Electronic Journal of Mathematics
www.shahindp.com/locate/ejm

Electron. J. Math. **10** (2025) 78–85
DOI: 10.47443/ejm.2025.058

Research Article

Hilbert-type integral inequalities based on the hyperbolic tangent function

Christophe Chesneau*

Department of Mathematics, LMNO, University of Caen-Normandie, 14032 Caen, France

(Received: 16 September 2025. Received in revised form: 19 October 2025. Accepted: 28 October 2025. Published online: 18 November 2025.)

Abstract

The Hilbert integral inequality is a well-established result in mathematical analysis and has given rise to an interesting line of research. In this article, two new variants of the Hilbert integral inequality are presented, both derived from the hyperbolic arctangent function. In the first variant, an appropriate upper bound is employed so that a comprehensive formulation is obtained in terms of a product of weighted integral norms of the main functions. In contrast, an exact formulation is provided in the second variant, although this is achieved at the expense of greater complexity. This variant involves sums and products of weighted integral norms of the main functions.

Keywords: Hilbert integral inequality; change of variable techniques; hyperbolic tangent function.

2020 Mathematics Subject Classification: 26D15.

1. Introduction

The Hilbert integral inequality is a well-known and widely recognized result in analysis, with deep connections to operator theory, harmonic analysis and function space theory. Since its original formulation, this inequality has inspired extensive research, resulting in numerous refinements, generalizations and extensions. These modifications often involve introducing new kernel functions, applying different bounding techniques, or adapting the inequality to various functional settings. The standard Hilbert integral inequality is formally described below. Let $f, g : (0, +\infty) \to (0, +\infty)$ be two functions (or $f, g : [0, +\infty) \to [0, +\infty)$). Then, we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y} dxdy \leq \pi \left(\int_0^{+\infty} f^2(x)dx \right)^{1/2} \left(\int_0^{+\infty} g^2(x)dx \right)^{1/2},$$

provided that the two integrals on the right-hand side converge. The constant factor π is optimal. Further information can be found in the books [3, 9, 10], the survey [1] and the recent articles [2, 4–8].

A famous example of extension of the Hilbert integral inequality was established in [7, Theorem 1]. It can be presented as a bounded-domain version of the Hilbert integral inequality, as formally stated below. Let $f, g : (0, +\infty) \to (0, +\infty)$ be two functions and $\alpha > 0$. Then we have

$$\int_0^{\alpha} \int_0^{\alpha} \frac{f(x)g(y)}{x+y} dxdy \leq \pi \left(\int_0^{\alpha} \left(1 - \frac{1}{2}\sqrt{\frac{x}{\alpha}}\right) f^2(x)dx \right)^{1/2} \left(\int_0^{\alpha} \left(1 - \frac{1}{2}\sqrt{\frac{x}{\alpha}}\right) g^2(x)dx \right)^{1/2}, \tag{1}$$

provided that the two integrals on the right-hand side converge. The additional weight function

$$w(x) = 1 - \frac{1}{2}\sqrt{\frac{x}{\alpha}}$$

reflects the influence of the boundary α and ensures the validity of the upper bound. Such extensions illustrate the richness of the Hilbert integral inequality framework and its flexibility in accommodating different analytical settings. In the present study, we contribute to this line of development by introducing two new variants of the Hilbert integral inequality derived from the hyperbolic arctangent function. To be more precise, they are based on the following double integral:

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{\tanh(x+y)} f(x)g(y)dxdy.$$

where $f, g : (0, +\infty) \to (0, +\infty)$ are two functions and, for any $a \in [0, +\infty)$,

$$\tanh(a) = \frac{e^a - e^{-a}}{e^a + e^{-a}}.$$

*E-mail address: christophe.chesneau@gmail.com

The first variant employs an appropriate upper bound, leading to a simpler yet approximate inequality involving weighted integral norms of f and g, while the second one offers an exact formulation at the expense of greater analytical complexity. The proofs of both variants rely on suitable changes of variables and the inequality (1). In addition, we establish series analogues of our results, based on the following double series:

$$\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}\frac{a_m b_n}{\tanh(m+n)},$$

where $(a_m)_{m\in\mathbb{N}\setminus\{0\}}$ and $(b_n)_{n\in\mathbb{N}\setminus\{0\}}$ are two sequences of non-negative real numbers. Beyond their intrinsic novelty, these new forms reveal interesting structural features and suggest potential applications in operator theory.

The article is organized as follows: Section 2 introduces a key integral transformation. Section 3 presents the first hyperbolic tangent Hilbert-type integral inequality, while Section 4 develops the second such inequality. Section 5 concludes the article.

2. An integral transformation

The theorem given below establishes a fundamental integral transformation that will play a central role in the proofs of the main results established in the subsequent sections.

Theorem 2.1. *Let $f, g : (0, +\infty) \to (0, +\infty)$ be two functions. Then,*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{\tanh(x+y)}dxdy = \int_0^1\int_0^1 f(\tanh^{-1}(u))g(\tanh^{-1}(v))\frac{1+uv}{(u+v)(1-u^2)(1-v^2)}dudv,$$

provided that the integrals converge.

Proof. We consider the change of variables

$$u = \tanh(x) \quad \text{and} \quad v = \tanh(y).$$

Then, $x = 0$ when $u = 0$, $x \to +\infty$ when $u \to 1$, $y = 0$ when $v = 0$, and $y \to +\infty$ when $v \to 1$. Also, we have the differentials

$$dx = \frac{1}{1-u^2}du \quad \text{and} \quad dy = \frac{1}{1-v^2}dv.$$

Furthermore, for $x, y \in (0, +\infty)$, we have

$$\tanh(x+y) = \frac{\tanh(x)+\tanh(y)}{1+\tanh(x)\tanh(y)} = \frac{u+v}{1+uv},$$

so that

$$\frac{1}{\tanh(x+y)} = \frac{1+uv}{u+v}.$$

Considering this change of variables, we obtain

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{\tanh(x+y)}dxdy &= \int_0^1\int_0^1 f(\tanh^{-1}(u))g(\tanh^{-1}(v))\frac{1+uv}{u+v}\times\frac{1}{1-u^2}du\times\frac{1}{1-v^2}dv\\ &= \int_0^1\int_0^1 f(\tanh^{-1}(u))g(\tanh^{-1}(v))\frac{1+uv}{(u+v)(1-u^2)(1-v^2)}dudv.\end{aligned}$$

This completes the proof. □

As will become clear in the proofs of the main results, the transformed integral is more straightforward to handle than the original form. This allows (1) to be applied more directly, which will be a key aspect of the proofs that follow.

3. First Hilbert-type integral inequality

The theorem given below presents our first hyperbolic tangent Hilbert-type integral inequality. It is based on Theorem 2.1, a specific bound, inequality (1) and suitable changes of variables.

Theorem 3.1. *Let $f, g : (0, +\infty) \to (0, +\infty)$ be two functions. Then,*

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{\tanh(x+y)} dxdy \le \pi \left(\int_0^{+\infty} \left(1 - \frac{1}{2}\sqrt{\tanh(x)}\right) e^{2x} f^2(x)dx \right)^{1/2} \left(\int_0^{+\infty} \left(1 - \frac{1}{2}\sqrt{\tanh(x)}\right) e^{2x} g^2(x)dx \right)^{1/2},$$

provided that the two integrals on the right-hand side converge.

Proof. Applying Theorem 2.1, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{\tanh(x+y)} dxdy = \int_0^1\int_0^1 f(\tanh^{-1}(u))g(\tanh^{-1}(v)) \frac{1+uv}{(u+v)(1-u^2)(1-v^2)} dudv. \tag{2}$$

For any $u, v \in (0, 1)$, we have

$$1 + uv \le 1 + u + v + uv = (1 + u)(1 + v). \tag{3}$$

Therefore, we obtain

$$\begin{aligned} &\int_0^1\int_0^1 f(\tanh^{-1}(u))g(\tanh^{-1}(v)) \frac{1+uv}{(u+v)(1-u^2)(1-v^2)} dudv \\ &\le \int_0^1\int_0^1 f(\tanh^{-1}(u))g(\tanh^{-1}(v)) \frac{(1+u)(1+v)}{(u+v)(1-u^2)(1-v^2)} dudv \\ &= \int_0^1\int_0^1 \frac{f_\dagger(u)g_\dagger(v)}{u+v} dudv, \end{aligned} \tag{4}$$

where

$$f_\dagger(u) = f(\tanh^{-1}(u)) \frac{1+u}{1-u^2} \quad \text{and} \quad g_\dagger(v) = g(\tanh^{-1}(v)) \frac{1+v}{1-v^2}.$$

Applying (1) to $f_\dagger$ and $g_\dagger$, and considering $\alpha = 1$, we obtain

$$\int_0^1\int_0^1 \frac{f_\dagger(u)g_\dagger(v)}{u+v} dudv \le \pi \left(\int_0^1 \left(1 - \frac{1}{2}\sqrt{u}\right) f_\dagger^2(u)du \right)^{1/2} \left(\int_0^1 \left(1 - \frac{1}{2}\sqrt{u}\right) g_\dagger^2(u)du \right)^{1/2}. \tag{5}$$

For the first integral of the upper bound, we use the expression of $f_\dagger$ and the change of variables $u = \tanh(x)$, so that $x = 0$ when $u = 0$, $x \to +\infty$ when $u \to 1$, and $du = (1 - u^2)dx$, and hence, we have

$$\begin{aligned} &\int_0^1 \left(1 - \frac{1}{2}\sqrt{u}\right) f_\dagger^2(u)du = \int_0^1 \left(1 - \frac{1}{2}\sqrt{u}\right) f^2(\tanh^{-1}(u)) \frac{(1+u)^2}{(1-u^2)^2} du \\ &= \int_0^{+\infty} \left(1 - \frac{1}{2}\sqrt{\tanh(x)}\right) f^2(x) \frac{(1+\tanh(x))^2}{1-\tanh^2(x)} dx \\ &= \int_0^{+\infty} \left(1 - \frac{1}{2}\sqrt{\tanh(x)}\right) e^{2x} f^2(x)dx. \end{aligned} \tag{6}$$

For the second integral, we proceed similarly, and consequently, we obtain

$$\begin{aligned} &\int_0^1 \left(1 - \frac{1}{2}\sqrt{u}\right) g_\dagger^2(u)du = \int_0^1 \left(1 - \frac{1}{2}\sqrt{u}\right) g^2(\tanh^{-1}(u)) \frac{(1+u)^2}{(1-u^2)^2} du \\ &= \int_0^{+\infty} \left(1 - \frac{1}{2}\sqrt{\tanh(x)}\right) g^2(x) \frac{(1+\tanh(x))^2}{1-\tanh^2(x)} dx \\ &= \int_0^{+\infty} \left(1 - \frac{1}{2}\sqrt{\tanh(x)}\right) e^{2x} g^2(x)dx. \end{aligned} \tag{7}$$

From (2), (4), (5), (6) and (7), it follows that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{\tanh(x+y)}dxdy \le \pi\left(\int_0^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(x)}\right)e^{2x}f^2(x)dx\right)^{1/2}\left(\int_0^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(x)}\right)e^{2x}g^2(x)dx\right)^{1/2}.$$

This completes the proof. □

To the best of the author's knowledge, this is the first hyperbolic tangent variant of the Hilbert integral inequality to appear in the literature. A notable feature of this inequality is the constant factor π, together with the original weight function

$$w_\circ(x)=\left(1-\frac{1}{2}\sqrt{\tanh(x)}\right)e^{2x}.$$

An inspection of the proof shows that it relies on the bound given in Equation (3). However, a more refined approach can be pursued that avoids this bound, as this will be developed in Theorem 4.1. Note also that the inequality (1) is in fact strict, which implies the strict inequality in Theorem 3.1.

We now present a series analogue of Theorem 3.1.

Proposition 3.1. *Let $(a_m)_{m\in\mathbb{N}\backslash\{0\}}$ and $(b_n)_{n\in\mathbb{N}\backslash\{0\}}$ be two sequences of non-negative real numbers. Then,*

$$\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}\frac{a_m b_n}{\tanh(m+n)} \le \pi\left(\sum_{m=1}^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(m-1)}\right)e^{2m}a_m^2\right)^{1/2}\left(\sum_{m=1}^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(m-1)}\right)e^{2m}b_m^2\right)^{1/2},$$

provided that the two series on the right-hand side converge.

Proof. For any $m,n\in\mathbb{N}\backslash\{0\}$, set $f_\triangleleft(x)=a_m$ for $x\in[m-1,m)$ and $g_\triangleleft(x)=b_n$ for $x\in[n-1,n)$. Note that $1/\tanh(x+y)$ is a non-increasing function with respect to x and y. Therefore, we have

$$\frac{a_m b_n}{\tanh(m+n)} \le \int_{n-1}^{n}\int_{m-1}^{m}\frac{f_\triangleleft(x)g_\triangleleft(y)}{\tanh(x+y)}dxdy.$$

Applying Theorem 3.1 to the functions $f_\triangleleft$ and $g_\triangleleft$, and using the fact that $\tanh(x)$ and e^{2x} are non-decreasing, we have

$$\begin{aligned}\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}\frac{a_m b_n}{\tanh(m+n)} &\le \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}\int_{n-1}^{n}\int_{m-1}^{m}\frac{f_\triangleleft(x)g_\triangleleft(y)}{\tanh(x+y)}dxdy\\
&= \int_0^{+\infty}\int_0^{+\infty}\frac{f_\triangleleft(x)g_\triangleleft(y)}{\tanh(x+y)}dxdy\\
&\le \pi\left(\int_0^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(x)}\right)e^{2x}f_\triangleleft^2(x)dx\right)^{1/2}\left(\int_0^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(x)}\right)e^{2x}g_\triangleleft^2(x)dx\right)^{1/2}\\
&= \pi\left(\sum_{m=1}^{+\infty}\int_{m-1}^{m}\left(1-\frac{1}{2}\sqrt{\tanh(x)}\right)e^{2x}a_m^2dx\right)^{1/2}\left(\sum_{m=1}^{+\infty}\int_{m-1}^{m}\left(1-\frac{1}{2}\sqrt{\tanh(x)}\right)e^{2x}b_m^2dx\right)^{1/2}\\
&\le \pi\left(\sum_{m=1}^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(m-1)}\right)e^{2m}a_m^2\right)^{1/2}\left(\sum_{m=1}^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(m-1)}\right)e^{2m}b_m^2\right)^{1/2}.\end{aligned}$$

This concludes the proof. □

To the best of the author's knowledge, this is one of the few series inequalities in the literature that features the hyperbolic tangent function in the denominator. The remainder of the article develops alternative formulations corresponding to Theorem 3.1 and Proposition 3.1.

4. Second Hilbert-type integral inequality

The second hyperbolic tangent Hilbert-type integral inequality is presented in the theorem given below. It can be viewed as a refined version of Theorem 3.1. It is based on Theorem 2.1 and a suitable change of variables.

Theorem 4.1. *Let $f, g : (0, +\infty) \to (0, +\infty)$ be two functions. Then,*

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{\tanh(x+y)}dxdy \leq \pi\left[\left(\int_0^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(x)}\right)\cosh^2(x)f^2(x)dx\right)^{1/2}\right.$$
$$\times\left(\int_0^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(x)}\right)\cosh^2(x)g^2(x)dx\right)^{1/2}$$
$$+\left(\int_0^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(x)}\right)\sinh^2(x)f^2(x)dx\right)^{1/2}$$
$$\left.\times\left(\int_0^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(x)}\right)\sinh^2(x)g^2(x)dx\right)^{1/2}\right],$$

provided that the four integrals on the right-hand side converge, where, for any $a \in [0, +\infty)$,

$$\cosh(a) = \frac{e^a + e^{-a}}{2} \quad \textit{and} \quad \sinh(a) = \frac{e^a - e^{-a}}{2}.$$

Proof. Applying Theorem 2.1, we have

$$\begin{aligned}
\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{\tanh(x+y)}dxdy &= \int_0^1\int_0^1 f(\tanh^{-1}(u))g(\tanh^{-1}(v))\frac{1+uv}{(u+v)(1-u^2)(1-v^2)}dudv \\
&= \int_0^1\int_0^1 f(\tanh^{-1}(u))g(\tanh^{-1}(v))\frac{1}{(u+v)(1-u^2)(1-v^2)}dudv \\
&+ \int_0^1\int_0^1 f(\tanh^{-1}(u))g(\tanh^{-1}(v))\frac{uv}{(u+v)(1-u^2)(1-v^2)}dudv \\
&= \int_0^1\int_0^1 \frac{f_\diamond(u)g_\diamond(v)}{u+v}dudv + \int_0^1\int_0^1 \frac{f_*(u)g_*(v)}{u+v}dudv, \qquad (8)
\end{aligned}$$

where

$$f_\diamond(u) = f(\tanh^{-1}(u))\frac{1}{1-u^2}, \quad g_\diamond(u) = g(\tanh^{-1}(v))\frac{1}{1-v^2},$$

and

$$f_*(u) = f(\tanh^{-1}(u))\frac{u}{1-u^2}, \quad g_*(u) = g(\tanh^{-1}(v))\frac{v}{1-v^2}.$$

Applying (1) to $f_\diamond$ and $g_\diamond$, and considering $\alpha = 1$, we obtain

$$\int_0^1\int_0^1 \frac{f_\diamond(u)g_\diamond(v)}{u+v}dudv \leq \pi\left(\int_0^1\left(1-\frac{1}{2}\sqrt{u}\right)f_\diamond^2(u)du\right)^{1/2}\left(\int_0^1\left(1-\frac{1}{2}\sqrt{u}\right)g_\diamond^2(u)du\right)^{1/2}. \qquad (9)$$

Similarly, we have

$$\int_0^1\int_0^1 \frac{f_*(u)g_*(v)}{u+v}dudv \leq \pi\left(\int_0^1\left(1-\frac{1}{2}\sqrt{u}\right)f_*^2(u)du\right)^{1/2}\left(\int_0^1\left(1-\frac{1}{2}\sqrt{u}\right)g_*^2(u)du\right)^{1/2}. \qquad (10)$$

From (8), (9) and (10), it follows that

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{\tanh(x+y)}dxdy \leq \pi\left[\left(\int_0^1\left(1-\frac{1}{2}\sqrt{u}\right)f_\diamond^2(u)du\right)^{1/2}\left(\int_0^1\left(1-\frac{1}{2}\sqrt{u}\right)g_\diamond^2(u)du\right)^{1/2}\right.$$
$$\left.+\left(\int_0^1\left(1-\frac{1}{2}\sqrt{u}\right)f_*^2(u)du\right)^{1/2}\left(\int_0^1\left(1-\frac{1}{2}\sqrt{u}\right)g_*^2(u)du\right)^{1/2}\right]. \qquad (11)$$

For the first integral of the upper bound in (11), we use the expression of $f_\diamond$ and the change of variables $u = \tanh(x)$, so that $x = 0$ when $u = 0$, $x \to +\infty$ when $u \to 1$, and $du = (1-u^2)dx$, and hence, we obtain

$$\begin{aligned}\int_0^1 \left(1-\frac{1}{2}\sqrt{u}\right) f_\diamond^2(u)du &= \int_0^1 \left(1-\frac{1}{2}\sqrt{u}\right) f^2(\tanh^{-1}(u))\frac{1}{(1-u^2)^2}du\\ &= \int_0^{+\infty} \left(1-\frac{1}{2}\sqrt{\tanh(x)}\right) f^2(x)\frac{1}{1-\tanh^2(x)}dx\\ &= \int_0^{+\infty} \left(1-\frac{1}{2}\sqrt{\tanh(x)}\right) \cosh^2(x)f^2(x)dx. \qquad (12)\end{aligned}$$

Similarly, for the second integral of the bound in (11), we obtain

$$\begin{aligned}\int_0^1 \left(1-\frac{1}{2}\sqrt{u}\right) g_\diamond^2(u)dv &= \int_0^1 \left(1-\frac{1}{2}\sqrt{u}\right) g^2(\tanh^{-1}(u))\frac{1}{(1-u^2)^2}du\\ &= \int_0^{+\infty} \left(1-\frac{1}{2}\sqrt{\tanh(x)}\right) g^2(x)\frac{1}{1-\tanh^2(x)}dx\\ &= \int_0^{+\infty} \left(1-\frac{1}{2}\sqrt{\tanh(x)}\right) \cosh^2(x)g^2(x)dx. \qquad (13)\end{aligned}$$

For the third integral of the bound in (11), we proceed similarly and find that

$$\begin{aligned}\int_0^1 \left(1-\frac{1}{2}\sqrt{u}\right) f_*^2(u)du &= \int_0^1 \left(1-\frac{1}{2}\sqrt{u}\right) f^2(\tanh^{-1}(u))\frac{u^2}{(1-u^2)^2}du\\ &= \int_0^{+\infty} \left(1-\frac{1}{2}\sqrt{\tanh(x)}\right) f^2(x)\frac{\tanh^2(x)}{1-\tanh^2(x)}dx\\ &= \int_0^{+\infty} \left(1-\frac{1}{2}\sqrt{\tanh(x)}\right) \sinh^2(x)f^2(x)dx. \qquad (14)\end{aligned}$$

For the fourth integral of the bound in (11), we proceed similarly and obtain

$$\begin{aligned}\int_0^1 \left(1-\frac{1}{2}\sqrt{u}\right) g_*^2(u)du &= \int_0^1 \left(1-\frac{1}{2}\sqrt{u}\right) g^2(\tanh^{-1}(u))\frac{u^2}{(1-u^2)^2}du\\ &= \int_0^{+\infty} \left(1-\frac{1}{2}\sqrt{\tanh(x)}\right) g^2(x)\frac{\tanh^2(x)}{1-\tanh^2(x)}dx\\ &= \int_0^{+\infty} \left(1-\frac{1}{2}\sqrt{\tanh(x)}\right) \sinh^2(x)g^2(x)dx. \qquad (15)\end{aligned}$$

Now, from (11), (12), (13), (14) and (15), it follows that

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{\tanh(x+y)}dxdy \leq \pi \Bigg[&\left(\int_0^{+\infty} \left(1-\frac{1}{2}\sqrt{\tanh(x)}\right) \cosh^2(x)f^2(x)dx\right)^{1/2}\\ &\times \left(\int_0^{+\infty} \left(1-\frac{1}{2}\sqrt{\tanh(x)}\right) \cosh^2(x)g^2(x)dx\right)^{1/2}\\ &+ \left(\int_0^{+\infty} \left(1-\frac{1}{2}\sqrt{\tanh(x)}\right) \sinh^2(x)f^2(x)dx\right)^{1/2}\\ &\times \left(\int_0^{+\infty} \left(1-\frac{1}{2}\sqrt{\tanh(x)}\right) \sinh^2(x)g^2(x)dx\right)^{1/2}\Bigg].\end{aligned}$$

This completes the proof of the theorem. □

To the best of the author's knowledge, the representation of the upper bound in Theorem 4.1 as a sum of two principal terms is new within the context of Hilbert-type integral inequalities. It arises naturally from the sharp analysis, namely from the transformed integral in Theorem 2.1 together with the application of (1). We also mention that (1) is strict, implying that the inequality in Theorem 4.1 is also strict.

A series analogue of Theorem 4.1 is given in the proposition below. It may also be viewed as a refined version of Proposition 3.1.

Proposition 4.1. *Let $(a_m)_{m\in\mathbb{N}\backslash\{0\}}$ and $(b_n)_{n\in\mathbb{N}\backslash\{0\}}$ be two sequences of non-negative real numbers. Then,*

$$\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}\frac{a_m b_n}{\tanh(m+n)} \leq \pi\left[\left(\sum_{m=1}^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(m-1)}\right)\cosh^2(m)a_m^2\right)^{1/2}\left(\sum_{m=1}^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(m-1)}\right)\cosh^2(m)b_m^2\right)^{1/2}\right.$$
$$\left.+\left(\sum_{m=1}^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(m-1)}\right)\sinh^2(m)a_m^2\right)^{1/2}\left(\sum_{m=1}^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(m-1)}\right)\sinh^2(m)b_m^2\right)^{1/2}\right],$$

provided that the four series on the right-hand side converge.

Proof. For any $m, n \in \mathbb{N}\backslash\{0\}$, we set $f_\triangleright(x) = a_m$ for $x \in [m-1, m)$ and $g_\triangleright(x) = b_n$ for $x \in [n-1, n)$. Note that $1/\tanh(x+y)$ is a non-increasing function with respect to x and y. Therefore, we have

$$\frac{a_m b_n}{\tanh(m+n)} \leq \int_{n-1}^{n}\int_{m-1}^{m}\frac{f_\triangleright(x)g_\triangleright(y)}{\tanh(x+y)}dxdy.$$

Applying Theorem 4.1 to the functions $f_\triangleright$ and $g_\triangleright$, and using the fact that the functions $\tanh(x)$, $\sinh(x)$, and $\cosh(x)$ are non-decreasing on $(0, +\infty)$, we obtain

$$\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}\frac{a_m b_n}{\tanh(m+n)} \leq \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}\int_{n-1}^{n}\int_{m-1}^{m}\frac{f_\triangleright(x)g_\triangleright(y)}{\tanh(x+y)}dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty}\frac{f_\triangleright(x)g_\triangleright(y)}{\tanh(x+y)}dxdy$$
$$\leq \pi\left[\left(\int_0^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(x)}\right)\cosh^2(x)f^2(x)dx\right)^{1/2}\left(\int_0^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(x)}\right)\cosh^2(x)g^2(x)dx\right)^{1/2}\right.$$
$$\left.+\left(\int_0^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(x)}\right)\sinh^2(x)f^2(x)dx\right)^{1/2}\left(\int_0^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(x)}\right)\sinh^2(x)g^2(x)dx\right)^{1/2}\right]$$
$$= \pi\left[\left(\sum_{m=1}^{+\infty}\int_{m-1}^{m}\left(1-\frac{1}{2}\sqrt{\tanh(x)}\right)\cosh^2(x)a_m^2dx\right)^{1/2}\left(\sum_{n=1}^{+\infty}\int_{m-1}^{m}\left(1-\frac{1}{2}\sqrt{\tanh(x)}\right)\cosh^2(x)b_m^2dx\right)^{1/2}\right.$$
$$\left.+\left(\sum_{m=1}^{+\infty}\int_{m-1}^{m}\left(1-\frac{1}{2}\sqrt{\tanh(x)}\right)\sinh^2(x)a_m^2dx\right)^{1/2}\left(\sum_{m=1}^{+\infty}\int_{m-1}^{m}\left(1-\frac{1}{2}\sqrt{\tanh(x)}\right)\sinh^2(x)b_m^2dx\right)^{1/2}\right]$$
$$\leq \pi\left[\left(\sum_{m=1}^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(m-1)}\right)\cosh^2(m)a_m^2\right)^{1/2}\left(\sum_{m=1}^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(m-1)}\right)\cosh^2(m)b_m^2\right)^{1/2}\right.$$
$$\left.+\left(\sum_{m=1}^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(m-1)}\right)\sinh^2(m)a_m^2\right)^{1/2}\left(\sum_{m=1}^{+\infty}\left(1-\frac{1}{2}\sqrt{\tanh(m-1)}\right)\sinh^2(m)b_m^2\right)^{1/2}\right].$$

This concludes the proof. □

To the best of the author's knowledge, Proposition 4.1 is a new series result in the literature, derived with only minimal use of intermediary inequalities.

5. Conclusion

In this article, two new hyperbolic tangent variants of Hilbert integral inequalities have been established, together with their series analogues. These results enrich the existing theory by providing new structural forms and potential applications. Future work may include extending them to weighted or multidimensional settings and exploring possible connections with operator theory and related functional inequalities.

Acknowledgment

The author would like to thank the two reviewers for their constructive comments.

References

[1] Q. Chen, B. C. Yang, A survey on the study of Hilbert-type inequalities, *J. Inequal. Appl.* **2015** (2015) #302.
[2] C. Chesneau, Study of two three-parameter non-homogeneous variants of the Hilbert integral inequality, *Lobachevskii J. Math.* **45** (2024) 4931–4953.
[3] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1934.
[4] Z. Huang, B. C. Yang, A new Hilbert-type integral inequality with the combination kernel, *Int. Math. Forum* **6** (2011) 1363–1369.
[5] Y. Li, J. Wu, B. He, A new Hilbert-type integral inequality and the equivalent form, *Int. J. Math. Math. Sci.* **8** (2006) #045378.
[6] B. Sun, Best generalization of a Hilbert type inequality, *J. Inequal. Pure Appl. Math.* **7** (2006) #113.
[7] B. C. Yang, On Hilbert's integral inequality, *J. Math. Anal. Appl.* **220** (1998) 778–785.
[8] B. C. Yang, A basic Hilbert-type integral inequality with the homogeneous kernel of -1-degree and extensions, *J. Guangdong Educ. Inst.* **28** (2008) 5–10.
[9] B. C. Yang, *Hilbert-Type Integral Inequalities*, Bentham Science, Sharjah, 2009.
[10] B. C. Yang, *The Norm of Operator and Hilbert-Type Inequalities*, Science Press, Beijing, 2009.