Electronic Journal of Mathematics
www.shahindp.com/locate/ejm

Electron. J. Math. **10** (2025) 86–96
DOI: 10.47443/ejm.2025.066

Research Article

On Zagreb eccentricity indices of trees with perfect matchings

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(Received: 14 October 2025. Received in revised form: 23 November 2025. Accepted: 29 November 2025. Published online: 31 December 2025.)

Abstract

For a connected graph G with vertex set $V(G)$ and edge set $E(G)$, the first Zagreb eccentricity index and the second Zagreb eccentricity index are defined as $\xi_1(G) = \sum_{v\in V(G)} e^2(v)$ and $\xi_2(G) = \sum_{vu\in E(G)} e(v)e(u)$, respectively, where $e(u)$ is the eccentricity of vertex u in G. In this paper, we identify the trees that minimize the first and second Zagreb eccentricity indices among all trees with $2n$ vertices and a perfect matching.

Keywords: Zagreb eccentricity index; eccentricity; perfect matching.

2020 Mathematics Subject Classification: 05C09.

1. Introduction

All graphs in this paper are finite and simple. Let G be a graph of order n with vertex $V(G)$ and edge set $E(G)$. For $u \in V(G)$, $d_G(u)$ denotes the degree of u in G and $N_G(u)$ denotes the set of neighbors of u in G. The distance between vertices u and v in G, denoted by $d_G(u,v)$, is the length of a shortest path between u and v in G. For a vertex $v \in V(G)$, its eccentricity $e_G(v)$ is given as $e_G(v) = \max\{d_G(u,v) : u \in V(G)\}$. The diameter of G, denoted by $diam(G)$, is the maximum of the eccentricities of all vertices of G. A diametrical path of G is a shortest path (between two vertices) whose length is equal to the diameter of G.

Topological indices are graph invariants that take numerical values and are used to predict certain properties of chemical compounds. Among these indices, the first and second Zagreb indices are well-known and extensively researched. The first Zagreb index of a graph G is defined [3] as

$$M_1(G) = \sum_{v\in G} d_G^2(v),$$

and the second Zagreb index of G is defined [3] as

$$M_2(G) = \sum_{vu\in E(G)} d_G(v)d_G(u).$$

In an analogy with the first and the second Zagreb indices, Zagreb eccentricity indices were introduced in [2, 10]. The first Zagreb eccentricity index and the second Zagreb eccentricity index of G are defined as

$$\xi_1(G) = \sum_{v\in G} e^2(v) \quad \text{and} \quad \xi_2(G) = \sum_{vu\in E(G)} e(v)e(u).$$

Recently, plenty of results on the Zagreb eccentricity indices have been obtained. Das and Lee [1] figured out some lower and upper bounds on the first and second Zagreb eccentricity indices of trees and graphs, and characterized the corresponding extremal graphs. Qi and Du [7] determined the trees with minimum Zagreb eccentricity indices when the domination number, maximum degree, and bipartition size are respectively given. Qi and Zhou [8] characterized the unicyclic graphs minimizing and maximizing the first and second Zagreb eccentricity indices. Li and Zhang [6] characterized the graphs with the maximum and second maximum values of the second Zagreb eccentricity index among all n-vertex bicyclic graphs. Song, Li and He [9] obtained sharp lower and upper bounds on the Zagreb eccentricity indices for cacti with a fixed order and the number of cycles, and identified the graphs that attain these bounds. Hayat, Xu and Qi [4] determined the graphs that minimize the second Zagreb eccentricity index among n-vertex bipartite graphs with a fixed number of edges and diameter.

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A matching in G is a set of edges without common vertices. The maximum matching is a matching with the maximum size in G. The matching number of G, denoted by $\alpha(G)$, is the size of a maximum matching in G. If $2\alpha(G) = |V(G)|$, then the maximum matching is a perfect matching.

In this paper, we determine the trees having the minimum values of the first and second Zagreb eccentricity indices among $2n$-vertex trees with perfect matching. The rest of this paper is organized as follows. In Section 2, we present some transformations of trees that decrease the first and second Zagreb eccentricity indices. In Section 3, we determine the $2n$-vertex trees with perfect matching having the minimum first and second Zagreb eccentricity indices.

2. Some transformations of trees

In this section, we propose four graph transformations that decrease the first and second Zagreb eccentricity indices under certain constraints. For $V' \subseteq V(G)$, let $G[V']$ be the induced subgraph of G with vertex set V' and two vertices being adjacent if and only if they are adjacent in G, and $G - V'$ be the induced subgraph of G obtained by deleting all vertices in V' and all edges incident with them. In particular, we write $G - v$ instead of $G - \{v\}$ if $V' = \{v\}$. For a subset E_1 of $E(G)$, $G - E_1$ denotes the graph obtained from G by deleting all the edges in E_1, and in particular, we write $G - xy$ instead of $G - \{xy\}$ if $E_1 = \{xy\}$. Let $\overline{G}$ be the complement of G. For a subset E_2 of $E(\overline{G})$, $G + E_2$ denotes the graph obtained from G by adding all edges in E_2, and in particular, we write $G + xy$ instead of $G + \{xy\}$ if $E_2 = \{xy\}$. For disjoint subsets U and W of $V(G)$, $E(U, W)$ denotes the set of edges between U and W.

Let $\mathcal{T}_{2n,d}$ denote the set of trees of order $2n$ with diameter d. Let $\mathcal{T}^*_{2n,d}$ be the subset of those trees in $\mathcal{T}_{2n,d}$ that have a perfect matching. Let $T \in \mathcal{T}_{2n,d}$ and $P_{d+1} = v_0v_1 \ldots v_d$ be a diametrical path of T. Denote by T_j the connected component of $T - E(P_{d+1})$ containing v_j for $j \in \{0, 1, \ldots, d\}$.

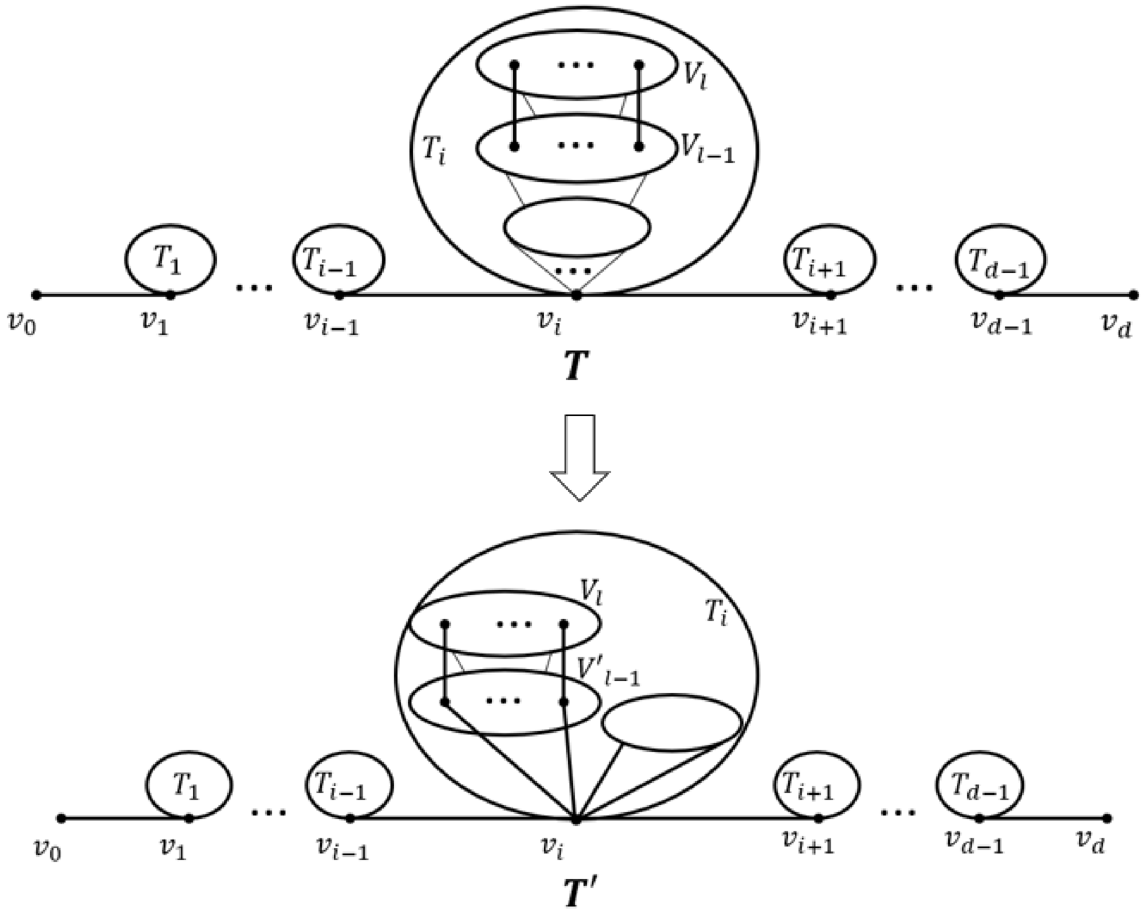

Figure 2.1: Trees T and T' used in Lemma 2.1.

Lemma 2.1. *Let $T \in \mathcal{T}_{2n,d}$ be a tree with a diametrical path $P_{d+1} = v_0v_1 \cdots v_d$ such that $d \geq 6$. Assume that there exists a component T_i with $e_{T_i}(v_i) = l \geq 3$. Denote $V_j = \{u \in V(T_i) : d_{T_i}(u, v_i) = j \leq l\}$. Denote $V'_{l-1} = \bigcup_{u \in V_l} N(u) \cap V_{l-1}$, $V'_{l-2} = \bigcup_{u \in V'_{l-1}} N(u) \cap V_{l-2}$ and $E = E(V'_{l-1}, V'_{l-2})$. Let*

$$T' = T - E + \{wv_i : w \in V'_{l-1}\},$$

where T and T' are depicted in Figure 2.1. Then, $\xi_1(T') < \xi_1(T)$, $\xi_2(T') < \xi_2(T)$ and $\alpha(T') = \alpha(T)$.

Proof. Note that $e_T(v_i) = \max\{d(v_i, v_0), d(v_i, v_d)\}$ for $v_i \in V(P_{d+1})$. Let $e_T(v_i) = l'$, then for $u \in V_l$, we have

$$e_T(u) = l + e(v_i) = l + l' \quad \text{and} \quad e_{T'}(u) = 2 + e(v_i) = 2 + l'.$$

For $u \in V_{l-1}$,

$$e_T(u) = l - 1 + e(v_i) = l + l' - 1 \quad \text{and} \quad e_{T'}(u) = 1 + e(v_i) = 1 + l'.$$

For $u \in V_{l-2}$,

$$e_T(u) = l - 2 + e(v_i) = l + l' - 2.$$

Note that $e_T(v) = e_{T'}(v)$ for all $v \in V(T) \setminus (V_l \cup V'_{l-1})$. Let $|V_l| = a$, $|V'_{l-1}| = b$, $|E(V_l, V'_{l-1})| = k_1$ and $|E| = k_2$. By direct calculation, we have

$$\begin{aligned}\xi_1(T) - \xi_1(T') &= \sum_{u \in V_l} e_T^2(u) + \sum_{u \in V'_{l-1}} e_T^2(u) - \sum_{u \in V_l} e_{T'}^2(u) - \sum_{u \in V'_{l-1}} e_{T'}^2(u) \\ &= a(l + l')^2 + b(l + l' - 1)^2 - a(2 + l')^2 - b(1 + l')^2 > 0\end{aligned}$$

and

$$\begin{aligned}\xi_2(T) - \xi_2(T') &= \sum_{uv \in E(V_l, V'_{l-1})} e_T(u)e_T(v) + \sum_{uv \in E} e_T(u)e_T(v) - \sum_{uv \in E(V_l, V'_{l-1})} e_{T'}(u)e_{T'}(v) - \sum_{u \in V'_{l-1}} e_{T'}(u)e_{T'}(v_i) \\ &= k_1(l + l')(l + l' - 1) + k_2(l + l' - 1)(l + l' - 2) - k_1(2 + l')(1 + l') - k_2(1 + l')l' > 0.\end{aligned}$$

By Lemma 3.1 of [5], we have $\alpha(T') = \alpha(T)$. □

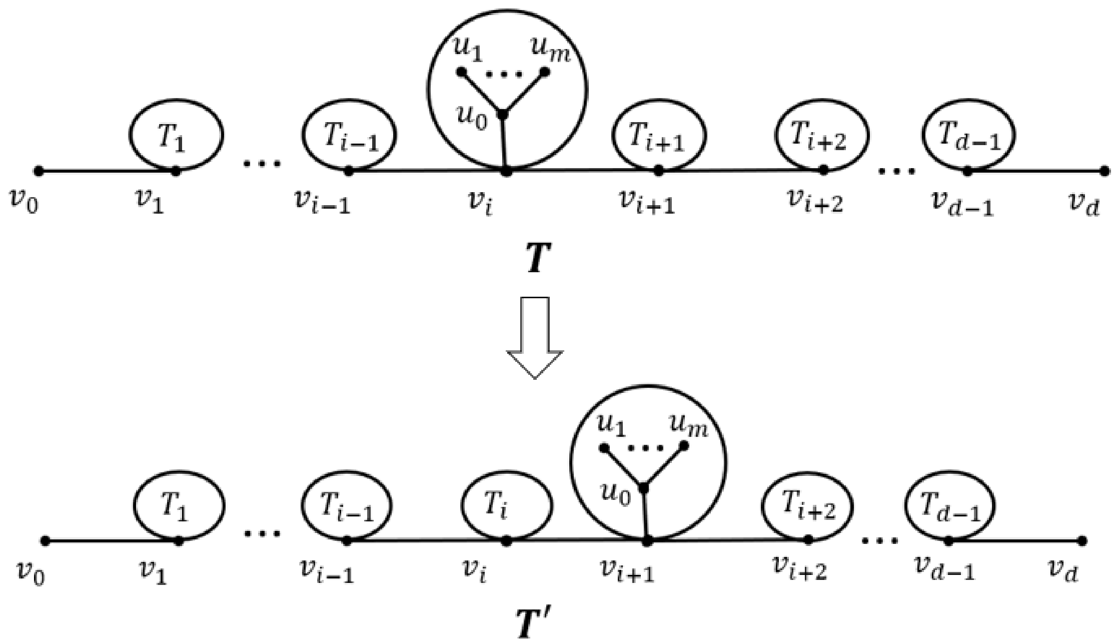

Figure 2.2: Trees T and T' used in Lemma 2.2.

Lemma 2.2. *Let $T \in \mathcal{T}_{2n,d}$ be a tree with a diametrical path $P_{d+1} = v_0v_1 \cdots v_d$ such that $d \geq 6$. Assume that there exists a component T_i with $e_{T_i}(v_i) = 2$ and $2 \leq i \leq \lfloor \frac{d-2}{2} \rfloor$. Let $u_0 \in N(v_i)$ and $u_1, u_2, \ldots, u_m$ be pendant neighbors of u_0. Let*

$$T' = T - u_0v_i + u_0v_{i+1},$$

where T and T' are depicted in Figure 2.2. Then, $\xi_1(T') < \xi_1(T)$, $\xi_2(T') < \xi_2(T)$ and $\alpha(T') = \alpha(T)$.

Proof. Note that $e_T(v) = e_{T'}(v)$ for all $v \in V(T) \setminus \{u_0, u_1, \ldots, u_m\}$. Furthermore, $e_{T'}(u_0) = e_T(u_0) - 1 = d - i$, and for $j = 1, 2, \ldots, m$, we have

$$e_{T'}(u_j) = e_T(u_j) - 1 = d - i + 1.$$

Thus,

$$\xi_1(T) - \xi_1(T') = e_T^2(u_0) - e_{T'}^2(u_0) + m \cdot (e_T^2(u_1) - e_{T'}^2(u_1)) > 0$$

and

$$\begin{aligned} \xi_2(T) - \xi_2(T') &= e_T(u_0)e_T(v_i) - e_{T'}(u_0)e_{T'}(v_{i+1}) + m \cdot (e_T(u_1)e_T(u_0) - e_T(u_1)e_T(u_0)) \\ &> m \cdot ((d-i+1)(d-i) - (d-i)(d-i-1)) > 0. \end{aligned}$$

By Lemma 3.2 of [5], we have $\alpha(T') = \alpha(T)$. □

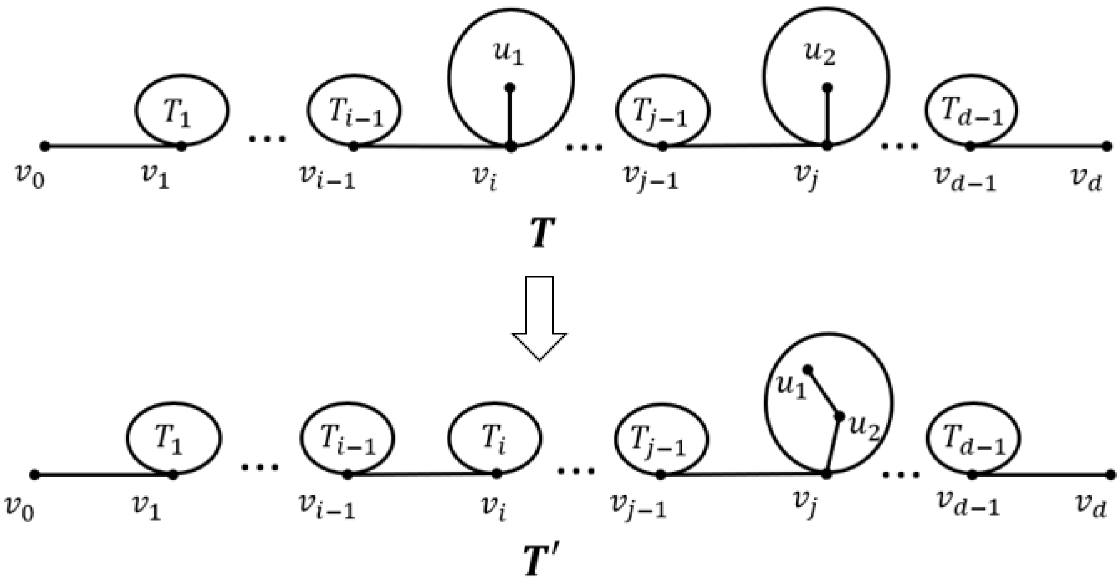

Figure 2.3: Trees T and T' considered in Lemma 2.3.

Lemma 2.3. *Let $T \in \mathcal{T}_{2n,d}$ be a tree with a diametrical path $P_{d+1} = v_0v_1 \cdots v_d$ such that $d \geq 4$. Assume that there are pendant edges u_1v_i, u_2v_j in T with $1 \leq i < j \leq \lfloor \frac{d}{2} \rfloor$ and $u_1, u_2 \notin V(P_{d+1})$. Let*

$$T' = T - u_1v_i + u_1u_2,$$

where T and T' are depicted in Figure 2.3. Then, each of the following statements holds true.

(i) *If $j = i + 1$, then $\xi_1(T') = \xi_1(T)$ and $\xi_2(T') = \xi_2(T)$.*

(ii) *If $j > i + 1$, then $\xi_1(T') < \xi_1(T)$ and $\xi_2(T') < \xi_2(T)$.*

(iii) [5] *Assume that $e_{T_k}(v_k) \leq 2$ for any $0 \leq k \leq d$ and there are no pendant edges incident with v_l for each $i+1 \leq l \leq j-1$. If T has a perfect matching, then T' has a perfect matching.*

Proof. Note that $e_T(v) = e_{T'}(v)$ for all $v \in V(T) \setminus \{u_1\}$. Moreover,

$$e_T(u_1) = d - i + 1, \quad e_T(v_i) = d - i, \quad e_{T'}(u_1) = d - j + 2 \quad \text{and} \quad e_{T'}(u_2) = d - j + 1.$$

If $j = i + 1$, then
$$\xi_1(T) - \xi_1(T') = e_T^2(u_1) - e_{T'}^2(u_1) = (d - i + 1)^2 - (d - i + 1)^2 = 0$$
and
$$\xi_2(T) - \xi_2(T') = e_T(u_1)e_T(v_i) - e_{T'}(u_1)e_{T'}(u_2) = (d - i + 1)(d - i) - (d - i + 1)(d - i) = 0.$$
If $j > i + 1$, then
$$\xi_1(T) - \xi_1(T') = e_T^2(u_1) - e_{T'}^2(u_1) = (d - i + 1)^2 - (d - j + 2)^2 > 0$$
and
$$\xi_2(T) - \xi_2(T') = e_T(u_1)e_T(v_i) - e_{T'}(u_1)e_{T'}(u_2) = (d - i + 1)(d - i) - (d - j + 1)(d - j + 2) > 0.$$

□

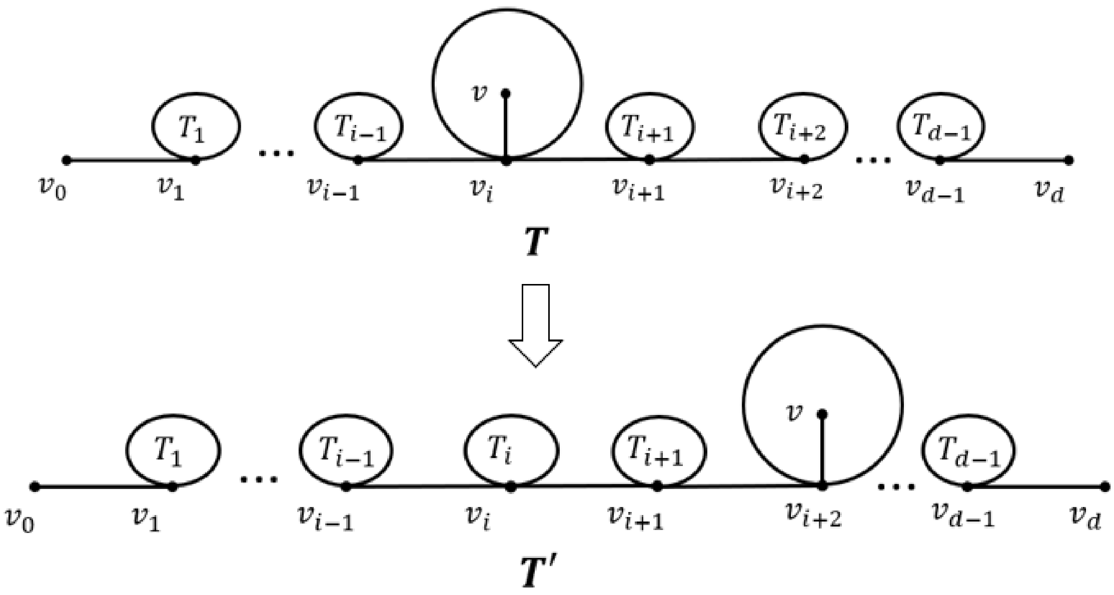

Figure 2.4: Tree T and T' used in Lemma 2.4.

Lemma 2.4. *Let $T \in \mathcal{T}_{2n,d}$ be a tree with a diametrical path $P_{d+1} = v_0v_1 \cdots v_d$ such that $d \geq 6$. Assume that there exists a pendant edge $vv_i \in E(T_i)$ with $1 \leq i \leq \lfloor \frac{d-4}{2} \rfloor$. Let*
$$T' = T - vv_i + vv_{i+2},$$
where T and T' are depicted in Figure 2.4. Then, the following statements hold true.

(i) $\xi_1(T') < \xi_1(T)$ *and* $\xi_2(T') < \xi_2(T)$.

(ii) [5] *Assume that there are no pendant edges incident with v_{i+1} and v_{i+2}. If T has a perfect matching, then T' has a perfect matching.*

Proof. Note that $e_T(u) = e_{T'}(u)$ for all $u \in V(T) \setminus \{v\}$. Moreover,
$$e_T(v_i) = d - i, \quad e_T(v_{i+2}) = d - i - 2, \quad e_T(v) = d - i + 1 \quad \text{and} \quad e_{T'}(v) = d - i - 1.$$
Thus
$$\xi_1(T) - \xi_1(T') = e_T^2(v) - e_{T'}^2(v) = (d - i + 1)^2 - (d - i - 1)^2 > 0$$
and
$$\xi_2(T) - \xi_2(T') = e_T(v)e_T(v_i) - e_{T'}(v)e_{T'}(v_{i+2}) = (d - i)(d - i + 1) - (d - i - 1)(d - i - 2) > 0.$$

□

Choose $T \in \mathcal{T}^*_{2n,d}$ such that $\xi_2(T)$ (or $\xi_1(T)$) is minimized. Let $P_{d+1} = v_0v_1 \dots v_n$ be a diametrical path of T. By using Lemma 2.1, we have $e_{T_i}(v_i) \le 2$ for every i satisfying $0 \le i \le d$. By Lemma 2.2, if $e_{T_i}(v_i) = 2$, then $i = \frac{d-1}{2}$ or $\frac{d+1}{2}$ for odd d and $i = \frac{d}{2}$ for even d. Furthermore, by Lemma 2.4, if d is odd and $e_{T_i}(v_i) = 1$, then $i \in \{\frac{d-3}{2}, \frac{d-1}{2}, \frac{d+1}{2}, \frac{d+3}{2}\}$; if d is even and $e_{T_i}(v_i) = 1$, then $i \in \{\frac{d-2}{2}, \frac{d}{2}, \frac{d+2}{2}\}$. In summary, the following corollaries hold.

Corollary 2.1. *Let $T \in \mathcal{T}^*_{2n,d}$ be a tree that minimizes ξ_1 or ξ_2, where $d \ge 5$ is odd. Suppose that T has the maximum possible number of non-pendant vertices. Then, the following statements hold true.*

(i) *$e_{T_i}(v_i) \le 2$ for every i satisfying $0 \le i \le d$.*

(ii) *If $e_{T_i}(v_i) = 2$, then $i \in \{\frac{d-1}{2}, \frac{d+1}{2}\}$.*

(iii) *If $e_{T_i}(v_i) = 1$, then $i \in \{\frac{d-3}{2}, \frac{d-1}{2}, \frac{d+1}{2}, \frac{d+3}{2}\}$.*

(iv) *The total number of the pendant edges incident with v_i is at most one, where $i \in \{\frac{d-3}{2}, \frac{d-1}{2}\}$.*

Corollary 2.2. *Let $T \in \mathcal{T}^*_{2n,d}$ be a tree that minimizes ξ_1 or ξ_2, where $d \ge 4$ is even. Suppose that T has the maximum possible number of non-pendant vertices. Then, the following statements hold true.*

(i) *$e_{T_i}(v_i) \le 2$ for every i satisfying $0 \le i \le d$.*

(ii) *If $e_{T_i}(v_i) = 2$, then $i = \frac{d}{2}$.*

(iii) *If $e_{T_i}(v_i) = 1$, then $i \in \{\frac{d-2}{2}, \frac{d}{2}, \frac{d+2}{2}\}$.*

(iv) *The total number of the pendant edges incident with v_i is at most one, where $i \in \{\frac{d-2}{2}, \frac{d}{2}\}$.*

3. Main results

In this section, we determine the trees with the minimum values of the first and second Zagreb eccentricity indices among $2n$-vertex trees with perfect matching. Suppose that $d \ge 5$ is odd. Let $T^{x,y}_{2n,d}$ be the tree obtained from $P_{d+1} = v_0v_1 \cdots v_d$ by attaching x paths of length 2 to $v_{\frac{d-1}{2}}$ and y paths of length 2 to $v_{\frac{d+1}{2}}$, where $2(x+y)+d+1 = 2n$, see Figure 3.1. Let $\dot{T}^{x,y}_{2n,d}$ be the tree obtained from $T^{x,y}_{2n-2,d}$ by attaching a pendant edge to $v_{\frac{d-3}{2}}$ and $v_{\frac{d+3}{2}}$, see Figure 3.2. Let $\ddot{T}^{x,y}_{2n,d}$ be the tree obtained from $T^{x,y}_{2n-2,d}$ by attaching a pendant edge to $v_{\frac{d-1}{2}}$ and $v_{\frac{d+1}{2}}$, see Figure 3.3.

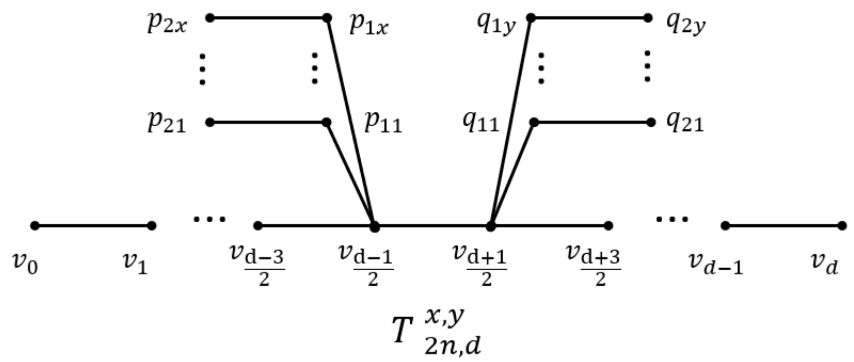

Figure 3.1: The tree $T^{x,y}_{2n,d}$.

Lemma 3.1. *Let $T \in \mathcal{T}^*_{2n,d}$ be a tree that minimizes ξ_1 or ξ_2, where $d \ge 5$ is odd. Suppose that the number of non-pendant vertices in T is maximum among all such trees. Then, T has one of the following structures.*

(i) *If $7 \le d \le 2n-3$ and $d \equiv 3 \pmod 4$, then*

$$T \cong T^{x,y}_{2n,d} \text{ or } T \cong \dot{T}^{x,y}_{2n,d}.$$

(ii) *If $5 \le d \le 2n-3$ and $d \equiv 1 \pmod 4$, then*

$$T \cong T^{x,y}_{2n,d} \text{ or } T \cong \ddot{T}^{x,y}_{2n,d}.$$

(iii) *If $d = 2n-1$, then $T \cong P_{2n}$.*

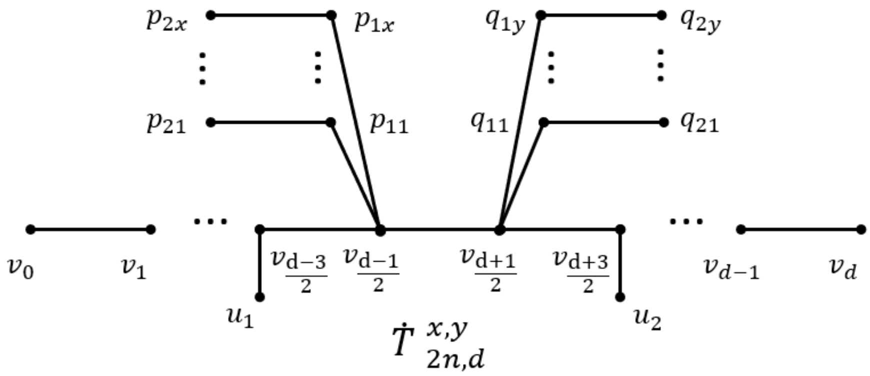

Figure 3.2: The tree $\dot{T}^{x,y}_{2n,d}$.

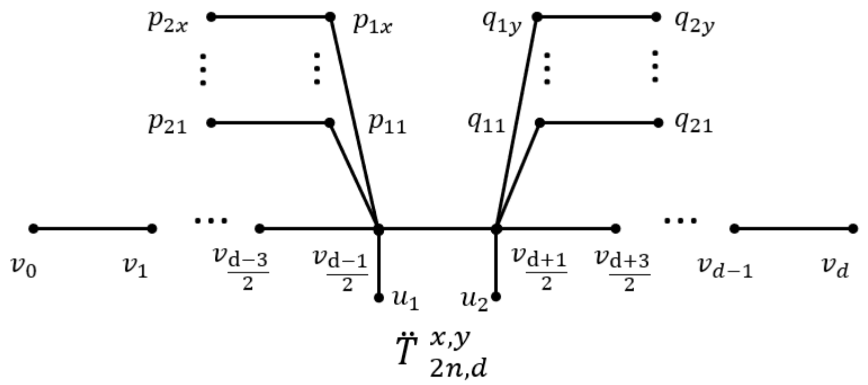

Figure 3.3: The tree $\ddot{T}^{x,y}_{2n,d}$.

Proof. Denote by a_1, a_2, a_3, a_4 the number of pendant vertices adjacent to $v_{\frac{d-3}{2}}, v_{\frac{d-1}{2}}, v_{\frac{d+1}{2}}, v_{\frac{d+3}{2}}$, respectively. As T has $2n$ vertices, $\sum_{i=1}^{4} a_i$ must be even. By Corollary 2.1(iv), we obtain $0 \le a_1 + a_2 = a_3 + a_4 \le 1$.

Suppose that $d \equiv 3 \pmod 4$. If $a_1 + a_2 = 0$, then $T \cong T^{x,y}_{2n,d}$. If $a_1 + a_2 = 1$, then $T \cong \dot{T}^{x,y}_{2n,d}$.

Suppose that $d \equiv 1 \pmod 4$. If $a_1 + a_2 = 0$, then $T \cong T^{x,y}_{2n,d}$. If $a_1 + a_2 = 1$, then $T \cong \ddot{T}^{x,y}_{2n,d}$.

If $d = 2n - 1$, then P_{2n} is the unique tree in $\mathcal{T}^*_{2n,d}$. Hence, the result follows. □

Suppose that $d \ge 4$ is even. Let $T^z_{n,d}$ be the tree obtained from $P_{d+1} = v_0 v_1 \cdots v_d$ by attaching z paths of length 2 to $v_{\frac{d}{2}}$, where $2z + d + 1 = n$, see Figure 3.4. Let $\dot{T}^z_{2n,d}$ be the tree obtained from $T^z_{2n-1,d}$ by attaching a pendant edge to $v_{\frac{d}{2}}$, see Figure 3.5. Let $\ddot{T}^z_{2n,d}$ be the tree obtained from $T^{x,y}_{2n-1,d}$ by attaching a pendant edge to $v_{\frac{d-2}{2}}$, see Figure 3.6.

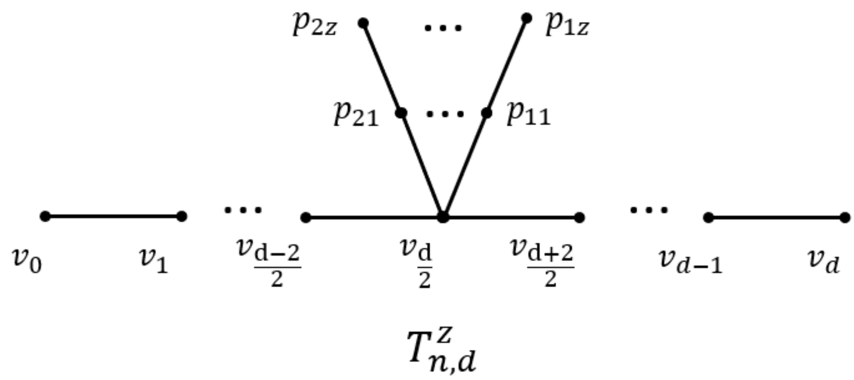

Figure 3.4: The tree $T^z_{n,d}$.

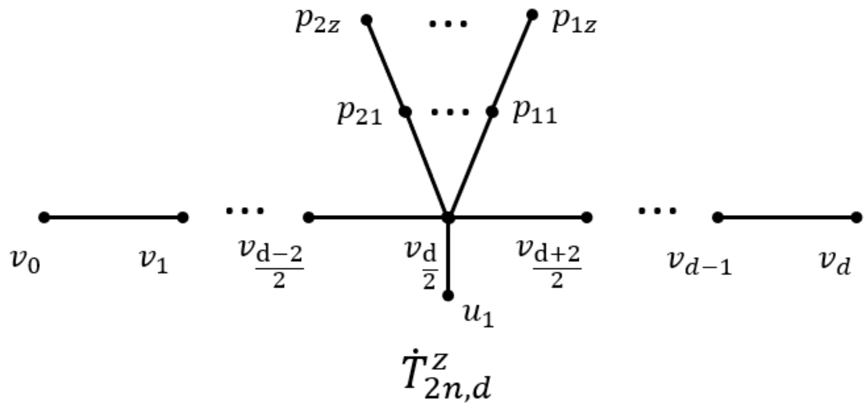

Figure 3.5: The tree $\dot{T}^{z}_{2n,d}$.

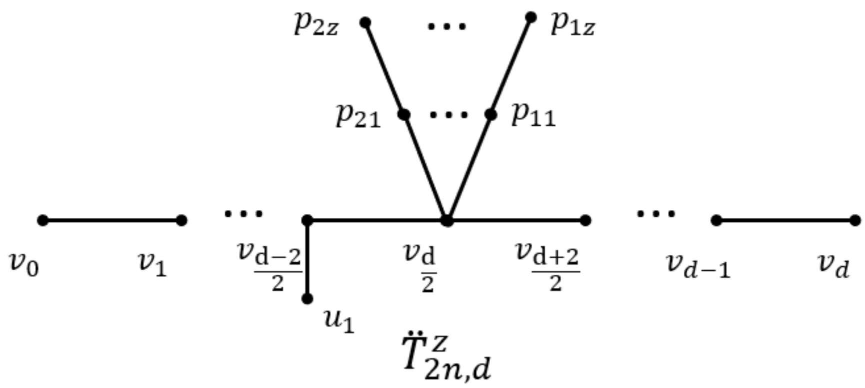

Figure 3.6: The tree $\ddot{T}^{z}_{2n,d}$.

Lemma 3.2. *Let $T \in \mathcal{T}^{*}_{2n,d}$ be a tree that minimizes ξ_1 or ξ_2, where $d \geq 4$ is even. Suppose that the number of non-pendant vertices in T is maximum among all such trees. Then, T has one of the following structures.*

(i) *If $d \geq 4$ and $d \equiv 0 \pmod 4$, then*

$$T \cong \dot{T}^{z}_{2n,d}.$$

(ii) *If $d \geq 6$ and $d \equiv 2 \pmod 4$, then*

$$T \cong \ddot{T}^{z}_{2n,d}.$$

Proof. Denote by a_1, a_2, a_3 the number of pendant vertices adjacent to $v_{\frac{d-2}{2}}, v_{\frac{d}{2}}, v_{\frac{d+2}{2}}$, respectively. As T has $2n$ vertices, $\sum_{i=1}^{3} a_i$ must be odd. If $\sum_{i=1}^{3} a_i = 3$, then $a_1 = a_2 = a_3 = 1$, and hence by the transformation of Lemma 2.3, we can obtain a new tree $T^* \in \mathcal{T}^{*}_{2n,d}$ with $\xi_1(T^*) = \xi_1(T)$ and $\xi_2(T^*) = \xi_2(T)$. Note that the number of non-pendant vertices in T^* is greater than that in T, which is a contradiction. Thus, $\sum_{i=1}^{3} a_i = 1$.

Suppose that $d \equiv 0 \pmod 4$. If $a_1 = 1$, then T doesn't have a perfect matching. If $a_2 = 1$, then $T \cong \dot{T}^{z}_{2n,d}$.

Suppose that $d \equiv 2 \pmod 4$. If $a_1 = 1$, then $T \cong \ddot{T}^{z}_{2n,d}$. If $a_2 = 1$, then T doesn't have a perfect matching. □

Lemma 3.3. *Let $T \in \mathcal{T}^{*}_{2n,d}$ be a tree that minimizes ξ_1 or ξ_2, where $d \geq 7$ is odd. Suppose that the number of non-pendant vertices in T is maximum among all such trees. Then, there exists a tree $T' \in \mathcal{T}^{*}_{2n,d-2}$ such that*

$$\xi_1(T') < \xi_1(T) \quad \text{and} \quad \xi_2(T') < \xi_2(T).$$

Proof. Suppose that $d \equiv 1 \pmod 4$. By Lemma 3.1, $T \cong T^{x,y}_{2n,d}$ or $\ddot{T}^{x,y}_{2n,d}$. Suppose that $T \cong T^{x,y}_{2n,d}$. Let

$$T' = T - v_0v_1 - v_{d-1}v_d + v_0v_{\frac{d-3}{2}} + v_dv_{\frac{d+3}{2}}.$$

Then, $T' \cong \ddot{T}^{x,y}_{2n,d-2}$. It is obvious that $e_{T'}(v) < e_T(v)$ for all $v \in V(T)$. Thus, $\xi_1(T') < \xi_1(T)$ and

$$\begin{aligned} \xi_2(T') - \xi_2(T) &< e_{T'}(v_0)e_{T'}(v_{\frac{d-3}{2}}) + e_{T'}(v_d)e_{T'}(v_{\frac{d+3}{2}}) - e_T(v_0)e_T(v_1) - e_T(v_{d-1})e_T(v_d) \\ &= 2 \cdot \frac{d+3}{2} \cdot \frac{d+1}{2} - 2d(d-1) < 0. \end{aligned}$$

Since $d-2 \equiv 3 \pmod 4$ and $T' \in \mathcal{T}^*_{2n,d-2}$, the result follows in the considered case.

Suppose now that $T \cong \ddot{T}^{x,y}_{2n,d}$. Let $T' = T - v_0v_1 - v_{d-1}v_d + v_0u_1 + v_du_2$. Then $T' \cong T^{x+1,y+1}_{2n,d-2}$. By a similar argument, we have that $\xi_1(T') < \xi_1(T)$ and $\xi_2(T') < \xi_2(T)$. Since $d-2 \equiv 3 \pmod 4$ and $T' \in \mathcal{T}^*_{2n,d-2}$, the result follows in the considered case.

Suppose that $d \equiv 3 \pmod 4$. By Lemma 3.1, $T \cong T^{x,y}_{2n,d}$ or $\dot{T}^{x,y}_{2n,d}$. Suppose that $T \cong T^{x,y}_{2n,d}$. Let

$$T' = T - v_0v_1 - v_{d-1}v_d + v_0v_{\frac{d-1}{2}} + v_dv_{\frac{d+1}{2}}.$$

Then, $T' \cong \ddot{T}^{x,y}_{2n,d-2}$. By a similar argument, we have that

$$\xi_1(\ddot{T}^{x,y}_{2n,d-2}) < \xi_1(T) \quad \text{and} \quad \xi_2(\ddot{T}^{x,y}_{2n,d-2}) < \xi_2(T).$$

Since $d-2 \equiv 1 \pmod 4$ and $T' \in \mathcal{T}^*_{2n,d-2}$, the result follows in the considered case.

Suppose now that $T \cong \dot{T}^{x,y}_{2n,d}$. Let

$$T_1 = T - v_0v_1 - v_{d-1}v_d + v_0v_{\frac{d-1}{2}} + v_dv_{\frac{d+1}{2}}$$

and

$$T_2 = T_1 - u_1v_{\frac{d-3}{2}} - u_2v_{\frac{d+3}{2}} + v_0u_1 + v_du_2.$$

By a similar argument, we have that

$$\xi_1(T_1) < \xi_1(T) \quad \text{and} \quad \xi_2(T_1) < \xi_2(T).$$

By Lemma 2.3, we have that $\xi_1(T_2) = \xi_1(T_1)$ and $\xi_2(T_2) = \xi_2(T_1)$. Note that $T_2 \cong T^{x+1,y+1}_{2n,d-2}$ and $d-2 \equiv 1 \pmod 4$. Thus, $T_2 \in \mathcal{T}^*_{2n,d-2}$, and hence, the desired result follows. □

Lemma 3.4. *Let $T \in \mathcal{T}^*_{2n,d}$ be a tree that minimizes ξ_1 or ξ_2, where $d \geq 6$ is even. Suppose that the number of non-pendant vertices in T is maximum among all such trees. Then, there exists a tree $T' \in \mathcal{T}^*_{2n,d-2}$ such that $\xi_1(T') < \xi_1(T)$ and $\xi_2(T') < \xi_2(T)$.*

Proof. Suppose that $d \equiv 2 \pmod 4$. By Lemma 3.1, $T \cong \ddot{T}^{z}_{2n,d}$. Let

$$T_1 = T - v_0v_1 - v_{d-1}v_d + v_0v_{\frac{d}{2}} + v_dv_{\frac{d+2}{2}}$$

and

$$T_2 = T_1 - u_1v_{\frac{d-2}{2}} + u_1v_0.$$

It is obvious that $e_{T'}(v) < e_T(v)$ for all $v \in V(T)$. Thus, $\xi_1(T_1) < \xi_1(T)$ and

$$\begin{aligned} \xi_2(T_1) - \xi_2(T) &< e_{T'}(v_0)e_{T'}(v_{\frac{d}{2}}) + e_{T'}(v_d)e_{T'}(v_{\frac{d+2}{2}}) - e_T(v_0)e_T(v_1) - e_T(v_{d-1})e_T(v_d) \\ &= \frac{d}{2} \cdot \frac{d-2}{2} + \frac{d-2}{2} \cdot \frac{d-4}{2} - 2(d-1)d < 0. \end{aligned}$$

By Lemma 2.3, we have that $\xi_1(T_1) = \xi_1(T_2)$ and $\xi_2(T_1) = \xi_2(T_2)$. So $\xi_1(T_2) < \xi_1(T)$ and $\xi_2(T_2) < \xi_2(T)$. Note that $T_2 \cong \dot{T}^{z+1}_{2n,d-2}$ and $d-2 \equiv 0 \pmod 4$. Thus $T_2 \in \mathcal{T}^*_{2n,d-2}$ and the result follows in the considered case.

Suppose that $d \equiv 0 \pmod 4$. By Lemma 3.1, $T \cong \dot{T}^{z}_{2n,d}$. Let

$$T_1 = T - v_0v_1 - v_{d-1}v_d + v_0v_{\frac{d-2}{2}} + v_dv_{\frac{d+2}{2}}$$

and

$$T_2 = T_1 - v_0v_{\frac{d-2}{2}} + v_0v_{\frac{d}{2}}.$$

By a similar argument, we have that $\xi_1(T_1) < \xi_1(T)$ and $\xi_2(T_1) < \xi_2(T)$. By Lemma 2.3, $\xi_1(T_1) = \xi_1(T_2)$ and $\xi_2(T_1) = \xi_2(T_2)$. Thus $\xi_1(T_2) < \xi_1(T)$ and $\xi_2(T_2) < \xi_2(T)$. Note that $T_2 \cong \ddot{T}^{z+1}_{2n,d-2}$ and $d-2 \equiv 2 \pmod 4$. Thus, $T_2 \in \mathcal{T}^*_{2n,d-2}$, and hence, the required result follows. □

Theorem 3.1. *Let $T \in \mathcal{T}^*_{2n,d}$ be a tree, where $d \geq 5$ is odd. Then*

$$\xi_1(T) \geq \xi_1(\ddot{T}^{a,b}_{2n,5}) \quad \textit{and} \quad \xi_2(T) \geq \xi_2(\ddot{T}^{a,b}_{2n,5})$$

where any of the equalities holds if and only if $T \cong \ddot{T}^{a,b}_{2n,5}$.

Proof. Let $T \in \mathcal{T}^*_{2n,d}$ be a tree that minimizes ξ_1 or ξ_2, where $d \geq 5$ is odd. Suppose that the number of non-pendant vertices in T is maximum among all such trees. From Lemmas 3.1 and 3.3, it follows that $T \cong T^{x,y}_{2n,5}$ with $2(x+y)+6 = 2n$ or $T \cong \ddot{T}^{a,b}_{2n,5}$ with $2(a+b)+8 = 2n$. Here, we have

$$\begin{aligned} \xi_1(T^{x,y}_{2n,5}) &= (x+1)\cdot 5^2 + (x+1)\cdot 4^2 + 3^2 + 3^2 + (y+1)\cdot 4^2 + (x+1)\cdot 5^2 \\ &= 41n - 23, \end{aligned}$$

$$\begin{aligned} \xi_1(\ddot{T}^{a,b}_{2n,5}) &= (a+1)\cdot 5^2 + (a+2)\cdot 4^2 + 3^2 + 3^2 + (b+2)\cdot 4^2 + (b+1)\cdot 5^2 \\ &= 41n - 32, \end{aligned}$$

$$\begin{aligned} \xi_2(T^{x,y}_{2n,5}) &= (x+1)\cdot 5\cdot 4 + (x+1)\cdot 4\cdot 3 + 3\cdot 3 + (y+1)\cdot 4\cdot 3 + (y+1)\cdot 5\cdot 4 \\ &= 32n - 23 \end{aligned}$$

and

$$\begin{aligned} \xi_2(\ddot{T}^{a,b}_{2n,5}) &= (a+1)\cdot 5\cdot 4 + (a+2)\cdot 4\cdot 3 + 3^2 + (b+2)\cdot 4\cdot 3 + (b+1)\cdot 5\cdot 4 \\ &= 32n - 31. \end{aligned}$$

Thus $\xi_1(\ddot{T}^{a,b}_{2n,5}) < \xi_1(T^{x,y}_{2n,5})$ and $\xi_2(\ddot{T}^{a,b}_{2n,5}) < \xi_2(T^{x,y}_{2n,5})$. □

From Lemmas 3.2 and 3.4, the following theorem follows immediately.

Theorem 3.2. *Let $T \in \mathcal{T}^*_{2n,d}$ be a tree such that $d \geq 4$ is even. Then,*

$$\xi_1(T) \geq \xi_1(\dot{T}^{z}_{2n,4}) \quad \textit{and} \quad \xi_2(T) \geq \xi_2(\dot{T}^{z}_{2n,4})$$

where any of the equalities holds if and only if $T \cong \dot{T}^{z}_{2n,4}$.

Theorem 3.3. *Let $T \in \mathcal{T}^*_{2n,d}$ be a tree such that $d \geq 4$. Then,*

$$\xi_1(T) \geq \xi_1(\dot{T}^{z}_{2n,4}) \quad \textit{and} \quad \xi_2(T) \geq \xi_2(\dot{T}^{z}_{2n,4})$$

where any of the equalities holds if and only if $T \cong \dot{T}^{z}_{2n,4}$.

Proof. By direct calculation, we have that

$$\xi_1(\dot{T}^{z}_{2n,4}) = (z+2)\cdot 4^2 + (z+3)\cdot 3^2 + 2^2 = 25n - 12 < \xi_1(\ddot{T}^{a,b}_{2n,5})$$

and

$$\xi_2(\dot{T}^{z}_{2n,4}) = (z+2)\cdot 4\cdot 3 + (z+3)\cdot 3\cdot 2 = 18n - 12 < \xi_2(\ddot{T}^{a,b}_{2n,5}),$$

where $2(a+b)+8 = 2n$ and $2z+6 = 2n$. Now, from Theorems 3.1 and 3.2, the desired result follows. □

4. Concluding remarks

In this paper, we have identified the trees that minimize the first and second Zagreb eccentricity indices among all trees with $2n$ vertices and a perfect matching. To obtain these results, we propose four graph transformations that decrease the first and the second Zagreb eccentricity indices under certain constraints. We believe that these transformations can be applied to some other graph invariants.

Acknowledgment

This work was supported by Natural Science Foundation of Guangdong Province (Grant No. 2025A1515011889).

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