groups and their homomorphisms is a monoidal category, with the tensor product of abeian group being the monoidal product, denoted by $\otimes_{\mathbb{Z}}$, and with the abelian group of integers $\mathbb{Z}$ as the unit. However, this structure is not cartesian (see [7, Sec. 1.1]).

Lemma 2.1.1 (see [5]). *Let $\mathcal{V}$ be a monoidal category with underlying ordinary category $\mathcal{V}_0$. The following diagrams commute for all objects $A, B \in \mathcal{V}_0$:*

$$\begin{array}{ccc} (A \otimes B) \otimes I & \xrightarrow{r_{A \otimes B}} & A \otimes B \\ & {\scriptstyle a_{ABI}} \searrow \quad \nearrow {\scriptstyle 1 \otimes r_B} & \\ & A \otimes (B \otimes I) & \end{array}$$

$$\begin{array}{ccc} I \otimes (A \otimes B) & \xrightarrow{l_{A \otimes B}} & A \otimes B \\ & {\scriptstyle a^{-1}_{IAB}} \searrow \quad \nearrow {\scriptstyle l_A \otimes 1} & \\ & (I \otimes A) \otimes B & \end{array}$$

A *symmetry* s for a monoidal category $\mathcal{V}$ is a natural isomorphism $s_{AB} : A \otimes B \to B \otimes A$ satisfying the coherence axioms expressed by the commutativity of

$$\begin{array}{ccccc} (A \otimes B) \otimes C & \xrightarrow{a_{ABC}} & A \otimes (B \otimes C) & \xrightarrow{s_{A,B\otimes C}} & (B \otimes C) \otimes A \\ {\scriptstyle s_{AB} \otimes 1} \downarrow & & & & \downarrow {\scriptstyle a_{BCA}} \\ (B \otimes A) \otimes C & \xrightarrow[a_{BAC}]{} & B \otimes (A \otimes C) & \xrightarrow[1 \otimes s_{AC}]{} & B \otimes (C \otimes A) \end{array}$$

together with the identities $s_{BA} \circ s_{AB} = 1_{A \otimes B}$ and $r_A \circ s_{IA} = l_A$.

The monoidal category (symmetric or not) is said to be *closed* if each functor $- \otimes B : \mathcal{V}_0 \to \mathcal{V}_0$ has a right adjoint $[B, -]$, so that we have a natural bijection $\mathcal{V}_0(A \otimes B, C) \cong \mathcal{V}_0(A, [B, C])$ with *unit* and *counit* (the latter is called evaluation), say $d_A : A \to [B, A \otimes B]$ and $ev_{B,C} : [B, C] \otimes B \to C$. Monoidal categories Set, Ab, Top, SL and 2 are symmetric, and all are closed except Top (see [4, 7]).

2.2. Enriched Categories

An *enriched category* over a monoidal category $\mathcal{V}$ or a $\mathcal{V}$-category $\mathcal{A}$ consists of a class $ob\mathcal{A}$, a hom-object $\mathcal{A}(A, B) \in \mathcal{V}_0$ for each pair of objects of $\mathcal{A}$, a composition law $C_{ABC} : \mathcal{A}(A, B) \otimes \mathcal{A}(B, C) \to \mathcal{A}(A, C)$ for each triple of objects, and an identity element $j_A : I \to \mathcal{A}(A, A)$ for each object; subject to the associativity axiom expressed by the commutativity of

$$\begin{array}{ccc} (\mathcal{A}(A, B) \otimes \mathcal{A}(B, C)) \otimes \mathcal{A}(C, D) & \xrightarrow{C_{ABC} \otimes 1} & \mathcal{A}(A, C) \otimes \mathcal{A}(C, D) \\ {\scriptstyle a_{\mathcal{A}(A,B),\mathcal{A}(B,C),\mathcal{A}(C,D)}} \downarrow & & \\ \mathcal{A}(A, B) \otimes (\mathcal{A}(B, C) \otimes \mathcal{A}(C, D)) & & \downarrow {\scriptstyle C_{ACD}} \\ {\scriptstyle 1 \otimes C_{BCD}} \downarrow & & \\ \mathcal{A}(A, B) \otimes \mathcal{A}(B, D) & \xrightarrow[C_{ABD}]{} & \mathcal{A}(A, D) \end{array}$$

and unit axioms expressed by the identities

$$C_{ABB} \circ (1_{\mathcal{A}(A,B)} \otimes j_B) = r_{\mathcal{A}(A,B)} \text{ and } C_{AAB} \circ (j_A \otimes 1_{\mathcal{A}(A,B)}) = l_{\mathcal{A}(A,B)}.$$

A $\mathcal{V}$-category $\mathcal{A}$ is called small when $ob\mathcal{A}$ is a set. Enriched categories over Set are ordinary categories, Ab-categories are preadditive categories and enriched categories over 2 are preordered sets. Any symmetric monoidal closed category $\mathcal{V}$ can itself be endowed with a structure of a $\mathcal{V}$-category. More precisely, there is a $\mathcal{V}$-category, also denoted by $\mathcal{V}$, whose objects are those of $\mathcal{V}_0$, and whose hom-object $\mathcal{V}(A, B)$ is $[A, B]$. Its composition law $C_{ABC} : [A, B] \otimes [B, C] \to [A, C]$ corresponds, under adjunction, to the composite morphism

$$\begin{array}{c}
([A,B]\otimes[B,C])\otimes A \\
{\scriptstyle s_{[A,B],[B,C]}\otimes 1}\big\downarrow \\
([B,C]\otimes[A,B])\otimes A \\
{\scriptstyle a_{[B,C],[A,B],A}}\big\downarrow \\
[B,C]\otimes([A,B]\otimes A) \\
{\scriptstyle 1\otimes ev_{AB}}\big\downarrow \\
[B,C]\otimes B \\
{\scriptstyle ev_{BC}}\big\downarrow \\
C
\end{array}$$

and its identity element $j_A : I \to [A,A]$ corresponds, under adjunction, to $l_A : I \otimes A \to A$.

For any commutative ring R, the category Mod_R of (left) modules over R and their homomorphisms is a monoidal category with the monoidal product given by the tensor product of modules, denoted as $\otimes_R$, and the unit object by the trivial module R. Also, Mod_R is self-enriched as a symmetric monoidal closed category. Enriched categories over Mod_R are R-linear categories or R-algebroids.

Given $\mathcal{V}$ a symmetric monoidal category, every $\mathcal{V}$-category $\mathcal{A}$ gives rise to the dual of a $\mathcal{V}$-category $\mathcal{A}^\star$ whose objects are those of $ob\mathcal{A}$, and whose hom-object $\mathcal{A}^\star(A,B)$ is $\mathcal{A}(B,A)$. Its composition law $C^\star_{ABC} : \mathcal{A}^\star(A,B)\otimes\mathcal{A}^\star(B,C) \to \mathcal{A}^\star(A,C)$ is the composite morphism

$$\begin{array}{c}
\mathcal{A}(B,A)\otimes\mathcal{A}(C,B) \\
{\scriptstyle s_{\mathcal{A}(B,A),\mathcal{A}(C,B)}}\big\downarrow \\
\mathcal{A}(C,B)\otimes\mathcal{A}(B,A) \\
{\scriptstyle C_{CBA}}\big\downarrow \\
\mathcal{A}(C,A)
\end{array}$$

and its identity element is $j_A : I \to \mathcal{A}(A,A)$.

Lemma 2.2.1. *Let $\mathcal{A}$ be an enriched category over a monoidal category $\mathcal{V}$. Then, for any objects A, B, and C in $ob\mathcal{A}$, the diagrams*

$$\begin{array}{ccc}
\mathcal{A}(A,B)\otimes(\mathcal{A}(B,C)\otimes I) & \xrightarrow{a^{-1}_{\mathcal{A}(A,B),\mathcal{A}(B,C),I}} & (\mathcal{A}(A,B)\otimes\mathcal{A}(B,C))\otimes I \\
{\scriptstyle 1\otimes r^{-1}_{\mathcal{A}(B,C)}}\big\uparrow & \nearrow{\scriptstyle r^{-1}_{\mathcal{A}(A,B)\otimes\mathcal{A}(B,C)}} & \big\downarrow{\scriptstyle C_{ABC}\otimes 1} \\
\mathcal{A}(A,B)\otimes\mathcal{A}(B,C) & & \mathcal{A}(A,C)\otimes I \\
 & {\scriptstyle C_{ABC}}\searrow \quad \mathcal{A}(A,C) \quad \nearrow{\scriptstyle r^{-1}_{\mathcal{A}(A,C)}} &
\end{array}$$

and

$$\begin{array}{ccc}
(I\otimes\mathcal{A}(A,B))\otimes\mathcal{A}(B,C) & \xrightarrow{a_{I,\mathcal{A}(A,B),\mathcal{A}(B,C)}} & I\otimes(\mathcal{A}(A,B)\otimes\mathcal{A}(B,C)) \\
{\scriptstyle l^{-1}_{\mathcal{A}(A,B)}\otimes 1}\big\uparrow & \nearrow{\scriptstyle l^{-1}_{\mathcal{A}(A,B)\otimes\mathcal{A}(B,C)}} & \big\downarrow{\scriptstyle 1\otimes C_{ABC}} \\
\mathcal{A}(A,B)\otimes\mathcal{A}(B,C) & & I\otimes\mathcal{A}(A,C) \\
 & {\scriptstyle C_{ABC}}\searrow \quad \mathcal{A}(A,C) \quad \nearrow{\scriptstyle l^{-1}_{\mathcal{A}(A,C)}} &
\end{array}$$

commute.

Proof. By Lemma 2.1.1, the two diagrams commute, respectively, by the naturality of r^{-1} in the first case and by the naturality of l^{-1} in the second case. □

For $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, a $\mathcal{V}$*-functor* $T : \mathcal{A} \to \mathcal{B}$ consists of the assignment of an object $TA \in ob\mathcal{B}$ to each object $A \in ob\mathcal{A}$, together with, for each pair $A, B \in ob\mathcal{A}$, a morphism $T_{AB} : \mathcal{A}(A,B) \to \mathcal{B}(TA,TB)$, subject to the compatibility with composition expressed by the commutativity of

$$\begin{array}{ccc}
\mathcal{A}(A,B) \otimes \mathcal{A}(B,C) & \xrightarrow{C_{ABC}} & \mathcal{A}(A,C) \\
{\scriptstyle T_{AB}\otimes T_{BC}}\downarrow & & \downarrow{\scriptstyle T_{AC}} \\
\mathcal{B}(TA,TB) \otimes \mathcal{B}(TB,TC) & \xrightarrow[C_{TATBTC}]{} & \mathcal{B}(TA,TC)
\end{array}$$

and with the identities expressed by $T_{AA} \circ j_A = j_{TA}$. Taking $\mathcal{V} = Set$, 2, Ab, we recover the classical notions of, respectively, an ordinary functor, an increasing function, and a preadditive functor. Enriched functors over Mod_R are R-linear functors, and hence additive functors because Mod_R is an additive category (see [6, Examples 2.2.2]). Given $N \in Mod_R$, the functor $(-) \otimes_R N : Mod_R \to Mod_R$ that forms the tensor product of modules is additive (see [12]). Further details about additive functors can be found in [8,11]. For $\mathcal{V}$-functors $T, S : \mathcal{A} \to \mathcal{B}$, a $\mathcal{V}$*-natural transformation* $\alpha : T \Rightarrow S$ is an $ob\mathcal{A}$-indexed family of components $\alpha_A : I \to \mathcal{B}(TA, SA)$ satisfying the $\mathcal{V}$-naturality condition expressed by the commutativity of

$$\begin{array}{ccccc}
 & & \mathcal{A}(A,B) & & \\
 & \swarrow{\scriptstyle l^{-1}_{\mathcal{A}(A,B)}} & & \searrow{\scriptstyle r^{-1}_{\mathcal{A}(A,B)}} & \\
I \otimes \mathcal{A}(A,B) & & & & \mathcal{A}(A,B) \otimes I \\
{\scriptstyle \alpha_A \otimes S_{AB}}\downarrow & & & & \downarrow{\scriptstyle T_{AB}\otimes \alpha_B} \\
\mathcal{B}(TA,SA) \otimes \mathcal{B}(SA,SB) & & & & \mathcal{B}(TA,TB) \otimes \mathcal{B}(TB,SB) \\
 & \searrow{\scriptstyle C_{TASASB}} & & \swarrow{\scriptstyle C_{TATBSB}} & \\
 & & \mathcal{B}(TA,SB) & &
\end{array}$$

Proposition 2.2.2 (see [3])**.** *Let $\mathcal{V}$ be a symmetric monoidal closed category. If $\mathcal{A}$ is a $\mathcal{V}$-category and $T, S : \mathcal{A} \to \mathcal{V}$ are $\mathcal{V}$-functors, then giving a $\mathcal{V}$-natural transformation $\alpha : T \Rightarrow S$ is equivalent to giving a family of morphisms $\alpha_A : TA \to SA$ satisfying the $\mathcal{V}$-naturality condition expressed by the commutativity of*

$$\begin{array}{ccc}
\mathcal{A}(A,B) & \xrightarrow{T_{AB}} & [TA,TB] \\
{\scriptstyle S_{AB}}\downarrow & & \downarrow{\scriptstyle [1,\alpha_B]} \\
[SA,SB] & \xrightarrow[{[\alpha_A,1]}]{} & [TA,SB]
\end{array}$$

Assume that $\mathcal{V}$ is a complete symmetric monoidal closed category. For two $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A}$ is small, the category of $\mathcal{V}$-functors $\mathcal{A} \to \mathcal{B}$ and $\mathcal{V}$-natural transformations can be provided with the structure of a $\mathcal{V}$-category (see [3, Proposition 6.3.1]).

Definition 2.2.3. *Let $\mathcal{V}$ be a symmetric monoidal closed category. Given a $\mathcal{V}$-category $\mathcal{A}$ and two $\mathcal{V}$-functors $T, S : \mathcal{A} \to \mathcal{V}$, an object $N \in \mathcal{V}_0$ is called the object of $\mathcal{V}$-natural transformations from T to S, and one writes $N \cong \mathcal{V} - Nat(T,S)$, if for all $V \in \mathcal{V}_0$ there exist bijections, natural in the variable $V \in \mathcal{V}_0$ between*

(i) *the set of morphisms $V \to N$;*

(ii) *the class of $\mathcal{V}$-natural transformations $T \Rightarrow [V, S-]$.*

Lemma 2.2.4 (see [3])**.** *Let $\mathcal{V}$ be a complete symmetric monoidal closed category. Given a small $\mathcal{A}$ and two $\mathcal{V}$-functors $T, S : \mathcal{A} \to \mathcal{V}$, an object $N \in \mathcal{V}_0$ is isomorphic to the object of $\mathcal{V}$-natural transformations $\mathcal{V} - Nat(T, S)$, if and only if, for every object $V \in \mathcal{V}_0$, there is a bijective correspondence between*

(i) *the morphisms $V \to N$;*

(ii) *the $\mathcal{V}$-natural transformations $T \Rightarrow [V, S-]$.*

2.3. Weighted Limits

Weighted limits enjoy all the formal properties expected of a notion of limit and include the classical cone-type limits as a special case. In particular, a weighted limit in a Set-category can be expressed as an ordinary limit (see [3, 7]).

Definition 2.3.1. *Let $\mathcal{V}$ be a symmetric monoidal closed category. Given $\mathcal{V}$-functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{A} \to \mathcal{V}$, the $\mathcal{V}$-limit of F weighted by G exists when*

(i) *for every $B \in ob\mathcal{B}$, the object $\mathcal{V} - Nat(G, \mathcal{B}(B, F-))$ of $\mathcal{V}$-natural transformations exists;*

(ii) *there exists an object $L \in ob\mathcal{B}$ and isomorphisms in $\mathcal{V}$*

$$\lambda_B : \mathcal{V} - Nat(G, \mathcal{B}(B, F-)) \cong \mathcal{B}(B, L)$$

which are $\mathcal{V}$-natural in B.

The dual notion of weighted $\mathcal{V}$-limit is that of weighted $\mathcal{V}$-colimit.

Definition 2.3.2. *Let $\mathcal{V}$ be a symmetric monoidal closed category. Given $\mathcal{V}$-functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{A}^{\star} \to \mathcal{V}$, the $\mathcal{V}$-colimit of F weighted by G exists when*

(i) *for every $B \in ob\mathcal{B}$, the object $\mathcal{V} - Nat(G, \mathcal{B}(F-, B))$ of $\mathcal{V}$-natural transformations exists;*

(ii) *there exists an object $L \in ob\mathcal{B}$ and isomorphisms in $\mathcal{V}$*

$$\lambda_B : \mathcal{V} - Nat(G, \mathcal{B}(F-, B)) \cong \mathcal{B}(L, B)$$

which are $\mathcal{V}$-natural in B.

We write in general $(co)lim_G F$ for this weighted (co)limit, when it exists. When $(co)lim_G F$ exists for all choices of F and G, $\mathcal{A}$ small, $\mathcal{B}$ is said to be $\mathcal{V}$-(co)complete.

3. Coalgebras for an Enriched Functor

First, in this section, we introduce the notion of coalgebra for an enriched functor. Then, we prove that coalgebras for an enriched functor form an enriched category. This section concludes with some results on the existence of weighted colimits and weighted limits in an enriched category of coalgebras.

3.1. Enriched Coalgebras

For any enriched endofunctor $T : \mathcal{A} \to \mathcal{A}$, we discuss the conditions under which the class of T-coalgebras can be endowed with an enriched category structure.

Definition 3.1.1. *Let $\mathcal{V}$ be a monoidal category and $T : \mathcal{A} \to \mathcal{A}$ a $\mathcal{V}$-endofunctor. A T-coalgebra or a coalgebra of type T is a pair $\underline{A} = (A, \tau_A)$ consisting of an object A in $ob\mathcal{A}$ together with a $\mathcal{V}_0$-morphism $\tau_A : I \to \mathcal{A}(A, TA)$. The object A is called the underlying object of the T-coalgebra (A, τ_A) and τ_A is called its coalgebra structure.*

Given a pair of T-coalgebras $\underline{A} = (A, \tau_A)$ and $\underline{B} = (B, \tau_B)$, the hom-object $\mathcal{A}_T(\underline{A}, \underline{B})$ is said to satisfy the *homomorphism condition* whenever it exists.

Proposition 3.1.2. *Let $T : \mathcal{A} \to \mathcal{A}$ be a $\mathcal{V}$-endofunctor, where $\mathcal{V}$ is a symmetric monoidal category admitting equalizers. Then, T-coalgebras form a $\mathcal{V}$-category, denoted by $\mathcal{A}_T$.*

Proof. By the assumptions, the hom-object $\mathcal{A}_T(\underline{A}, \underline{B})$ exists for any two T-coalgebras $\underline{A} = (A, \tau_A)$ and $\underline{B} = (B, \tau_B)$. It is given as the regular subobject of $\mathcal{A}(A, B)$ represented by the equalizer $e_{AB} : \mathcal{A}_T(\underline{A}, \underline{B}) \to \mathcal{A}(A, B)$ of the pair of composite morphisms

$$\mathcal{A}(A,B) \xrightarrow{r^{-1}_{\mathcal{A}(A,B)}} \mathcal{A}(A,B) \otimes I \xrightarrow{1 \otimes \tau_B} \mathcal{A}(A,B) \otimes \mathcal{A}(B,TB) \xrightarrow{C_{ABTB}} \mathcal{A}(A,TB)$$

and

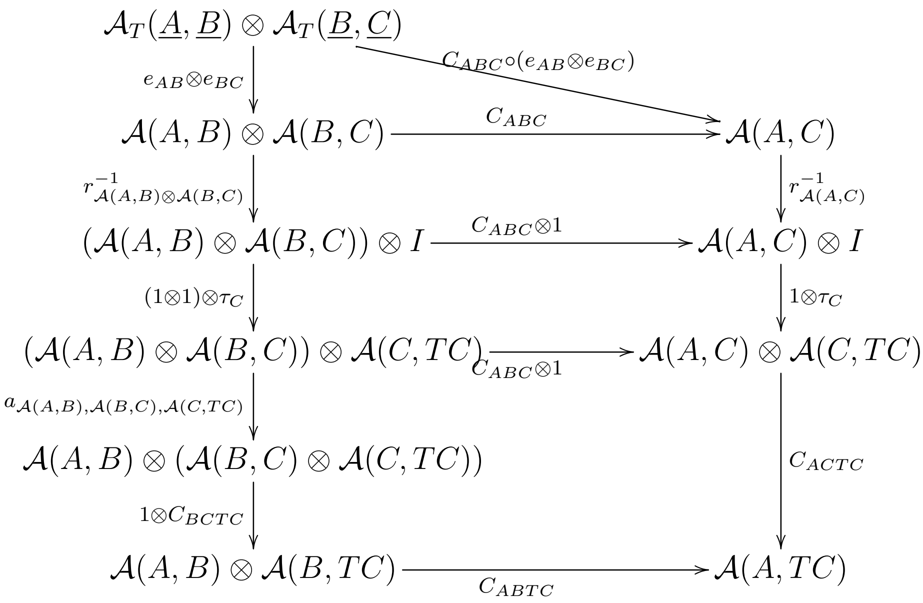
$$\mathcal{A}(A,B) \xrightarrow{l^{-1}_{\mathcal{A}(A,B)}} I \otimes \mathcal{A}(A,B) \xrightarrow{1 \otimes T_{AB}} I \otimes \mathcal{A}(TA,TB) \xrightarrow{\tau_A \otimes 1} \mathcal{A}(A,TA) \otimes \mathcal{A}(TA,TB) \xrightarrow{C_{ATATB}} \mathcal{A}(A,TB)$$

For any triple $\underline{A} = (A, \tau_A)$, $\underline{B} = (B, \tau_B)$ and $\underline{C} = (C, \tau_C)$ of T-coalgebras, let us prove that the composition law of $\mathcal{A}_T$ is the unique arrow $C_{\underline{A}\,\underline{B}\,\underline{C}}$ making the diagram

$$\begin{array}{ccc} \mathcal{A}_T(\underline{A},\underline{B}) \otimes \mathcal{A}_T(\underline{B},\underline{C}) & \xrightarrow{C_{\underline{A}\,\underline{B}\,\underline{C}}} & \mathcal{A}_T(\underline{A},\underline{C}) \\ {\scriptstyle e_{AB} \otimes e_{BC}} \downarrow & & \downarrow {\scriptstyle e_{AC}} \\ \mathcal{A}(A,B) \otimes \mathcal{A}(B,C) & \xrightarrow[C_{ABC}]{} & \mathcal{A}(A,C) \end{array}$$

commute with e_{AB}, e_{BC} and e_{AC}, the respective regular subobject witness.

By the naturality of r^{-1} together with the associativity axiom, the following diagram commutes as the monoidal product $\otimes$ is a bifunctorial correspondence:



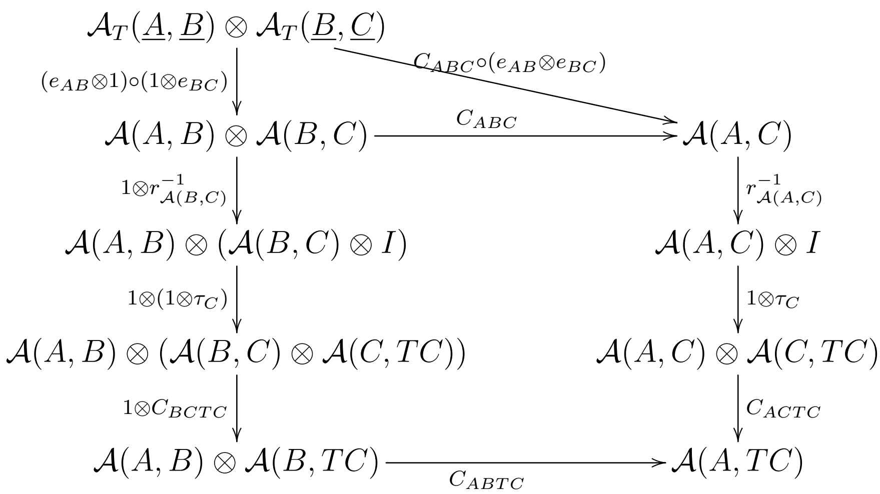
Also, it holds that

$$a_{\mathcal{A}(A,B),\mathcal{A}(B,C),\mathcal{A}(C,TC)} \circ ((1\otimes 1)\circ \tau_C) = (1\otimes(1\otimes\tau_C))\circ a_{\mathcal{A}(A,B),\mathcal{A}(B,C),I}$$

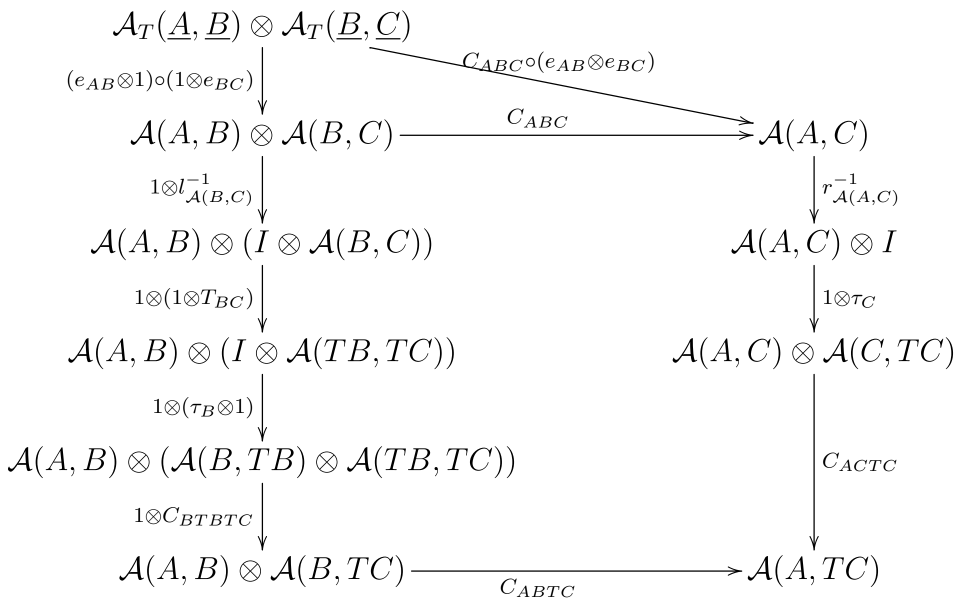
due to the naturality of a. Moreover,

$$a_{\mathcal{A}(A,B),\mathcal{A}(B,C),I}\circ r^{-1}_{\mathcal{A}(A,B)\otimes\mathcal{A}(B,C)} = 1\otimes r^{-1}_{\mathcal{A}(B,C)}$$

due to Lemma 2.1.1. As a consequence, the following diagram commutes:



The following diagram also commutes as $\mathcal{A}_T(\underline{B}, \underline{C})$ satisfies the homomorphism condition:



From the associativity axiom, it follows that

$$C_{ABTC} \circ (1 \otimes C_{BTBTC}) = C_{ATBTC} \circ (C_{ABTB} \otimes 1) \circ a^{-1}_{\mathcal{A}(A,B),\mathcal{A}(B,TB),\mathcal{A}(TB,TC)}.$$

In addition, it holds that

$$\begin{aligned} &a^{-1}_{\mathcal{A}(A,B),\mathcal{A}(B,TB),\mathcal{A}(TB,TC)} \circ [(1 \otimes (\tau_B \otimes 1)) \circ (1 \otimes (1 \otimes T_{BC}))] \\ &= [((1 \otimes 1) \otimes T_{BC}) \circ ((1 \otimes \tau_B) \otimes 1)] \circ a^{-1}_{\mathcal{A}(A,B),I,\mathcal{A}(B,C)} \end{aligned}$$

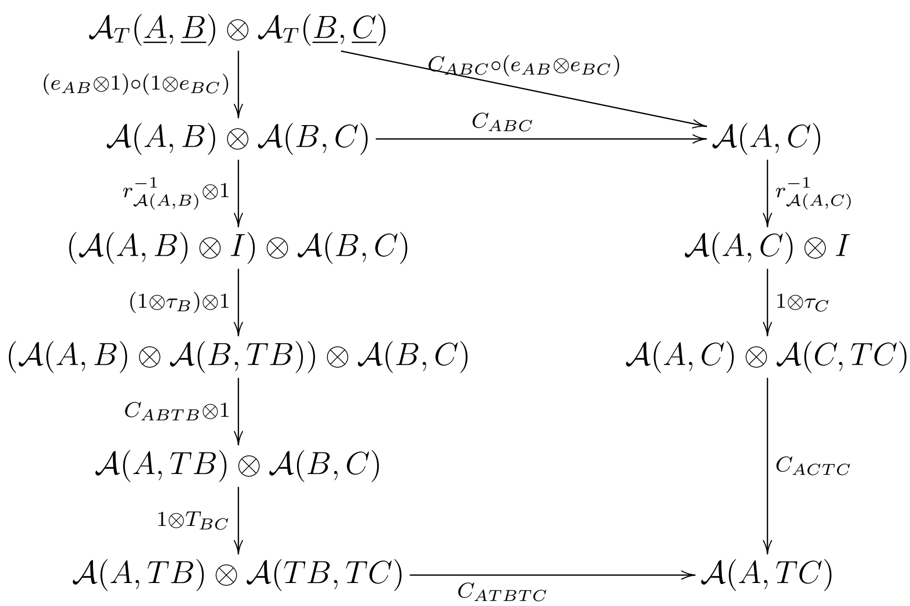
because a^{-1} is natural. By the second coherence axiom, one has

$$a^{-1}_{\mathcal{A}(A,B),I,\mathcal{A}(B,C)} \circ (1 \otimes l^{-1}_{\mathcal{A}(B,C)}) = r^{-1}_{\mathcal{A}(A,B)} \otimes 1.$$

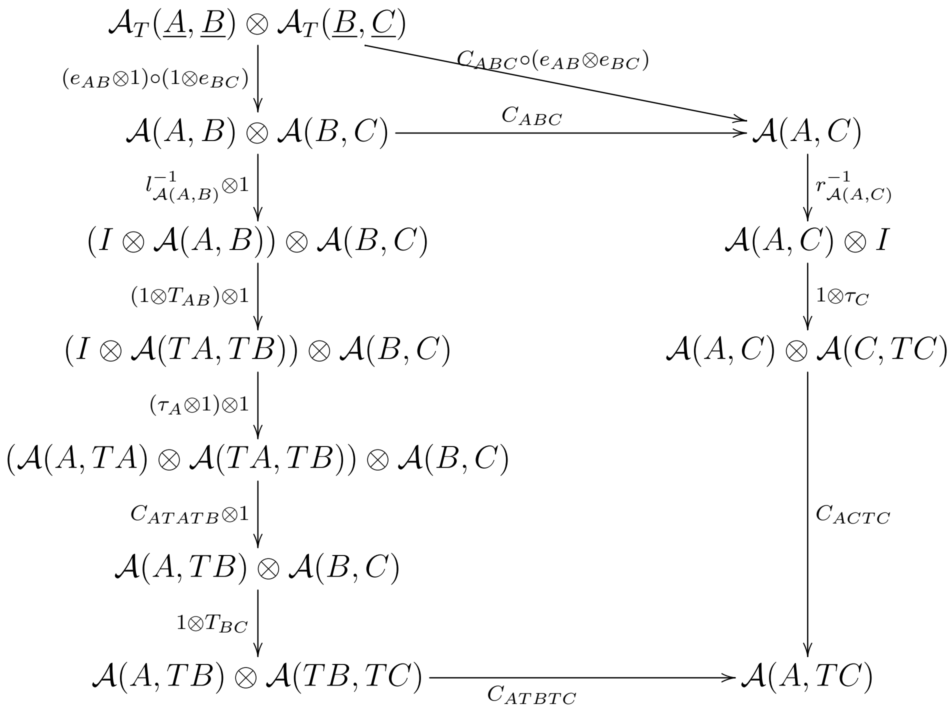
Also,

$$(C_{ABTB} \otimes 1) \circ ((1 \otimes 1) \otimes T_{BC}) = (1 \otimes T_{BC}) \circ (C_{ABC} \otimes 1)$$

as $\otimes$ is a bifunctor. Hence, the following commutative diagram is obtained:



Since $\mathcal{A}_T(\underline{A}, \underline{B})$ satisfies the homomorphism condition, the following diagram commutes:



However,

$$(1 \otimes T_{BC}) \otimes [(C_{ATATB} \otimes 1) \circ ((\tau_A \otimes 1) \otimes 1)] = [(C_{ATATB} \otimes 1) \circ ((\tau_A \otimes 1) \otimes 1)] \circ ((1 \otimes 1) \otimes T_{BC})$$

as $\otimes$ is a bifunctor. In a similar way, we have

$$((1 \otimes 1) \otimes T_{BC}) \circ ((1 \otimes T_{AB}) \otimes 1) = 1 \otimes (T_{AB} \otimes T_{BC}).$$

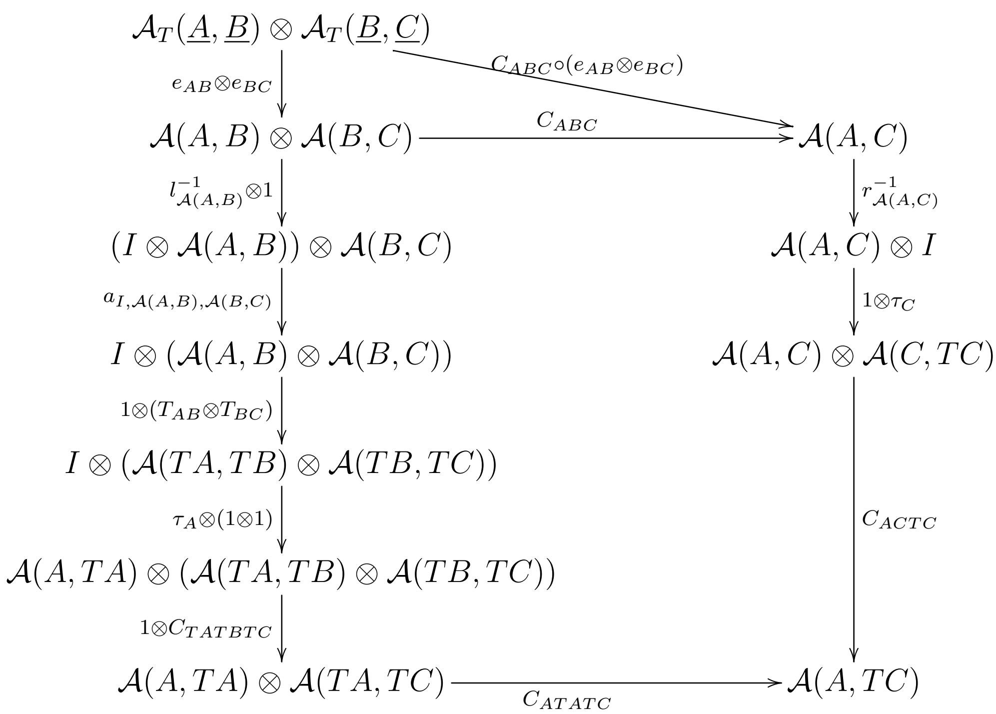
By the associativity axiom, it holds that

$$C_{ATBTC} \circ (C_{ATATB} \otimes 1) = C_{ATATC} \circ (1 \otimes C_{TATBTC}) \circ a_{\mathcal{A}(A,TA),\mathcal{A}(TA,TB),\mathcal{A}(TB,TC)}.$$

Furthermore, the naturality of a yields

$$a_{\mathcal{A}(A,TA),\mathcal{A}(TA,TB),\mathcal{A}(TB,TC)} \circ [((\tau_A \otimes 1) \otimes 1) \circ ((1 \otimes T_{AB}) \otimes T_{BC})] = [(\tau_A \otimes (1 \otimes 1)) \circ (1 \otimes (T_{AB} \otimes T_{BC}))] \circ a_{I,\mathcal{A}(A,B),\mathcal{A}(B,C)},$$

that is, the following diagram commutes:



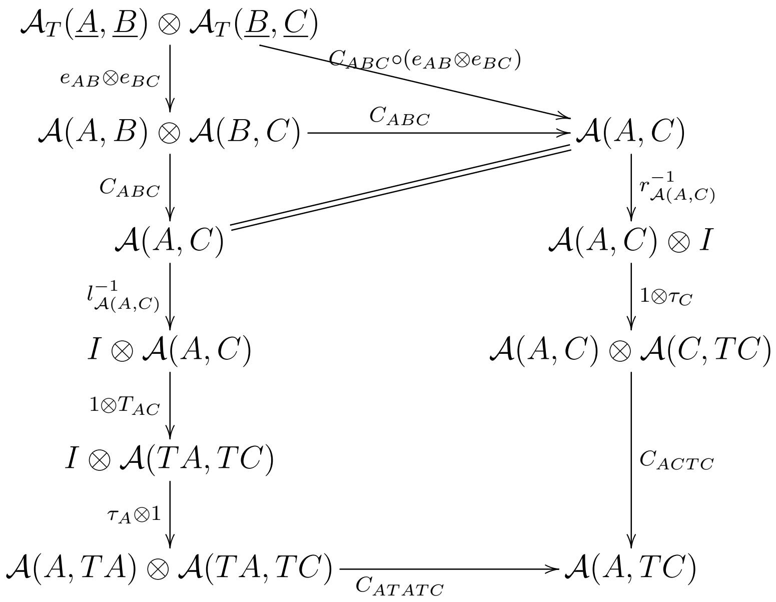
Since the monoidal product $\otimes$ is a bifunctor, one has

$$(1 \otimes C_{TATBTC}) \circ (\tau_A \otimes (1 \otimes 1)) = (\tau_A \otimes 1) \circ (1 \circ C_{TATBTC}).$$

According to the compatibility with composition,

$$(1 \otimes C_{TATBTC}) \circ (1 \otimes (T_{AB} \otimes T_{BC})) = (1 \otimes T_{AC}) \circ (1 \otimes C_{ABC}).$$

Also, it holds that $(1 \otimes C_{ABC}) \circ a_{I,\mathcal{A}(A,B),\mathcal{A}(B,C)} \circ (l^{-1}_{\mathcal{A}(A,B)} \otimes 1) = l^{-1}_{\mathcal{A}(A,C)} \circ C_{ABC}$ due to Lemma 2.1.1. Consequently, the following diagram commutes:

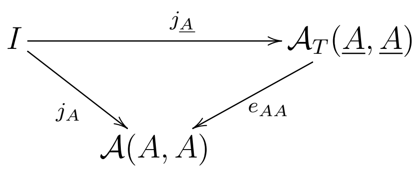


As, in addition, $\mathcal{A}_T(\underline{A},\underline{C})$ satisfies the homomorphism condition, there is a unique arrow

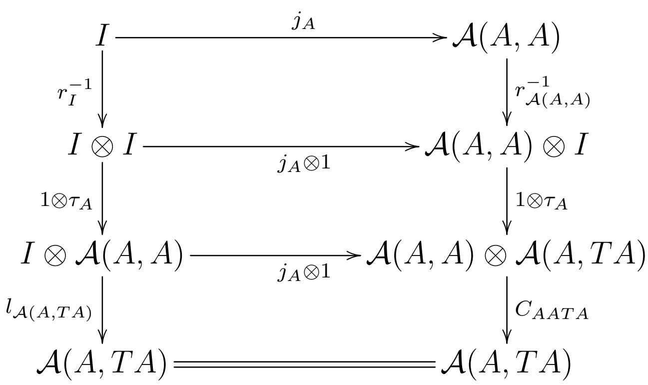
$$C_{\underline{A}\,\underline{B}\,\underline{C}} : \mathcal{A}_T(\underline{A},\underline{B}) \otimes \mathcal{A}_T(\underline{B},\underline{C}) \to \mathcal{A}_T(\underline{A},\underline{C})$$

such that $e_{AC} \circ C_{\underline{A}\,\underline{B}\,\underline{C}} = C_{ABC} \circ (e_{AB} \otimes e_{BC})$. This is the composition law of $\mathcal{A}_T$.

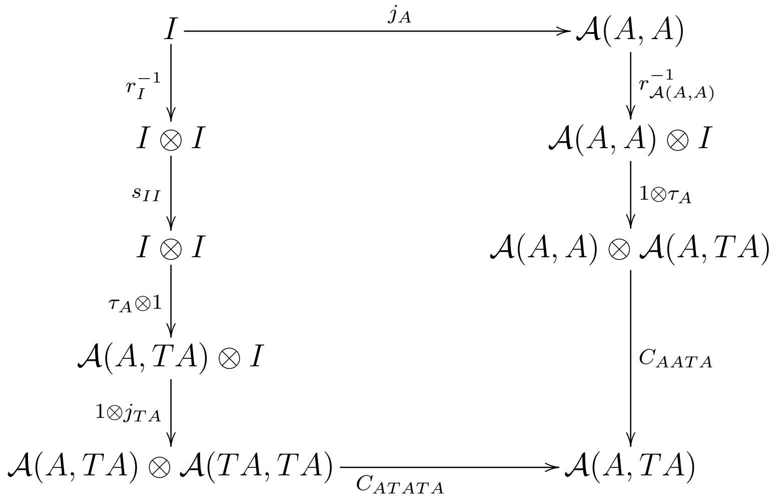
For each T-coalgebra $\underline{A} = (A, \tau_A)$, the identity element of $\mathcal{A}_T$ is the unique arrow $j_{\underline{A}} : I \to \mathcal{A}_T(\underline{A},\underline{A})$ making the following diagram commute:



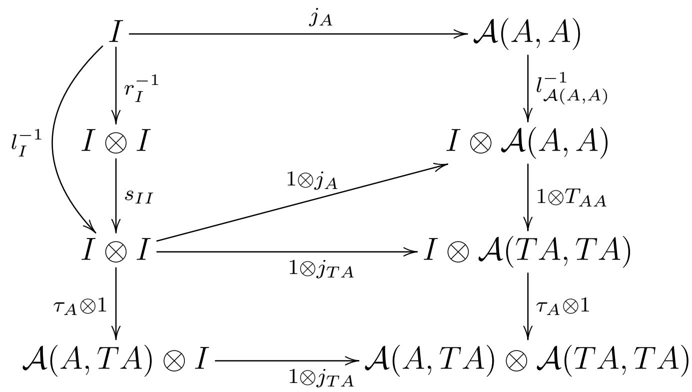
with e_{AA} the regular subobject witness. Indeed, the following diagram commutes as $\otimes$ is a bifunctor:



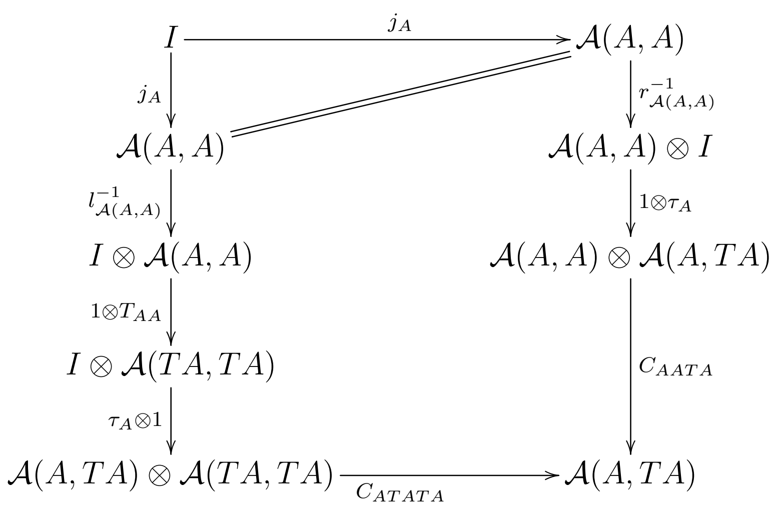
This results from the naturality of r^{-1} and the unit axiom $C_{AATA} \circ (j_A \otimes 1) = l_{\mathcal{A}(A,TA)}$. By the coherence axiom $l_{\mathcal{A}(A,TA)} = r_{\mathcal{A}(A,TA)} \circ s_{I,\mathcal{A}(A,TA)}$ and the unit axiom $C_{ATATA} \circ (1 \otimes j_{TA}) = r_{\mathcal{A}(A,TA)}$, it holds that $l_{\mathcal{A}(A,TA)} = C_{ATATA} \circ (1 \otimes j_{TA}) \circ s_{I,\mathcal{A}(A,TA)}$, and hence, the following diagram commutes as $s_{I,\mathcal{A}(A,TA)} \circ (1 \otimes \tau_A) = (\tau_A \otimes 1) \circ s_{II}$ because s is natural:



Since $\otimes$ is a bifunctor, $(1 \otimes j_{TA}) \circ (\tau_A \otimes 1) = (\tau_A \otimes 1) \circ (1 \otimes j_{TA})$. Also, the identity $T_{AA} \circ j_A = j_{TA}$ can be transformed into $(1 \otimes T_{AA}) \circ (1 \otimes j_A) = 1 \otimes j_{TA}$. Consequently, the following diagram commutes as l^{-1} is natural:



From this, we deduce the commutative diagram given below.



By the universal property of equalizers, there is a unique arrow $j_{\underline{A}}; I \to \mathcal{A}_T(\underline{A}, \underline{A})$ such that $e_{AA} \circ j_{\underline{A}} = j_A$. Hence, $j_{\underline{A}}$ is the identity element of $\mathcal{A}_T$ as indicated above. The associativity and unit axioms are easily verified. Thus, $\mathcal{A}_T$ is a $\mathcal{V}$-category. $\square$

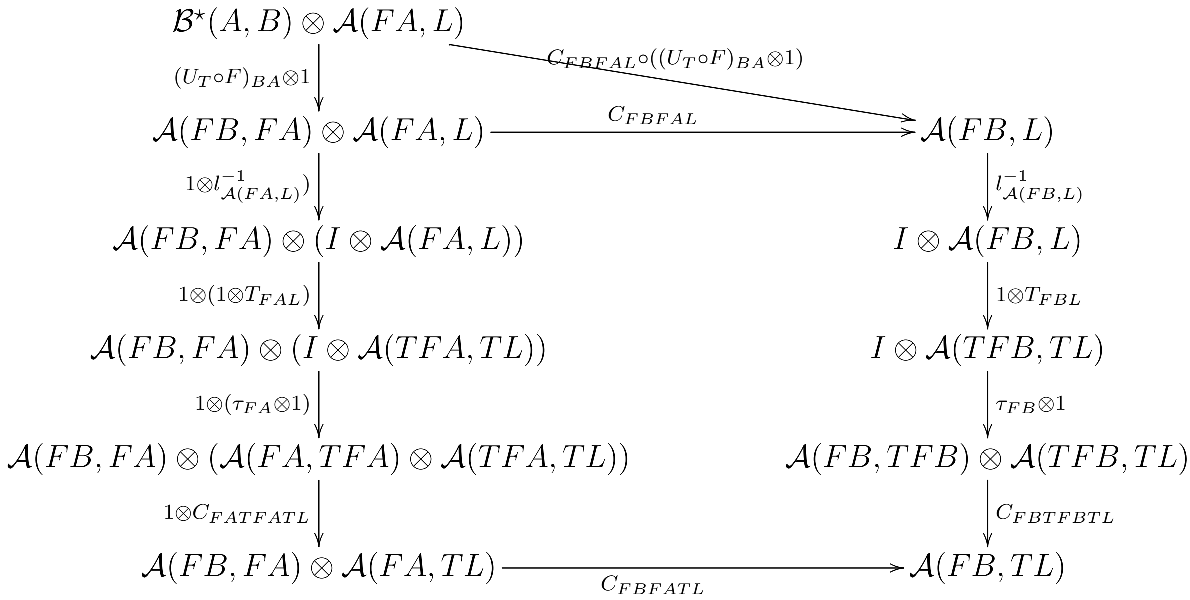
Corollary 3.1.3. *Let $T : \mathcal{A} \to \mathcal{A}$ be a $\mathcal{V}$-endofunctor, where $\mathcal{V}$ is a symmetric monoidal category admitting equalizers. The correspondence $U_T : ob\mathcal{A}_T \to ob\mathcal{A}$ that assigns each T-coalgebra (A, τ_A) to its underlying object A together with for each pair of T-coalgebras $\underline{A} = (A, \tau_A)$ and $\underline{B} = (B, \tau_B)$, the regular subobject witness $(U_T)_{\underline{A}\,\underline{B}} : \mathcal{A}_T(\underline{A}, \underline{B}) \to \mathcal{A}(A, B)$ is a $\mathcal{V}$-functor.*

Proof. It suffices to check the compatibility with composition and with the identities that, respectively, arise from the composition law and the identity element of $\mathcal{A}_T$. □

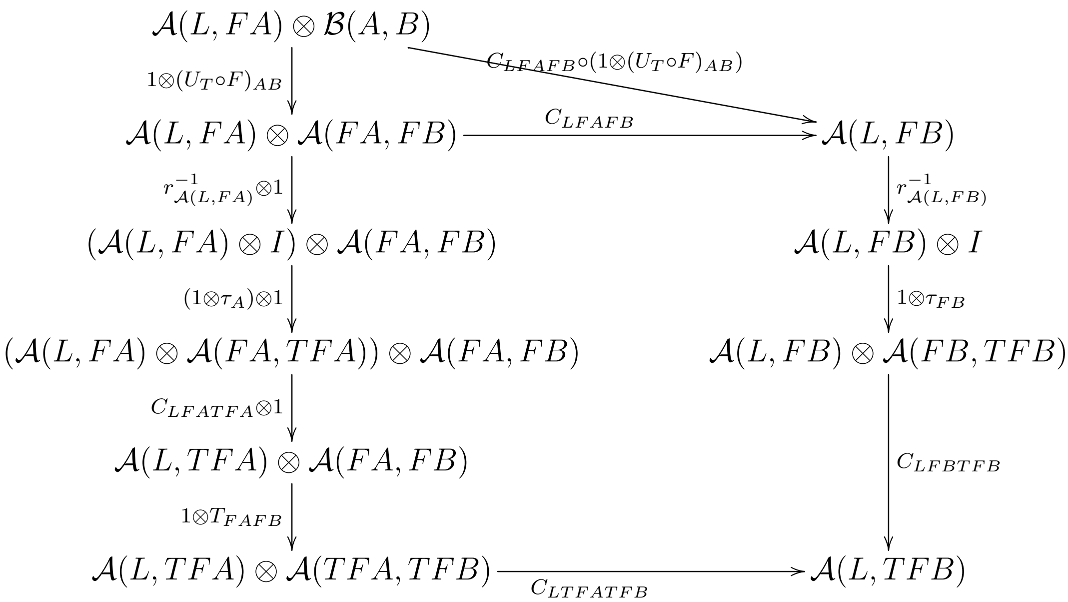
3.2. Weighted Limits in $\mathcal{A}_T$

We are interested in the creation of weighted limits by the underlying $\mathcal{V}$-functor $U_T : \mathcal{A}_T \to \mathcal{A}$. Before doing so, we recall the following fact.

Lemma 3.2.1. *Let $T : \mathcal{A} \to \mathcal{A}$ be a $\mathcal{V}$-endofunctor, where $\mathcal{V}$ is a symmetric monoidal category admitting equalizers. Given a $\mathcal{V}$-functor $F : \mathcal{B} \to \mathcal{A}_T$. For each pair $A, B \in \mathcal{B}$ and $L \in \mathcal{A}$, the diagrams*



and



commute.

Proof. First Diagram. By the associativity axiom, it holds that

$$C_{FBFATL} \circ (1 \otimes C_{FATFATL}) = C_{FBTFATL} \circ (C_{FBFATFA} \otimes 1) \circ a^{-1}_{\mathcal{A}(FB,FA),\mathcal{A}(FA,TFA),\mathcal{A}(TFA,TL)}.$$

Also, we have

$$\begin{aligned} &a^{-1}_{\mathcal{A}(FB,FA),\mathcal{A}(FA,TFA),\mathcal{A}(TFA,TL)} \circ [(1 \otimes (\tau_{FA} \otimes 1)) \circ (1 \otimes (1 \otimes T_{FAL}))] \\ &= [((1 \otimes \tau_{FA}) \otimes 1) \circ (1 \otimes 1) \otimes T_{FAL}))] \circ a^{-1}_{\mathcal{A}(FB,FA),I,\mathcal{A}(FA,L)} \end{aligned}$$

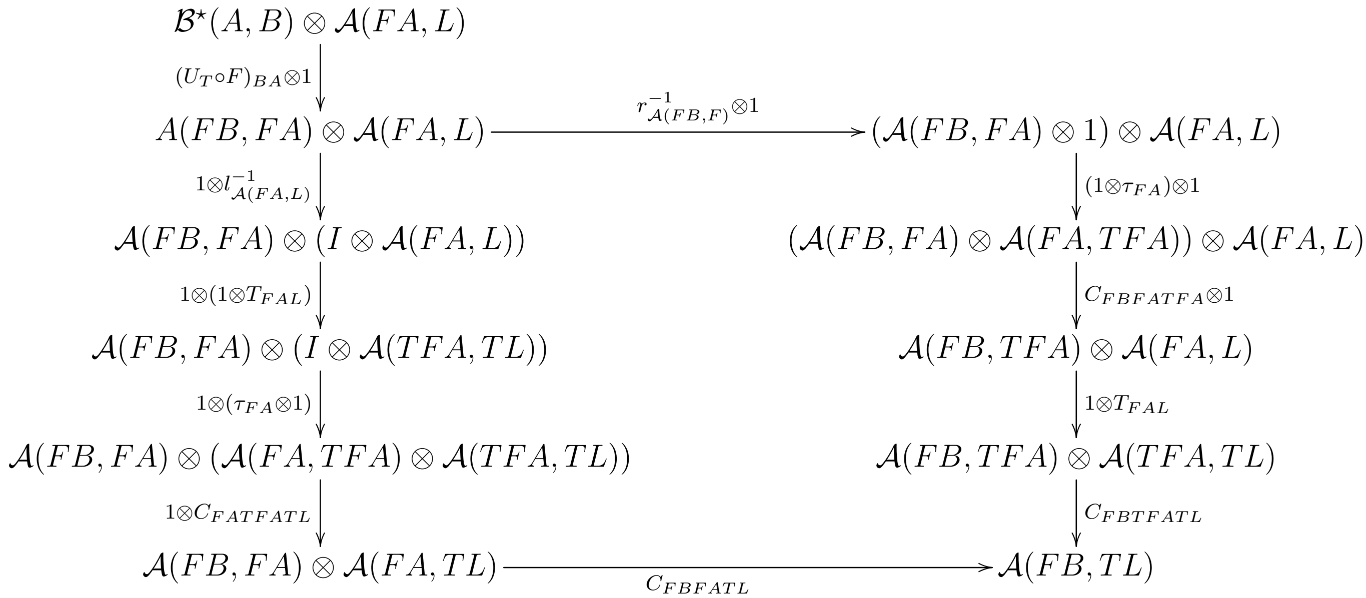
as a^{-1} is natural, and due to the second coherence axiom,

$$a^{-1}_{\mathcal{A}(FB,FA),I,\mathcal{A}(FA,L)} \circ (1 \otimes l^{-1}_{\mathcal{A}(FA,L)}) = r^{-1}_{\mathcal{A}(FB,FA)} \otimes 1.$$

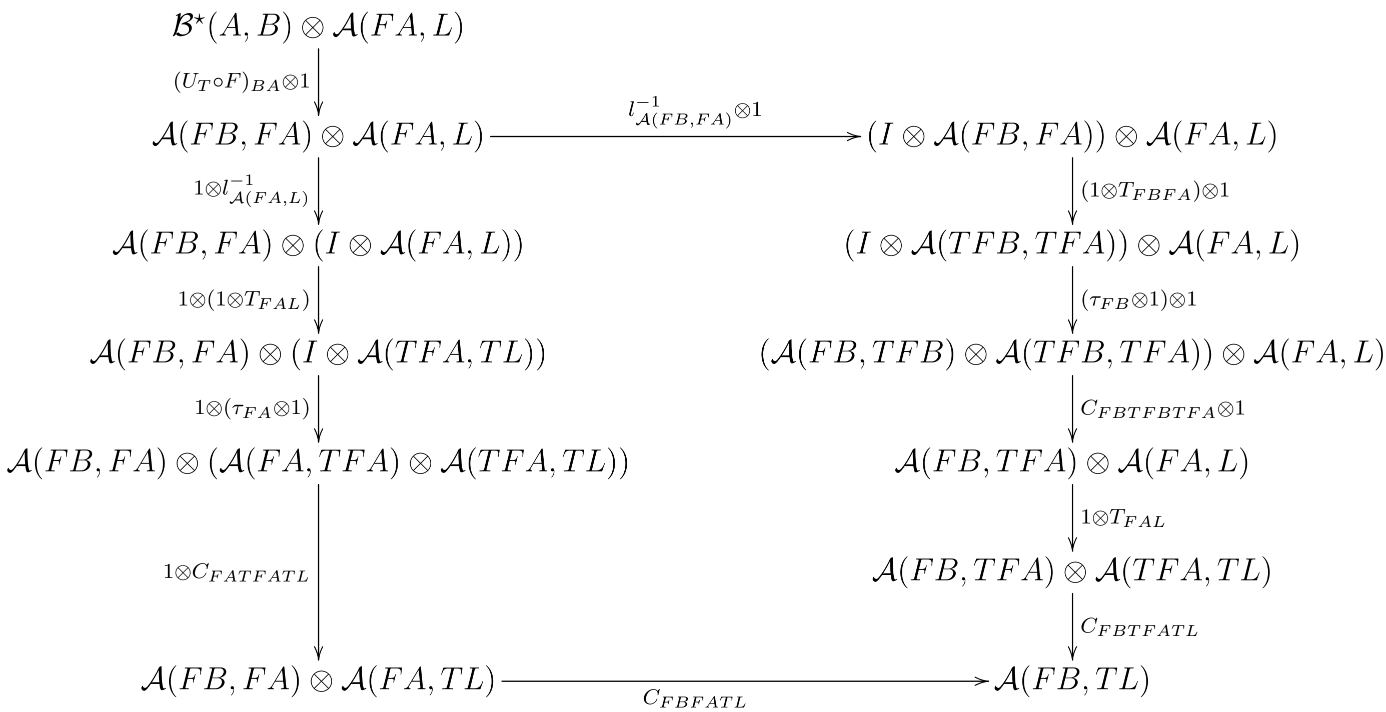
Furthermore, we have

$$[(C_{FBFATFA} \otimes 1) \circ ((1 \otimes \tau_{FA}) \otimes 1)] \circ ((1 \otimes 1) \otimes T_{FAL}) = (1 \otimes T_{FAL}) \circ [(C_{FBFATFA} \otimes 1) \circ ((1 \otimes \tau_{FA}) \otimes 1)]$$

as $\otimes$ is a bifunctor, and hence, the following diagram commutes:



Since $(U_T \circ F)_{BA} = (U_T)_{FBFA} \circ F_{BA}$, the following diagram also commutes because $\mathcal{A}_T(FB, FA)$ satisfies the homomorphism condition:



Notice that

$$
\begin{aligned}
&(1\otimes T_{FAL})\circ(C_{FBTFBTFA}\otimes 1)\circ((\tau_{FB}\otimes 1)\otimes 1)\circ((1\otimes T_{FBFA})\otimes 1)\\
&=(C_{FBTFBTFA}\otimes 1)\circ((\tau_{FB}\otimes 1)\otimes 1)\circ((1\otimes T_{FBFA})\otimes T_{FAL})
\end{aligned}
$$

as $\otimes$ is a bifunctor. By the associativity axiom, it holds that

$$
C_{FBTFATL}\circ(C_{FBTFBTL}\otimes 1)=C_{FBTFBTL}\circ(1\otimes C_{TFBTFATL})\circ a_{\mathcal{A}(FB,TFB),\mathcal{A}(TFB,TFA),\mathcal{A}(TFA,TL)}.
$$

The naturality of a yields

$$
\begin{aligned}
&a_{\mathcal{A}(FB,TFB),\mathcal{A}(TFB,TFA),\mathcal{A}(TFA,TL)}\circ[((\tau_{FB}\otimes 1)\otimes 1)\circ((1\otimes T_{FBFA})\otimes T_{FAL})]\\
&=[(\tau_{FB}\otimes(1\otimes 1))\circ(1\otimes(T_{FBFA}\otimes T_{FAL}))]\circ a_{I,\mathcal{A}(FB,FA),\mathcal{A}(FA,L)}.
\end{aligned}
$$

As a consequence, the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{B}^\star(A,B)\otimes\mathcal{A}(FA,L) & & \\
\downarrow{\scriptstyle (U_T\circ F)_{BA}\otimes 1} & & \\
\mathcal{A}(FB,FA)\otimes\mathcal{A}(FA,L) & \xrightarrow{l^{-1}_{\mathcal{A}(FB,FA)}\otimes 1} & (I\otimes\mathcal{A}(FB,FA))\otimes\mathcal{A}(FA,L)\\
\downarrow{\scriptstyle 1\otimes l^{-1}_{\mathcal{A}(FA,L)}} & & \downarrow{\scriptstyle a_{I,\mathcal{A}(FB,FA),\mathcal{A}(FA,L)}}\\
\mathcal{A}(FB,FA)\otimes(I\otimes\mathcal{A}(FA,L)) & & I\otimes(\mathcal{A}(FB,FA)\otimes\mathcal{A}(FA,L))\\
\downarrow{\scriptstyle 1\otimes(1\otimes T_{FAL})} & & \downarrow{\scriptstyle 1\otimes(T_{FBFA}\otimes T_{FAL})}\\
\mathcal{A}(FB,FA)\otimes(I\otimes\mathcal{A}(TFA,TL)) & & I\otimes(\mathcal{A}(TFB,TFA)\otimes\mathcal{A}(TFA,TL))\\
\downarrow{\scriptstyle 1\otimes(\tau_{FA}\otimes 1)} & & \downarrow{\scriptstyle \tau_{FB}\otimes(1\otimes 1)}\\
\mathcal{A}(FB,FA)\otimes(\mathcal{A}(FA,TFA)\otimes\mathcal{A}(TFA,TL)) & & \mathcal{A}(FB,TFB)\otimes(\mathcal{A}(TFB,TFA)\otimes\mathcal{A}(TFA,TL))\\
\downarrow{\scriptstyle 1\otimes C_{FATFATL}} & & \downarrow{\scriptstyle 1\otimes C_{TFBTFATL}}\\
 & & \mathcal{A}(FB,TFB)\otimes\mathcal{A}(TFB,TL)\\
 & & \downarrow{\scriptstyle C_{FBTFBTL}}\\
\mathcal{A}(FB,FA)\otimes\mathcal{A}(FA,TL) & \xrightarrow[C_{FBTFBTL}]{} & \mathcal{A}(FB,TL)
\end{array}
$$

Since $\otimes$ is a bifunctor, we have

$$
(1\otimes C_{TFBTFATL})\circ(\tau_{FB}\otimes(1\otimes 1))=(\tau_{FB}\otimes 1)\circ(1\otimes C_{TFBTFATL}).
$$

Also, we have $(1\otimes C_{TFBTFATL})\circ(1\otimes(T_{FBFA}\otimes T_{FAL}))=(1\otimes T_{FBL})\circ(1\otimes C_{FBFAL})$, as a consequence of the compatibility with composition. Moreover,

$$
(1\otimes C_{FBFAL})\circ a_{I,\mathcal{A}(FB,FA),\mathcal{A}(FA,L)}\circ(l^{-1}_{\mathcal{A}(FB,FA)}\otimes 1)=l^{-1}_{\mathcal{A}(FB,L)}\circ C_{FBFAL},
$$

by virtue of Lemma 2.2.1.

Second Diagram. We note that

$$
(1\otimes T_{FAFB})\circ[(C_{LFATFA}\otimes 1)\circ((1\otimes\tau_{FA})\otimes 1)]=[(C_{LFATFA}\otimes 1)\circ((1\otimes\tau_{FA})\otimes 1)]\circ((1\otimes 1)\otimes T_{FAFB})
$$

since $\otimes$ is a bifunctor. From the associativity axiom, it follows that

$$
C_{LTFATFB}\circ(C_{LFATFA}\otimes 1)=C_{LFATFB}\circ(1\otimes C_{FATFATFB})\circ a_{\mathcal{A}(L,FA),\mathcal{A}(FA,TFA),\mathcal{A}(TFA,TFB)}.
$$

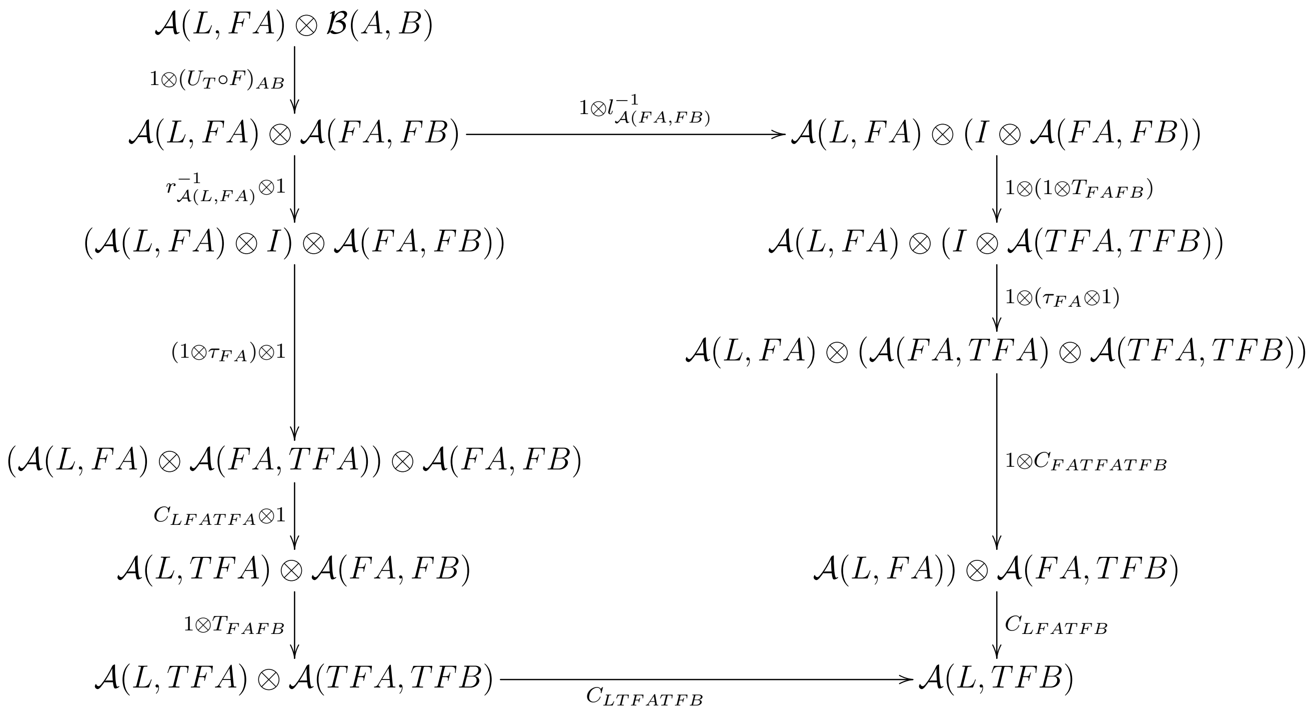
However,

$$a_{\mathcal{A}(L,FA),\mathcal{A}(FA,TFA),\mathcal{A}(TFA,TFB)} \circ [((1 \otimes \tau_{FA}) \otimes 1) \circ ((1 \otimes 1) \otimes T_{FAFB})]$$
$$= [(1 \otimes (\tau_{FA} \otimes 1)) \circ (1 \otimes (1 \otimes T_{FAFB}))] \circ a_{\mathcal{A}(L,FA),I,\mathcal{A}(FA,FB)}$$

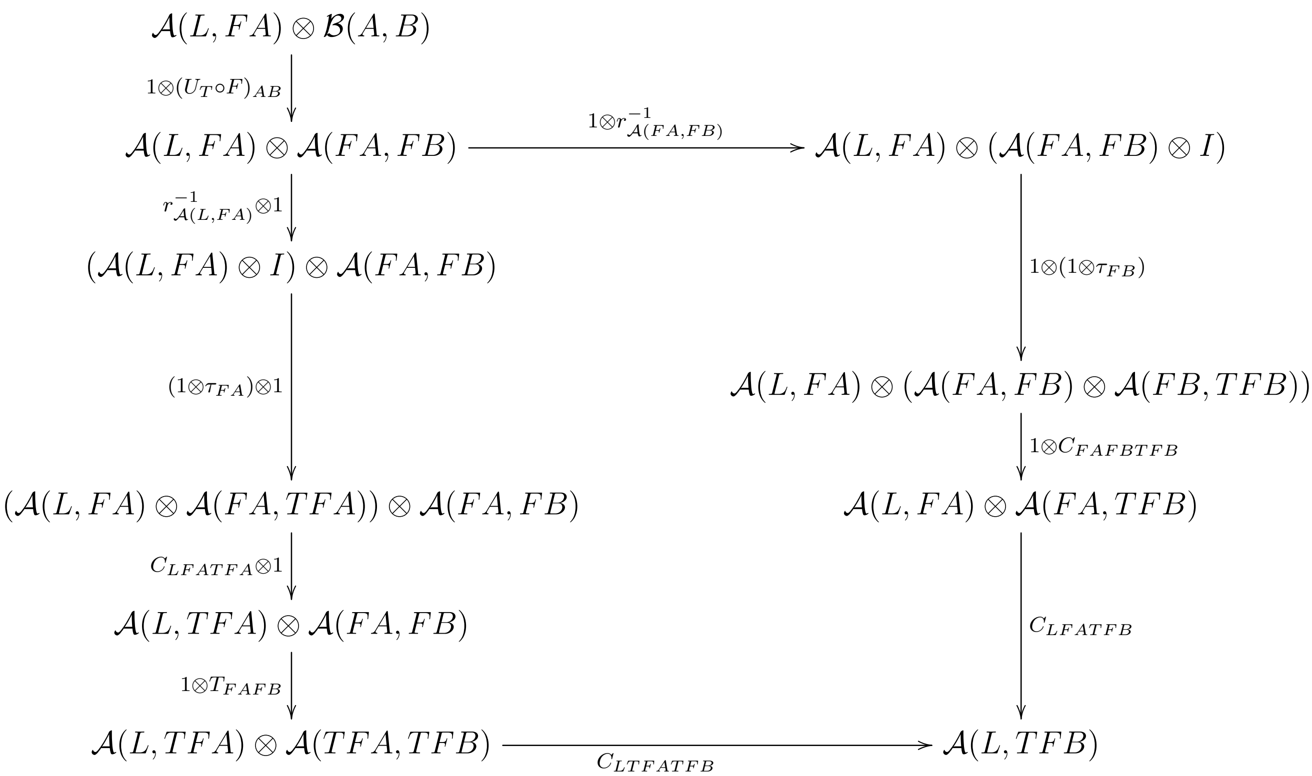
as a is natural. Also, we have

$$a_{\mathcal{A}(L,FA),I,\mathcal{A}(FA,FB)} \circ (r^{-1}_{\mathcal{A}(L,FA)} \otimes 1) = 1 \otimes l^{-1}_{\mathcal{A}(FA,FB)}$$

due to the second coherence axiom. Hence, the following diagram commutes:



The following diagram also commutes as $\mathcal{A}_T(FA, FB)$ satisfies the homomorphism condition (which follows from the fact that $(U_T \circ F)_{AB} = (U_T)_{FAFB} \circ F_{AB}$):



By the associativity axiom, we have

$$C_{LFATFB}\circ(1\otimes C_{FAFBTFB})=C_{LFBTFB}\circ(C_{LFAFB}\otimes 1)\circ a^{-1}_{\mathcal{A}(L,FA),\mathcal{A}(FA,FB),\mathcal{A}(FB,TFB)}.$$

Since a^{-1} is natural, we have

$$a^{-1}_{\mathcal{A}(L,FA),\mathcal{A}(FA,FB),\mathcal{A}(FB,TFB)}\circ(1\otimes(1\otimes\tau_{FB}))=((1\otimes 1)\otimes\tau_{FB})\circ a^{-1}_{\mathcal{A}(L,FA),\mathcal{A}(FA,FB),I}.$$

As a consequence, the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{A}(L,FA)\otimes\mathcal{B}(A,B) & & \\
\downarrow{\scriptstyle 1\otimes(U_T\circ F)_{AB}} & & \\
\mathcal{A}(L,FA)\otimes\mathcal{A}(FA,FB) & \xrightarrow{1\otimes r^{-1}_{\mathcal{A}(FA,FB)}} & \mathcal{A}(L,FA)\otimes(\mathcal{A}(FA,FB)\otimes I)\\
\downarrow{\scriptstyle r^{-1}_{\mathcal{A}(L,FA)}\otimes 1} & & \downarrow{\scriptstyle a^{-1}_{\mathcal{A}(L,FA),\mathcal{A}(F,FB),I}}\\
(\mathcal{A}(L,FA)\otimes I)\otimes\mathcal{A}(FA,FB) & & (\mathcal{A}(L,FA)\otimes\mathcal{A}(FA,FB))\otimes I\\
\downarrow{\scriptstyle (1\otimes\tau_{FA})\otimes 1} & & \downarrow{\scriptstyle (1\otimes 1)\otimes\tau_{FB}}\\
(\mathcal{A}(L,FA)\otimes\mathcal{A}(FA,TFA))\otimes\mathcal{A}(FA,FB) & & (\mathcal{A}(L,FA)\otimes\mathcal{A}(FA,FB))\otimes\mathcal{A}(FB,TFB)\\
\downarrow{\scriptstyle C_{LFATFA}\otimes 1} & & \downarrow{\scriptstyle C_{LFAFB}\otimes 1}\\
\mathcal{A}(L,TFA)\otimes\mathcal{A}(FA,FB) & & \mathcal{A}(L,FB)\otimes\mathcal{A}(B,TFB)\\
\downarrow{\scriptstyle 1\otimes T_{FAFB}} & & \downarrow{\scriptstyle C_{LFBTFB}}\\
\mathcal{A}(L,TFA)\otimes\mathcal{A}(TFA,TFB) & \xrightarrow[C_{LTFATFB}]{} & \mathcal{A}(L,TFB)
\end{array}
$$

However, $(C_{LFAFB}\otimes 1)\circ((1\otimes 1)\otimes\tau_{FB})=(1\otimes\tau_{FB})\circ(C_{LFAFB}\otimes 1)$ as $\otimes$ is a bifunctor. On the other hand, it holds that

$$(C_{LFAFB}\otimes 1)\circ a^{-1}_{\mathcal{A}(L,FA),\mathcal{A}(FA,FB),I}\circ(1\otimes r^{-1}_{\mathcal{A}(FA,FB)})=r^{-1}_{\mathcal{A}(L,FB)}\circ C_{LFAFB}$$

due to Lemma 2.2.1. □

3.2.1. The Case of Weighted Colimits

Definition 3.2.1.1. *Let $\mathcal{V}$ be a symmetric monoidal closed category. Given $\mathcal{V}$-functors $F:\mathcal{A}\to\mathcal{B}$ and $G:\mathcal{A}^{\star}\to\mathcal{V}$, a $\mathcal{V}$-functor $H:\mathcal{B}\to\mathcal{C}$ is said to create the $\mathcal{V}$-colimit of F weighted by G if $colim_G F$ exists whenever $colim_G(H\circ F)$ exists.*

Proposition 3.2.1.2. *Let $\mathcal{V}$ be a complete symmetric monoidal closed category, $\mathcal{A}$ a small $\mathcal{V}$-category and $T:\mathcal{A}\to\mathcal{A}$ a $\mathcal{V}$-endofunctor. Given $\mathcal{V}$-functors $F:\mathcal{B}\to\mathcal{A}_T$ and $G:\mathcal{B}^{\star}\to\mathcal{V}$. The underlying $\mathcal{V}$-functor $U_T:\mathcal{A}_T\to\mathcal{A}$ creates the $\mathcal{V}$-colimit of F weighted by G.*

Proof. Suppose that $colim_G(U_T\circ F)$ exists. Then,

(*i*) for every $C\in ob\mathcal{A}$, the object $\mathcal{V}-Nat(G,\mathcal{A}((U_T\circ F)-,C))$ of $\mathcal{V}$-natural transformations exists;

(*ii*) there exists an object $L\in ob\mathcal{A}$ and isomorphisms in $\mathcal{V}$,

$$\lambda_C:\mathcal{V}-Nat(G,\mathcal{A}((U_T\circ F)-,C))\cong\mathcal{A}(L,C),$$

which are $\mathcal{V}$-natural in C.

By Lemma 2.2.4, for every $V \in \mathcal{V}_0$, there exist bijective correspondences

$$\mathcal{V} - Nat(G, [V, \mathcal{A}((U_T \circ F)-, C)]) \cong \mathcal{V}_0(V, \mathcal{A}(L, C))$$

which are natural in C. This induces a bijection

$$\mathcal{V} - Nat(G, [I, \mathcal{A}((U_T \circ F)-, L)]) \cong \mathcal{V}_0(I, \mathcal{A}(L, L))$$

which is natural in L. Hence, the unit element $j_L : I \to \mathcal{A}(L, L)$ corresponds under bijection with a $\mathcal{V}$-natural transformation $\alpha : G \Rightarrow [I, \mathcal{A}((U_T \circ F)-, L)]$. For each pair $A, B \in ob\mathcal{B}$, the following diagram commutes due to Proposition 2.2.2:

$$\begin{array}{ccc}
\mathcal{B}^\star(A, B) & \xrightarrow{G_{BA}} & [GA, GB] \\
{\scriptstyle [I,\mathcal{A}((U_T\circ F)-,L)]_{BA}} \downarrow & & \downarrow {\scriptstyle [1,\alpha_B]} \\
[[I, \mathcal{A}(FA, L)], [I, \mathcal{A}(FB, L)]] & \xrightarrow[{[\alpha_A,1]}]{} & [GA, [I, \mathcal{A}(FB, L)]]
\end{array}$$

Each component $\alpha_A : GA \to [I, \mathcal{A}(FA, L)]$ corresponds, under adjunction, with a morphism

$$\beta_A : GA \otimes I \to \mathcal{A}(FA, L).$$

Consider the composite morphism $u_A : \mathcal{A}(FA, L) \to \mathcal{A}(FA, TL)$ defined as

$$\begin{array}{c}
\mathcal{A}(FA, L) \\
{\scriptstyle l^{-1}_{\mathcal{A}(FA,L)}} \downarrow \\
I \otimes \mathcal{A}(FA, L) \\
{\scriptstyle 1\otimes T_{FAL}} \downarrow \\
I \otimes \mathcal{A}(TFA, TL) \\
{\scriptstyle \tau_{FA}\otimes 1} \downarrow \\
\mathcal{A}(FA, TFA) \otimes \mathcal{A}(TFA, TL) \\
{\scriptstyle c_{FATFATL}} \downarrow \\
\mathcal{A}(FA, TL)
\end{array}$$

The composite $u_A \circ \beta_A$ corresponds, under adjunction, with a morphism $\gamma_A : GA \to [I, \mathcal{A}(FA, TL)]$. Let

$$d_{GA} : GA \to [I, GA \otimes I]$$

denote the unit of the adjunction $- \otimes I \dashv [I, -]$. Here, we have

$$\gamma_A = [1, u_A \circ \beta_A] \circ d_{GA} = [1, u_A] \circ [1, \beta_A] \circ d_{GA} = [1, u_A] \circ \alpha_A.$$

First, let us show that L can be equipped with a T-coalgebra structure. By Lemma 3.2.1, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{B}^\star(A, B) \otimes \mathcal{A}(FA, L) & \xrightarrow{c_{FBFAL}\circ((U_T\circ F)_{BA}\otimes 1)} & \mathcal{A}(FB, L) \\
{\scriptstyle (U_T\circ F)_{BA}\otimes 1} \downarrow & & \| \\
\mathcal{A}(FB, FA) \otimes \mathcal{A}(FA, L) & \xrightarrow{c_{FBFAL}} & \mathcal{A}(FB, L) \\
{\scriptstyle 1\otimes u_A} \downarrow & & \downarrow {\scriptstyle u_B} \\
\mathcal{A}(FB, FA) \otimes \mathcal{A}(FA, TL) & \xrightarrow[c_{FBFATL}]{} & \mathcal{A}(FB, TL)
\end{array}$$

Also, $C_{FBFAL} \circ ((U_T \circ F)_{BA} \otimes 1) = ev_{\mathcal{A}(FA,L),\mathcal{A}(FB,L)} \circ (\mathcal{A}((U_T \circ F)-,L)_{BA} \otimes 1)$ as $\mathcal{A}((U_T \circ F)-,L)$ is the composite of $\mathcal{V}$-functors $U_T \circ F$ and $\mathcal{A}(-,L)$. Since $\otimes$ is a bifunctor, we have

$$(1 \otimes u_A) \circ ((U_T \circ F)_{BA} \otimes 1) = ((U_T \circ F)_{BA} \otimes 1) \circ (1 \otimes u_A).$$

Hence, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{B}^\star(A,B) \otimes \mathcal{A}(FA,L) & \xrightarrow{\mathcal{A}((U_T\circ F)-,L)_{BA}\otimes 1} & [\mathcal{A}(FA,L),\mathcal{A}(FB,L)] \otimes \mathcal{A}(FA,L) \\
{\scriptstyle 1\otimes u_A}\downarrow & & \downarrow{\scriptstyle ev_{\mathcal{A}(FA,L),\mathcal{A}(FB,L)}} \\
\mathcal{B}^\star(A,B) \otimes \mathcal{A}(FA,TL) & & \mathcal{A}(FB,L) \\
{\scriptstyle \mathcal{A}((U_T\circ F)-,TL)_{BA}\otimes 1}\downarrow & & \downarrow{\scriptstyle u_B} \\
[\mathcal{A}(FA,TL),\mathcal{A}(FB,TL)] \otimes \mathcal{A}(FA,TL) & \xrightarrow[ev_{\mathcal{A}(FA,TL),\mathcal{A}(FB,TL)}]{} & \mathcal{A}(FB,TL)
\end{array}$$

Subsequently, the composite morphisms

$$\begin{array}{c}
(\mathcal{B}^\star(A,B) \otimes [I,\mathcal{A}(FA,L)]) \otimes I \\
{\scriptstyle a_{\mathcal{B}^\star(A,B),[I,\mathcal{A}(FA,L)],I}}\downarrow \\
\mathcal{B}^\star(A,B) \otimes ([I,\mathcal{A}(FA,L)] \otimes I) \\
{\scriptstyle 1\otimes ev_{I,\mathcal{A}(FA,L)}}\downarrow \\
\mathcal{B}^\star(A,B) \otimes \mathcal{A}(FA,L) \\
{\scriptstyle \mathcal{A}((U_T\circ F)-,TL)_{BA}\otimes 1}\downarrow \\
[\mathcal{A}(FA,TL),\mathcal{A}(FB,TL)] \otimes \mathcal{A}(FA,L) \\
{\scriptstyle [u_A,1]\otimes 1}\downarrow \\
[\mathcal{A}(FA,L),\mathcal{A}(FB,TL)] \otimes \mathcal{A}(FA,L) \\
{\scriptstyle ev_{\mathcal{A}(FA,L),\mathcal{A}(FB,TL)}}\downarrow \\
\mathcal{A}(FB,TL)
\end{array}$$

(which corresponds, under adjunction, with $[[1,u_A],1] \circ [I,\mathcal{A}((U_T \circ F)-,TL)]_{BA}$) and

$$\begin{array}{c}
(\mathcal{B}^\star(A,B) \otimes [I,\mathcal{A}(FA,L)]) \otimes I \\
{\scriptstyle a_{\mathcal{B}(A,B),[I,\mathcal{A}(FA,L)],I}}\downarrow \\
\mathcal{B}^\star(A,B) \otimes ([I,\mathcal{A}(FA,L)] \otimes I) \\
{\scriptstyle 1\otimes ev_{I,\mathcal{A}(FA,L)}}\downarrow \\
\mathcal{B}^\star(A,B) \otimes \mathcal{A}(FA,L) \\
{\scriptstyle \mathcal{A}((U_T\circ F)-,L)_{BA}\otimes 1}\downarrow \\
[\mathcal{A}(FA,L),\mathcal{A}(FB,L)] \otimes \mathcal{A}(FA,L) \\
{\scriptstyle ev_{\mathcal{A}(FA,L),\mathcal{A}(FB,L)}}\downarrow \\
\mathcal{A}(FB,L) \\
{\scriptstyle u_B}\downarrow \\
\mathcal{A}(FB,TL)
\end{array}$$

(which corresponds, under adjunction, with $[1,[1,u_B]] \circ [I,\mathcal{A}((U_T \circ F)-,L)]_{BA}$) are equal. It follows that

$$[[1,u_A],1] \circ [I,\mathcal{A}((U_T \circ F)-,TL)]_{BA} = [1,[1,u_B]] \circ [I,\mathcal{A}((U_T \circ F)-,L)]_{BA}.$$

Consequently, we have

$$\begin{aligned}[\gamma_A, 1] \circ [I, \mathcal{A}((U_T \circ F)-, TL)]_{BA} &= [\alpha_A, 1] \circ [[1, u_A], 1] \circ [I, \mathcal{A}((U_T \circ F)-, TL)]_{BA} \\ &= [\alpha_A, 1] \circ [1, [1, u_B]] \circ [I, \mathcal{A}((U_T \circ F)-, L)]_{BA} \\ &= [1, [1, u_B]] \circ [\alpha_A, 1] \circ [I, \mathcal{A}((U_T \circ F)-, L)]_{BA} \\ &= [1, [1, u_B]] \circ [1, \alpha_B] \circ G_{BA} = [1, \gamma_B] \circ G_{BA}.\end{aligned}$$

That is, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{B}^\star(A, B) & \xrightarrow{G_{BA}} & [GA, GB] \\ {\scriptstyle [I,\mathcal{A}((U_T\circ F)-,TL)]_{BA}}\downarrow & & \downarrow{\scriptstyle [1,\gamma_B]} \\ [[I, \mathcal{A}(FA, TL)], [I, \mathcal{A}(FB, TL)]] & \xrightarrow[{[\gamma_A,1]}]{} & [GA, [I, \mathcal{A}(FB, TL)]] \end{array}$$

Hence, the $\mathcal{B}$-indexed family of morphisms $\gamma_A : GA \to [I, \mathcal{A}(FA, TL)]$ defines a $\mathcal{V}$-natural transformation $\gamma : G \Rightarrow [I, \mathcal{A}((U_T \circ F)-, TL)]$, which corresponds under bijection with a T-coalgebra structure $\tau_L : I \to \mathcal{A}(L, TL)$ on L.

Next, we show that $\underline{L} = (L, \tau_L)$ is the $\mathcal{V}$-colimit of F weighted by G. By hypothesis, $\mathcal{A}$ is a small $\mathcal{V}$-category. Therefore, $\mathcal{A}_T$ is also a small $\mathcal{V}$-category. Given $\underline{C} = (C, \tau_C)$ in $ob\mathcal{A}_T$, the object $\mathcal{V} - Nat(G, \mathcal{A}_T(F-, \underline{C}))$ of $\mathcal{V}$-natural transformations exists because $\mathcal{V}$ is complete (see [3, Sec. 6.6]). For every $V \in \mathcal{V}_0$ and a pair $A, B \in ob\mathcal{B}$, the following diagram commutes as $[-, -]$ is a bifunctor:

$$\begin{array}{ccc} \mathcal{B}^\star(A, B) & \xrightarrow{[V,\mathcal{A}_T(F-,\underline{C})]_{BA}} & [[V, \mathcal{A}_T(FA, \underline{C})], [V, \mathcal{A}_T(FB, \underline{C})]] \\ {\scriptstyle [V,\mathcal{A}((U_T\circ F)-,C)]_{BA}}\downarrow & & \downarrow{\scriptstyle [1,[1,(U_T)_{FB\underline{C}}]]} \\ [[V, \mathcal{A}(FA, C)], [V, \mathcal{A}(FB, C)]] & \xrightarrow[{[[1,(U_T)_{FA\underline{C}}],1]}]{} & [[V, \mathcal{A}_T(FA, C)], [V, \mathcal{A}(FB, C)]] \end{array}$$

Given a $\mathcal{V}$-natural transformation $\bar{\alpha} : G \Rightarrow [V, \mathcal{A}_T(F-, \underline{C})]$, the following diagram commutes due to the $\mathcal{V}$-naturality condition:

$$\begin{array}{ccc} \mathcal{B}^\star(A, B) & \xrightarrow{G_{BA}} & [GA, GB] \\ {\scriptstyle [V,\mathcal{A}_T(F-,C)]_{BA}}\downarrow & & \downarrow{\scriptstyle [1,\bar{\alpha}_B]} \\ [[V, \mathcal{A}_T(FA, \underline{C})], [V, \mathcal{A}_T(FB, \underline{C})]] & \xrightarrow[{[\bar{\alpha}_A,1]}]{} & [GA, [V, \mathcal{A}_T(FB, \underline{C})]] \\ {\scriptstyle [1,[1,(U_T)_{FB\underline{C}}]]}\downarrow & & \downarrow{\scriptstyle [1,[1,(U_T)_{FB\underline{C}}]]} \\ [[V, \mathcal{A}_T(FA, \underline{C})], [V, \mathcal{A}(FB, C)]] & \xrightarrow[{[\bar{\alpha}_A,1]}]{} & [GA, [V, \mathcal{A}(FB, C)]] \end{array}$$

Putting together the last two diagrams, the following one commutes:

$$\begin{array}{ccc} \mathcal{B}^\star(A, B) & \xrightarrow{G_{BA}} & [GA, GB], [V, \mathcal{A}_T(FB, \underline{C})]] \\ {\scriptstyle [V,\mathcal{A}((U_T\circ F)-,C)]_{BA}}\downarrow & & \downarrow{\scriptstyle [1,\varphi_B]} \\ [[V, \mathcal{A}(FA, C)], [V, \mathcal{A}(FB, C)]] & \xrightarrow[{[\varphi_A,1]}]{} & [GA, [V, \mathcal{A}(FB, C)]] \end{array}$$

As a consequence, the $\mathcal{B}$-indexed family of morphisms φ_A

$$\begin{array}{c} GA \\ {\scriptstyle \alpha_A}\downarrow \\ [V, \mathcal{A}_T(FA, \underline{C})] \\ {\scriptstyle [1,(U_T)_{FA\underline{C}}]}\downarrow \\ [V, \mathcal{A}(FA, C)] \end{array}$$

satisfies the $\mathcal{V}$-naturality condition, and hence, it defines a $\mathcal{V}$-natural transformation

$$\varphi : G \Rightarrow [V, \mathcal{A}((U_T \circ F)-, C)].$$

Furthermore, the $\mathcal{B}$-indexed families of morphisms

$$\begin{array}{c} GA \\ \downarrow{\scriptstyle \varphi_A} \\ [V,\mathcal{A}(FA,C)] \\ \downarrow{\scriptstyle [1,l^{-1}_{\mathcal{A}(FA,C)}]} \\ [V,I\otimes\mathcal{A}(FA,C)] \\ \downarrow{\scriptstyle [1,1\otimes T_{FAC}]} \\ [V,I\otimes\mathcal{A}(FA,C)] \\ \downarrow{\scriptstyle [1,\tau_{FA}\otimes 1]} \\ [V,\mathcal{A}(FA,TFA)\otimes\mathcal{A}(TFA,TC)] \\ \downarrow{\scriptstyle [1,C_{FA\,TFA\,TC}]} \\ [V,\mathcal{A}(FA,TC)] \end{array}$$

and

$$\begin{array}{c} GA \\ \downarrow{\scriptstyle \varphi_A} \\ [V,\mathcal{A}(FA,C)] \\ \downarrow{\scriptstyle [1,r^{-1}_{\mathcal{A}(FA,C)}]} \\ [V,\mathcal{A}(FA,C)\otimes I] \\ \downarrow{\scriptstyle [1,1\otimes\tau_C]} \\ [V,\mathcal{A}(FA,C)\otimes\mathcal{A}(C,TC)] \\ \downarrow{\scriptstyle [1,C_{FA\,C\,TC}]} \\ [V,\mathcal{A}(FA,TC)] \end{array}$$

satisfy the $\mathcal{V}$-naturality condition, and define a unique $\mathcal{V}$-natural transformation

$$\rho : G \Rightarrow [V,\mathcal{A}((U_T\circ F)-,TC)]$$

because, for each $A\in ob\mathcal{B}$, $\varphi_A=[1,(U_T)_{FA\underline{C}}]\circ\alpha_A$ and $\mathcal{A}_T(FA,\underline{C})$ satisfies the homomorphism condition. Since L is the $\mathcal{V}$-colimit of $U_T\circ F$ weighted by G, there exist bijective correspondences

$$\mu_C : \mathcal{V}-Nat(G,[V,\mathcal{A}((U_T\circ F)-,C)])\cong\mathcal{V}_0(V,\mathcal{A}(L,C))$$

which are natural in C. Therefore, φ corresponds under bijection with a $\mathcal{V}_0$-morphism $u:V\to\mathcal{A}(L,C)$. Also, u equalizes the morphisms

$$\begin{array}{c} \mathcal{A}(L,C) \\ \downarrow{\scriptstyle l^{-1}_{\mathcal{A}(L,C)}} \\ I\otimes\mathcal{A}(L,C) \\ \downarrow{\scriptstyle 1\otimes T_{LC}} \\ I\otimes\mathcal{A}(TL,TC) \\ \downarrow{\scriptstyle \tau_L\otimes 1} \\ \mathcal{A}(L,TL)\otimes\mathcal{A}(TL,TC)] \\ \downarrow{\scriptstyle C_{L\,TL\,TC}} \\ \mathcal{A}(L,TC) \end{array}$$

and

$$\begin{array}{c} \mathcal{A}(L,C) \\ \Big\downarrow{\scriptstyle r^{-1}_{\mathcal{A}(L,C)}} \\ \mathcal{A}(L,C)\otimes I \\ \Big\downarrow{\scriptstyle 1\otimes\tau_C} \\ \mathcal{A}(L,C)\otimes\mathcal{A}(C,TC) \\ \Big\downarrow{\scriptstyle c_{LCTC}} \\ \mathcal{A}(L,TC)] \end{array}$$

as composing u with each of these morphisms gives the unique $\mathcal{V}_0$-morphism $w : V \to \mathcal{A}(L,TC)$ such that $\mu_{TC}(\rho) = w$; this follows from the naturality of μ_C in C. By the universal property of equalizers, there is a unique $\mathcal{V}_0$-morphism

$$\bar{u} : V \to \mathcal{A}_T(\underline{L},\underline{C})$$

such that $(U_T)_{\underline{L}\,\underline{C}} \circ \bar{u} = u$. Conversely, let $v : V \to \mathcal{A}_T(\underline{L},\underline{C})$ be a $\mathcal{V}_0$-morphism. Then,

$$(U_T)_{\underline{L}\,\underline{C}} \circ v : V \to \mathcal{A}(L,C)$$

equalizes the morphisms

$$\begin{array}{c} \mathcal{A}(L,C) \\ \Big\downarrow{\scriptstyle l^{-1}_{\mathcal{A}(L,C)}} \\ I\otimes\mathcal{A}(L,C) \\ \Big\downarrow{\scriptstyle 1\otimes T_{LC}} \\ I\otimes\mathcal{A}(TL,TC) \\ \Big\downarrow{\scriptstyle \tau_L\otimes 1} \\ \mathcal{A}(L,TL)\otimes\mathcal{A}(TL,TC)] \\ \Big\downarrow{\scriptstyle c_{LTLTC}} \\ \mathcal{A}(L,TC) \end{array}$$

and

$$\begin{array}{c} \mathcal{A}(L,C) \\ \Big\downarrow{\scriptstyle r^{-1}_{\mathcal{A}(L,C)}} \\ \mathcal{A}(L,C)\otimes I \\ \Big\downarrow{\scriptstyle 1\otimes\tau_C} \\ \mathcal{A}(L,C)\otimes\mathcal{A}(C,TC) \\ \Big\downarrow{\scriptstyle c_{LCTC}} \\ \mathcal{A}(L,TC)] \end{array}$$

because $\mathcal{A}_T(\underline{L},\underline{C})$ satisfies the homomorphism condition.

Hence, it corresponds under bijection with a $\mathcal{V}$-natural transformation $\sigma : G \Rightarrow [V, \mathcal{A}((U_T \circ F)-, C)]$ for which each component $\sigma_A : GA \to [V, \mathcal{A}(FA, C)]$ equalizes the morphisms

$$\begin{array}{c}
[V, \mathcal{A}(FA, C)] \\
\downarrow {\scriptstyle [1, l^{-1}_{\mathcal{A}(FA,C)}]} \\
[V, I \otimes \mathcal{A}(FA, C)] \\
\downarrow {\scriptstyle [1, 1 \otimes T_{FA\,C}]} \\
[V, I \otimes \mathcal{A}(TFA, TC)] \\
\downarrow {\scriptstyle [1, \tau_{FA} \otimes 1]} \\
[V, \mathcal{A}(FA, TFA) \otimes \mathcal{A}(TFA, TC)] \\
\downarrow {\scriptstyle [1, C_{FA\,TFA\,TC}]} \\
[V, \mathcal{A}(FA, TC)]
\end{array}$$

and

$$\begin{array}{c}
[V, \mathcal{A}(FA, C)] \\
\downarrow {\scriptstyle [1, r^{-1}_{\mathcal{A}(FA,C)}]} \\
[V, \mathcal{A}(FA, C) \otimes I] \\
\downarrow {\scriptstyle [1, 1 \otimes \tau_C]} \\
[V, \mathcal{A}(FA, C) \otimes \mathcal{A}(C, TC)] \\
\downarrow {\scriptstyle [1, C_{FA\,C\,TC}]} \\
[V, \mathcal{A}(FA, TC)]
\end{array}$$

due to the naturality of μ_C^{-1} in C. From the universality of the equalizer, there is a unique arrow $\bar{\sigma}_A : GA \to [V, \mathcal{A}_T(FA, \underline{C})]$ such that $[1, (U_T)_{FA\underline{C}}] \circ \bar{\sigma}_A = \sigma_A$. As, in addition, the $\mathcal{B}$-indexed family of morphisms $\sigma_A : GA \to [V, \mathcal{A}(FA, C)]$ satisfies the $\mathcal{V}$-naturality condition, with $[-, -]$ being a bifunctor, the following diagram commutes for each pair $A, B \in ob\mathcal{B}$:

$$\begin{array}{ccc}
\mathcal{B}^\star(A, B) & \xrightarrow{\quad G_{BA} \quad} & [GA, GB] \\
\downarrow {\scriptstyle [V, \mathcal{A}_T(F-, C)]_{BA}} & & \downarrow {\scriptstyle [1, \bar{\sigma}_B]} \\
[[V, \mathcal{A}_T(FA, \underline{C})], [V, \mathcal{A}_T(FB, \underline{C})]] & \xrightarrow[{[\bar{\sigma}_A, 1]}]{} & [GA, [V, \mathcal{A}_T(FB, \underline{C})]] \\
\downarrow {\scriptstyle [1, [1, (U_T)_{FB\underline{C}}]]} & & \downarrow {\scriptstyle [1, [1, (U_T)_{FB\underline{C}}]]} \\
[[V, \mathcal{A}_T(FA, \underline{C})], [V, \mathcal{A}(FB, C)]] & \xrightarrow[{[\bar{\sigma}_A, 1]}]{} & [GA, [V, \mathcal{A}(FB, C)]]
\end{array}$$

It follows that the diagram

$$\begin{array}{ccc}
\mathcal{B}^\star(A, B) & \xrightarrow{\quad G_{BA} \quad} & [GA, GB], [V, \mathcal{A}_T(FB, \underline{C})]] \\
\downarrow {\scriptstyle [V, \mathcal{A}_T(F-, \underline{C})]_{BA}} & & \downarrow {\scriptstyle [1, \bar{\sigma}_B]} \\
[[V, \mathcal{A}_T(FA, \underline{C})], [V, \mathcal{A}_T(FB, \underline{C})]] & \xrightarrow[{[\bar{\sigma}_A, 1]}]{} & [GA, [V, \mathcal{A}_T(FB, \underline{C})]]
\end{array}$$

commutes because $[1, [1, (U_T)_{FB\underline{C}}]]$ is a monomorphism; that is, the $\mathcal{B}$-indexed family $\bar{\sigma}_A : GA \to [V, \mathcal{A}_T(FA, \underline{C})]$ defines a $\mathcal{V}$-natural transformation $\bar{\sigma} : G \Rightarrow [V, \mathcal{A}_T(F-, \underline{C})]$, and hence, there exist bijective correspondences $\hat{\mu}_{\underline{C}} : \mathcal{V} - Nat(G, [V, \mathcal{A}_T(F-, \underline{C})]) \cong \mathcal{V}_0(V, \mathcal{A}_T(\underline{L}, \underline{C}))$. which are natural in $\underline{C}$; their naturality arises from that of bijections μ_C in C.

As a consequence, there exist isomorphisms in $\mathcal{V}$,

$$\hat{\lambda}_{\underline{C}} : \mathcal{V} - Nat(G, \mathcal{A}_T(F-, \underline{C})) \cong \mathcal{A}_T(\underline{L}, \underline{C})$$

which are $\mathcal{V}$-natural in $\underline{C}$. Thus, the $\mathcal{V}$-colimit of F weighted by G exists and it is in fact $\underline{l} = (L, \tau_L)$. □

3.2.2. The Case of Weighted Limits

Definition 3.2.2.1. *Let $\mathcal{V}$ be a symmetric monoidal closed category. Given $\mathcal{V}$-functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{A} \to \mathcal{V}$ such that $lim_G F$ exists. A $\mathcal{V}$-functor $H : \mathcal{B} \to \mathcal{C}$ is said to preserve $lim_G F$ if $lim_G(H \circ F)$ exists and if the canonical morphism $H(lim_G F) \to lim_G(H \circ F)$ is an isomorphism.*

Proposition 3.2.2.2. *Let $\mathcal{V}$ be a complete symmetric monoidal closed category, $\mathcal{A}$ a small $\mathcal{V}$-category and $T : \mathcal{A} \to \mathcal{A}$ a $\mathcal{V}$-endofunctor. Given $\mathcal{V}$-functors $F : \mathcal{B} \to \mathcal{A}_T$ and $G : \mathcal{B} \to \mathcal{V}$ such that $lim_G(U_T \circ F)$ exists. Then, $lim_G F$ exists, provided that T preserves the $\mathcal{V}$-limit of $U_T \circ F$ weighted by G.*

Proof. Suppose that $lim_G(U_T \circ F)$ exists. Then,

- (i) for every $C \in ob\mathcal{A}$, the object $\mathcal{V} - Nat(G, \mathcal{A}(C, (U_T \circ F)-))$ of $\mathcal{V}$-natural transformations exists;
- (ii) there exists an object $L \in ob\mathcal{A}$ and isomorphisms in $\mathcal{V}$

$$\lambda_C : \mathcal{V} - Nat(G, \mathcal{A}(C, (U_T \circ F)-) \cong \mathcal{A}(C, L)$$

  which are $\mathcal{V}$-natural in C.

According to Lemma 2.2.4, for every $V \in \mathcal{V}_0$, there exist bijective correspondences

$$\mathcal{V} - Nat(G, [V, \mathcal{A}(C, (U_T \circ F)-)]) \cong \mathcal{V}_0(V, \mathcal{A}(C, L))$$

which are natural in C. Particularly, there exists a bijection

$$\mathcal{V} - Nat(G, [I, \mathcal{A}(L, (U_T \circ F)-)]) \cong \mathcal{V}_0(I, \mathcal{A}(L, L))$$

which is natural in L. This implies that the unit element $j_L : I \to \mathcal{A}(L, L)$ corresponds under bijection with a $\mathcal{V}$-natural transformation $\alpha : G \Rightarrow [I, \mathcal{A}(L, (U_T \circ F)-)]$. By Proposition 2.2.2, the following diagram commutes for each pair $A, B \in ob\mathcal{B}$:

$$\begin{array}{ccc}
\mathcal{B}(A, B) & \xrightarrow{G_{AB}} & [GA, GB] \\
{\scriptstyle [I, \mathcal{A}(L, (U_T \circ F)-)]_{AB}}\downarrow & & \downarrow{\scriptstyle [1, \alpha_B]} \\
[[I, \mathcal{A}(L, FA)], [I, \mathcal{A}(L, FB)]] & \xrightarrow[{[\alpha_A, 1]}]{} & [GA, [I, \mathcal{A}(L, FB)]]
\end{array}$$

Each component $\alpha_A : GA \to [I, \mathcal{A}(L, FA)]$ corresponds under adjunction with a morphism $\beta_A : GA \otimes I \to \mathcal{A}(L, FA)$. Denote by $v_A : \mathcal{A}(L, FA) \to \mathcal{A}(L, TFA)$ the composite morphism

$$\begin{array}{c}
\mathcal{A}(L, FA) \\
{\scriptstyle r^{-1}_{\mathcal{A}(L, FA)}}\downarrow \\
\mathcal{A}(L, FA) \otimes I \\
{\scriptstyle 1 \otimes \tau_{FA}}\downarrow \\
\mathcal{A}(L, FA) \otimes \mathcal{A}(FA, TFA) \\
{\scriptstyle c_{LFATFA}}\downarrow \\
\mathcal{A}(L, TFA)
\end{array}$$

Therefore, $v_A \circ \beta_A$ corresponds under adjunction with a morphism $\gamma_A : GA \to [I, \mathcal{A}(L, TFA)]$ written as

$$\gamma_A = [1, v_A \circ \beta_A] \circ d_{GA} = [1, v_A] \circ [1, \beta_A] \circ d_{GA} = [1, v_A] \circ \alpha_A.$$

Now, we find the $\mathcal{V}$-limit of F weighted by G. By Lemma 3.2.1, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{A}(L, FA) \otimes \mathcal{B}(A, B) & \xrightarrow{C_{LFAFB} \circ (1 \otimes (U_T \circ F)_{AB})} & \mathcal{A}(L, FB) \\
{\scriptstyle 1 \otimes (U_T \circ F)_{AB}} \Big\downarrow & & \\
\mathcal{A}(L, FA) \otimes \mathcal{A}(FA, FB) & \xrightarrow{C_{LFAFB}} & \mathcal{A}(L, FB) \\
{\scriptstyle v_A \otimes 1} \Big\downarrow & & \Big\downarrow {\scriptstyle v_B} \\
\mathcal{A}(L, TFA) \otimes \mathcal{A}(FA, FB) & & \\
{\scriptstyle 1 \otimes T_{FAFB}} \Big\downarrow & & \\
\mathcal{A}(L, TFA) \otimes \mathcal{A}(TFA, TFB) & \xrightarrow[C_{LTFATFB}]{} & \mathcal{A}(L, TFB)
\end{array}$$

Furthermore, $(1 \otimes T_{FAFB}) \circ (v_A \otimes 1) \circ (1 \otimes (U_T \circ F)_{AB}) = (v_A \otimes 1) \circ (1 \otimes T_{FAFB}) \circ (1 \otimes (U_T \circ F)_{AB})$ as $\otimes$ is a bifunctor. Since $\mathcal{A}(L, (U_T \circ F)-)$ is the composite of $\mathcal{V}$-functors $U_T \circ F$ and $\mathcal{A}(L, -)$,

$$C_{LFAFB} \circ (1 \otimes (U_T \circ F)_{AB}) \circ s_{\mathcal{B}(A,B), \mathcal{A}(L,FA)} = ev_{\mathcal{A}(L,FA), \mathcal{A}(l,FB)} \circ (\mathcal{A}(L, (U_T \circ F)-)_{AB} \otimes 1),$$

and hence,

$$\begin{aligned}
&C_{LTFATFB} \circ s_{\mathcal{A}(TFA,TFB), \mathcal{A}(L,TFA)} \circ (T_{FAFB} \otimes 1) \circ ((U_T \circ F)_{AB} \otimes 1) \\
&= ev_{\mathcal{A}(L,TFA), \mathcal{A}(L,TFB)} \circ (\mathcal{A}(L, T((U_T \circ F)-))_{AB} \otimes 1).
\end{aligned}$$

By the fact that s is a natural isomorphism, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{B}(A, B) \otimes \mathcal{A}(L, FA) & \xrightarrow{\mathcal{A}(L, (U_T \circ F)-)_{AB} \otimes 1} & [\mathcal{A}(L, FA), \mathcal{A}(L, FB)] \otimes \mathcal{A}(L, FA) \\
{\scriptstyle 1 \otimes v_A} \Big\downarrow & & \Big\downarrow {\scriptstyle ev_{\mathcal{A}(L,FA), \mathcal{A}(L,FB)}} \\
\mathcal{B}(A, B) \otimes \mathcal{A}(L, TFA) & & \mathcal{A}(L, FB) \\
{\scriptstyle \mathcal{A}(L, T((U_T \circ F)-))_{AB} \otimes 1} \Big\downarrow & & \Big\downarrow {\scriptstyle v_B} \\
[\mathcal{A}(L, TFA), \mathcal{A}(L, TFB)] \otimes \mathcal{A}(L, TFA) & \xrightarrow[ev_{\mathcal{A}(L,TFA), \mathcal{A}(L,TFB)}]{} & \mathcal{A}(L, TFB)
\end{array}$$

Hence, the composite morphisms

$$\begin{array}{c}
\mathcal{B}(A, B) \otimes [I, \mathcal{A}(L, FA)]) \otimes I \\
{\scriptstyle a_{\mathcal{B}(A,B), [I, \mathcal{A}(L,FA)], I}} \Big\downarrow \\
\mathcal{B}(A, B) \otimes ([I, \mathcal{A}(L, FA)] \otimes I) \\
{\scriptstyle 1 \otimes ev_{I, \mathcal{A}(L,FA)}} \Big\downarrow \\
\mathcal{B}(A, B) \otimes \mathcal{A}(L, FA) \\
{\scriptstyle \mathcal{A}(L, T((U_T \circ F)-))_{AB} \otimes 1} \Big\downarrow \\
[\mathcal{A}(L, TFA), \mathcal{A}(L, TFB)] \otimes \mathcal{A}(L, FA) \\
{\scriptstyle [v_A, 1] \otimes 1} \Big\downarrow \\
[\mathcal{A}(L, FA), \mathcal{A}(L, TFB)] \otimes \mathcal{A}(L, FA) \\
{\scriptstyle ev_{\mathcal{A}(L,FA), \mathcal{A}(L,TFB)}} \Big\downarrow \\
\mathcal{A}(L, TFB)
\end{array}$$

(which corresponds under adjunction with $[[1, v_A], 1] \circ [I, \mathcal{A}(L, T((U_T \circ F)-))]_{AB}$), and

$$\begin{array}{c}(\mathcal{B}(A,B) \otimes [I, \mathcal{A}(L, FA)]) \otimes I \\ \Big\downarrow{\scriptstyle a_{\mathcal{B}(A,B),[I,\mathcal{A}(L,FA)],I}} \\ \mathcal{B}(A,B) \otimes ([I, \mathcal{A}(L, FA)] \otimes I) \\ \Big\downarrow{\scriptstyle 1 \otimes ev_{I,\mathcal{A}(L,FA)}} \\ \mathcal{B}(A,B) \otimes \mathcal{A}(L, FA) \\ \Big\downarrow{\scriptstyle \mathcal{A}(L,(U_T \circ F)-)_{AB} \otimes 1} \\ [\mathcal{A}(L, FA), \mathcal{A}(L, FB)] \otimes \mathcal{A}(L, FA) \\ \Big\downarrow{\scriptstyle ev_{\mathcal{A}(L,FA),\mathcal{A}(L,FB)}} \\ \mathcal{A}(L, FB) \\ \Big\downarrow{\scriptstyle v_B} \\ \mathcal{A}(L, TFB)\end{array}$$

(which corresponds under adjunction with $[1, [1, v_B]] \circ [I, \mathcal{A}(L, (U_T \circ F)-)]_{AB}$) are equal. Thus, we have

$$[[1, v_A], 1] \circ [I, \mathcal{A}(L, T((U_T \circ F)-))]_{AB} = [1, [1, v_B]] \circ [I, \mathcal{A}(L, (U_T \circ F)-)]_{AB}.$$

It follows that

$$\begin{aligned}[\gamma_A, 1] \circ [I, \mathcal{A}(L, T((U_T \circ F)-))]_{AB} &= [\alpha_A, 1] \circ [[1, v_A], 1] \circ [I, \mathcal{A}(L, T((U_T \circ F)-))]_{AB} \\ &= [\alpha_A, 1] \circ [1, [1, v_B]] \circ [I, \mathcal{A}(L, (U_T \circ F)-)]_{AB} \\ &= [1, [1, v_B]] \circ [\alpha_A, 1] \circ [I, \mathcal{A}(L, (U_T \circ F)-)]_{AB} \\ &= [1, [1, v_B]] \circ [1, \alpha_B] \circ G_{AB} \\ &= [1, \gamma_B] \circ G_{AB}.\end{aligned}$$

Consequently, the $\mathcal{B}$-indexed family of morphisms $\gamma_A : GA \to [I, \mathcal{A}(L, TFA)]$ defines a $\mathcal{V}$-natural transformation

$$\gamma : G \Rightarrow [I, \mathcal{A}(L, T((U_T \circ F)-))].$$

Since T preserves the $\mathcal{V}$-limit of $U_T \circ F$ weighted by G, there exists a bijection

$$\mathcal{V} - Nat(G, [I, \mathcal{A}(L, T((U_T \circ F)-))]) \cong \mathcal{V}_0(I, \mathcal{A}(L, TL))$$

which is natural in L, and hence, γ corresponds under this bijection with a T-coalgebra structure $\tau_L : I \to \mathcal{A}(L, TL)$ on L.

For every $\underline{C} = (C, \tau_C)$ in $ob\mathcal{A}_T$, the object $\mathcal{V} - Nat(G, \mathcal{A}_T(\underline{C}, F-))$ exists as $\mathcal{V}$ is complete and $\mathcal{A}_T$ is a small $\mathcal{V}$-category. Given $\underline{L} = (L, \tau_L)$ in $ob\mathcal{A}_T$, there exist isomorphisms in $\mathcal{V}$,

$$\hat{\lambda}_{\underline{C}} : \mathcal{V} - Nat(G, \mathcal{A}_T(\underline{C}, F-)) \cong \mathcal{A}_T(\underline{C}, \underline{L})$$

which are $\mathcal{V}$-natural in $\underline{C}$. They are induced (as in the proof of Proposition 3.2.1.2) by the isomorphisms in $\mathcal{V}$,

$$\lambda_C : \mathcal{V} - Nat(G, \mathcal{A}(C, (U_T \circ F)-)) \cong \mathcal{A}(C, L)$$

which are $\mathcal{V}$-natural in C. Consequently, the $\mathcal{V}$-limit of F weighted by G exists and it is nothing else but $\underline{L} = (L, \tau_L)$. □

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