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Research Article

Characterizing weak fraction-dense and other coherent quantales by localization

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Abstract

In a recent work, Bhattacharjee and Dube [*Algebra Universalis* **83** (2022) #6] introduced the notion of fraction-dense frames as a generalization of fraction-dense f-rings, previously defined by Hager and Martinez [*Canad. J. Math.* **45** (1993) 977–996]. In this paper, the concept of fraction-dense frames is extended to the context of quantale theory. For this purpose, two notions are defined: weak fraction-dense quantales and fraction-dense quantales. (In the case of semiprime coherent quantales, these two notions coincide.) Weak fraction-dense coherent quantales are characterized, along with other classes of coherent quantales — such as complemented quantales and PP-quantales — by using localization theory.

Keywords: coherent quantales; localization; weak fraction-dense quantales; fraction-dense quantales; PP-quantales; complemented quantales.

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1. Introduction

The concept of localization of (commutative) multiplicative lattices was introduced by Anderson [1] as an abstraction of the localization of commutative rings. Anderson defined the main concepts and proved the fundamental results of the localization theory. This theory was later developed by numerous authors in connection with the abstract treatment of various topics in commutative algebra (see, e.g., [18,21]) or in relation to other classes of rings. For instance, starting from the work of Hager and Martinez [16], Bhattacharjee and Dube [3] studied the notion of fraction-dense algebraic frames. They also presented a frame-theoretic form of the total ring of quotients associated with an f-ring, which they used to characterize fraction-dense algebraic frames and other important classes of algebraic frames.

In [15], flat quantale morphisms were studied in relation to the localization of coherent quantales (which coincide with Anderson's localization theory). As a continuation of [15], the present paper extends the concept of fraction-dense frames to the context of quantale theory. For this purpose, two notions are defined: weak fraction-dense quantales and fraction-dense quantales. (In the case of semiprime coherent quantales, these two notions coincide.) Weak fraction-dense coherent quantales are characterized, along with other classes of coherent quantales — such as complemented quantales and PP-quantales — by using localization theory.

The rest of the paper is organized as follows. Section 2 presents certain definitions and some basic results on quantales and frames. Section 3 is concerned with some properties of quantale morphisms and quantale nuclei. Section 4 is devoted to the localization of coherent quantales. It includes some results from [1], with proofs based on Lemma 4.1, along with a few new results.

Section 5 deals with weak fraction-dense and fraction-dense quantales, two notions that generalize the concept of a fraction-dense quantale introduced in [3]. It is shown that a coherent quantale A is weak fraction-dense if and only if the frame $R(A)$ of radical elements of A is fraction-dense. This section also includes the construction of the quantale $Q(A)$ through which a quantale A is associated with the localization quantale $Q(A) = A_{S(A)}$, where $S(A)$ is the multiplicatively closed subset of dense compact elements of A. If A is a coherent frame, then $Q(A)$ coincides with the frame qA constructed in [3]. It is also shown that a semiprime coherent quantale is fraction-dense if and only if $Q(A)$ is strongly projectable.

Section 6 introduces complemented quantales as a natural generalization of the notion of complemented frames. This section characterizes the complemented coherent quantales and the coherent PP-quantales in terms of the localization quantale $Q(A)$.

Section 7 investigates functoriality of the assignment $A \mapsto Q(A)$. In general, this assignment is not functorial: not every coherent quantale morphism $A \to B$ gives rise to a coherent quantale morphism $Q(A) \to Q(B)$. This section's main result shows that such functorial behaviour occurs for those quantale morphisms that preserve dense compact elements.

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2. Preliminaries

This section presents some basic notions and results from quantale theory (see [7,23,25]) and frame theory (see [19,24]).

Recall from [25] that a *quantale* is defined as a structure $(A, \bigvee, \bigwedge, \cdot, 0, 1)$, whenever $(A, \bigvee, \bigwedge)$ is a complete lattice, $(A, \cdot)$ a semigroup and, for all $T \subseteq A$ and $x \in A$, the following infinite distributive law holds:

$$x \cdot \left(\bigvee T\right) = \bigvee\{x \cdot t : t \in T\}; \left(\bigvee T\right) \cdot x = \bigvee\{t \cdot x : t \in T\}.$$

Usually, we write xy instead of $x \cdot y$ and shortly denote by A the quantale $(A, \bigvee, \bigwedge, \cdot, 0, 1)$. The quantale A is said to be *integral* if $(A, \cdot, 1)$ is monoid, and *commutative* if the multiplication "$\cdot$" is a commutative operation. If the operations "$\cdot$" and "$\wedge$" coincide, then we obtain the notion of *frame* (see [19,24]). Throughout this paper, by the word "quantale" we mean an integral and commutative quantale.

Let A be a quantale. Then, $d \in A$ is *compact* if for any $T \subseteq A$ with $d \leq \bigvee T$, there exists a finite subset T_0 of T such that $d \leq \bigvee T_0$. We denote by $K(A)$ the set of all compact elements of A. Recall from [25] that for each ring R, $K(Id(R))$ is the set of finitely generated ideals in R.

A quantale in which any element is a join of compact elements is known as an *algebraic quantale*. For every element x of an algebraic quantale A, we have $x = \bigvee\{d \in K(A) : d \leq x\}$. An algebraic quantale A is said to be *coherent* if $1 \in K(A)$ and the set $K(A)$ is closed under the operation "$\cdot$". If R is a ring, then $Id(R)$ is a coherent quantale. Note that in several papers, coherent quantales are referred to as C-lattices (see, e.g., [18,21]).

Lemma 2.1 (see [4]). *If A is a quantale and $x, y, z \in A$, then the following hold:*

- **(i).** *If $x \vee y = 1$, then $xy = x \wedge y$.*
- **(ii).** *If $x \vee y = 1$, then $x^n \vee y^n = 1$ for every integer $n \geq 1$.*
- **(iii).** *If $x \vee y = 1$ and $x \vee z = 1$, then $x \vee (yz) = x \vee (y \wedge z) = 1$.*

According to [25], the following operations are defined on any quantale A:

- the *implication* or *residuation* "$\rightarrow$": for all $x, y \in A$,

$$x \rightarrow y = \bigvee\{a \in A : xa \leq y\};$$

- the *annihilator or polar operation* "$\perp$": for any $x \in A$,

$$x^\perp = x^{\perp_A} = x \rightarrow 0 = \bigvee\{a \in A : xa = 0\}.$$

According to [25], the following residuation rule holds: for all $x, y, z \in A$, $x \leq y \rightarrow z$ if and only if $xy \leq z$. Therefore, $(A, \vee, \wedge, \cdot, \rightarrow, 0, 1)$ has a structure of a (commutative) residuated lattice. For all $x, y \in A$, we have $x \leq x^{\perp_A \perp_A}$ and $(x \vee y)^{\perp_A} = x^{\perp_A} \vee y^{\perp_A}$. An element x of A is said to be dense if $x^{\perp_A} = 0$. A simple computation shows that $x \vee x^{\perp_A}$ is a dense element.

In what follows, some basic properties of residuated lattices will be used without explicit mention (cf. [10]). In particular, the set $Pol(A) = \{x^{\perp_A} : x \in A\}$ of polars has a canonical structure of a complete Boolean algebra.

Following [25], we say that an element $p < 1$ of an arbitrary quantale A is *m-prime* if for all $x, y \in A$, $xy \leq p$ implies $x \leq p$ or $y \leq p$. In an algebraic quantale A, the following characterization of m-prime elements holds: an element $p < 1$ is m-prime if and only if for all $e, f \in K(A)$, $ef \leq p$ implies $e \leq p$ or $f \leq p$. We say that an element $m < 1$ is *maximal* if for each $a \in A$ with $m \leq a < 1$, it holds that $m = a$. Following the standard notations [25], $Spec(A)$ is the set of m-prime elements of A and $Max(A)$ is the set of maximal elements of A. Keeping the terminology in ring theory [2], $Spec(A)$ is called the m-prime spectrum of A and $Max(A)$ is called the maximal spectrum of A. If $1 \in K(A)$, then $Max(A) \subseteq Spec(A)$ and for each $x < 1$ there exists $m \in Max(A)$ such that $x \leq m$.

Let x be an arbitrary element of a quantale A and $q \in Spec(A)$. We say that q is *minimal* over x if for any $r \in Spec(A)$, $x \leq r \leq q$ implies $q = r$. From Zorn's axiom, it follows that for any $x \in A$ there exists $q \in Spec(A)$ which is minimal over x. If $q \in Spec(A)$ is minimal over 0, then we say that q is a *minimal m-prime* element of A. The *minimal m-prime spectrum* of A is defined as the set $Min(A)$ of minimal m-prime elements of A.

Let us denote by $B(A)$ the set of complemented elements of a quantale A. Then $B(A)$ is a Boolean algebra (see [10]). If $x \in B(A)$, then $x^{\perp_A}$ is the complement of x in $B(A)$. It is well-known that for any $a \in A$, $a \in B(A)$ if and only if $a \vee a^{\perp_A} = 1$. In accordance with Lemma 15 of [5], we have $B(A) \subseteq K(A)$.

According to [5], a quantale A is *hyperarchimedean* if for each $c \in K(A)$ there exists an integer $n \geq 1$ such that $c^n \in B(A)$. Recall from [5] that a coherent quantale A is hyperarchimedean if and only if $Spec(A) = Max(A)$, that is, if and only if $Max(A) = Min(A)$.

Let x be an element of a quantale A. In accordance with [25], the *radical* $\rho(x)$ of x is defined by

$$\rho(x) = \rho_A(x) = \bigwedge\{q \in Spec(A) : x \leq q\}.$$

If $x = \rho(x)$, then we say that x is a *radical element* of A. Denote by $R(A)$ the set of radical elements of A. If $\rho(0) = 0$, then we say that A is a *semiprime* quantale. Recall from [25] that for all $x, y \in A$ the following properties hold:

(2.1) $x \leq \rho(x)$;

(2.2) $\rho(xy) = \rho(x) \wedge \rho(y)$;

(2.3) $\rho(\rho(x)) = \rho(x)$;

(2.4) $\rho(x \vee y) = \rho(\rho(x) \vee \rho(y))$;

(2.5) $\rho(x) = 1$ if and only if $x = 1$;

(2.6) $\rho(x) \vee \rho(y) = 1$ if and only if $x \vee y = 1$;

(2.7) for any integer $n \geq 1$, $\rho(x^n) = \rho(x)$;

(2.8) $\rho(\bigvee T) = \rho(\bigvee\{\rho(t) : t \in T\})$ for any $T \subseteq A$.

Since the set $R(A)$ of radical elements of A is closed under arbitrary meets, it forms a complete lattice. For any $T \subseteq R(A)$ we set $\dot{\bigvee} T = \rho(\bigvee T)$. Using the above properties of the map $\rho : A \to A$, it is easy to prove that $(R(A), \dot{\bigvee}, \wedge, 0, 1)$ is a frame (see [25]). By virtue of Lemma 8 of [5], for any coherent quantale A, we have $K(R(A)) = \rho(K(A))$, and $R(A)$ is a coherent frame.

Lemma 2.2 (see [13]). *If A is a coherent quantale and $a \in A$, then $a \to \rho(0) = \rho(a) \to \rho(0)$.*

The following lemma represents the quantale abstraction of a well-known result in commutative ring theory (see Proposition 1.4 of [2]).

Lemma 2.3 (see [1, 20]). *Let A be a coherent quantale. For any $x \in A$, the following properties hold:*

(i). $\rho(x) = \bigvee\{d \in K(A) : d^n \leq x$ *for some integer* $n \geq 1\}$.

(ii). *For any compact element d of A, $d \leq \rho(x)$ if and only if $d^n \leq x$ for some integer $n \geq 1$.*

(iii). *A is semiprime if and only if for all $d \in K(A)$ and $n \geq 1$, $d^n = 0$ implies $d = 0$.*

Lemma 2.4 (see [13]). *If A is a semiprime coherent quantale, then for all $x, y \in A$, $xy = 0$ if and only if $x \wedge y = 0$.*

3. Quantale morphisms and nuclei

Suppose that A and B are two quantales. According to [25], a function $u : A \to B$ is said to be a *quantale morphism* if it preserves the arbitrary joins and the multiplicative operation "$\cdot$". Therefore, any quantale morphism preserves the bottom element 0. A quantale morphism $u : A \to B$ is *integral* if it preserves the top element 1. An integral quantale morphism $u : A \to B$ is said to be *coherent* if $u(c) \in K(B)$ for any $c \in K(A)$.

Let $u : A \to B$ be a quantale morphism. Consider the function $u_\star : B \to A$, defined by $u_\star(y) = \bigvee\{x \in A : u(x) \leq y\}$, for any $y \in B$. The pair $(u, u_\star)$ verifies the following adjointness property: for all $x \in A$ and $y \in B$, $u(x) \leq y$ if and only if $x \leq u_*(y)$. Thus, u_* is a right adjoint of u.

Lemma 3.1. *Let $u : A \to B$ be a quantale morphism. Then the following hold:*

(i). $a \leq u_*(u(a))$ *for any* $a \in A$.

(ii). $u(u_*(b)) \leq b$ *for any* $b \in B$.

The following lemma is well-known in quantale theory.

Lemma 3.2. *Let A and B be two quantales. Let $u : A \to B$ be a surjective quantale morphism. If $u(K(A)) \subseteq K(B)$, then $u(K(A)) = K(B)$.*

Lemma 3.3 (see [14]). *If $u : A \to B$ is a coherent quantale morphism, then there exists a unique coherent frame morphism $u^\rho : R(A) \to R(B)$ such that $\rho_B \circ u = u^\rho \circ \rho_A$.*

We note that $u^\rho(a) = \rho_B(u(a))$ for any $a \in Rad(A)$.

Lemma 3.4 (see [15]). *Let A and B be two coherent quantales. Let $v : K(A) \to K(B)$ be a map that preserves multiplication, finite joins and top element. Then there exists a unique coherent quantale morphism $v^* : A \to B$ such that $v^*(a) = v(a)$ for any $a \in K(A)$.*

Let A be a coherent quantale and $a \in A$. An easy computation shows that the set $[a)_A = \{x \in A : a \leq x\}$ is closed under arbitrary joins of A. Given two elements $x, y \in [a)_A$, we set $x \cdot_a y = xy \vee a$. Therefore, $[a)_A$ is closed under the operation "$\cdot_a$". It is straightforward to verify that $([a)_A, \bigvee, \wedge, \cdot_a, a, 1)$ is a quantale. Consider the map $u_a^A : A \to [a)_A$ defined by $u_a^A(x) = a \vee x$ for every $x \in A$.

Lemma 3.5 (see [5]). *For any element a of a coherent quantale A, the following properties hold:*

(i). *$[a)_A$ is a coherent quantale.*

(ii). *$u_a^A : A \to [a)_A$ is a coherent quantale morphism.*

(iii). *$K([a)_A) = \{a \vee c : c \in K(A)\}$.*

(iv). *$\rho_{[a)_A}(x) = \rho_A(x)$, for any $x \in [a)_A$.*

(v). *$Spec([a)_A) = \{p \in Spec(A) : a \leq p\}$.*

Lemma 3.6. *Let A be a coherent quantale and $a \in A$. Then $(u_a^A)_*(x) = x$ for any $x \in [a)_A$.*

Definition 3.1. *A quantale morphism $u : A \to B$ is a flat quantale morphism if $u(c \to a) = u(c) \to u(a)$ for all $c \in K(A)$ and $a \in A$.*

Lemma 3.7. *If $u : A \to B$ is a quantale morphism, then the following statements are equivalent:*

(i). *u is a flat quantale morphism.*

(ii). *For all $c \in K(A)$ and $a \in A$, $u(c) \to u(a) \leq u(c \to a)$.*

Definition 3.2 (see [17, 25]). *A function $j : A \to A$ is said to be a nucleus of the quantale A if the following conditions are fulfilled:*

(i). *j is a closure operator on A;*

(ii). *$j(a)j(b) \leq j(ab)$ for all $a, b \in A$.*

For any nucleus j of A, we denote $A_j = \{a \in A : j(a) = a\}$.

Lemma 3.8 (see [17, 25]). *If $j : A \to A$ is a nucleus, then, for all $a, b \in A$, the following hold:*

(i). *$j(ab) = j(aj(b)) = j(j(a)b) = j(j(a)j(b))$.*

(ii). *$j(a) \to j(b) = a \to j(b)$.*

(iii). *A_j is closed under "$\wedge$" and "$\to$".*

For all $a, b \in A$ and $S \subseteq A$, we set $a \cdot_j b = j(ab)$ and $\bigvee_j S = j(\bigvee S)$.

Lemma 3.9 (see [17, 25]). *$(A_j, \bigvee_j, \wedge, \cdot_j, 0, 1)$ is a quantale.*

Remark 3.1. *Assume that $j : A \to A$ is a nucleus. Then we can consider the map $j : A \to A_j$. It is obvious that $j : A \to A_j$ is a quantale morphism.*

Lemma 3.10. *Assume that A is a coherent quantale. If $j : A \to A$ is a nucleus, then the following statements are equivalent:*

(i). *$j : A \to A_j$ is a flat quantale morphism.*

(ii). *For all $c \in K(A)$ and $a \in A$, $j(c) \to j(a) \leq j(c \to a)$.*

(iii). *For all $c \in K(A)$ and $a \in A$, $c \to j(a) \leq j(c \to a)$.*

(iv). *For all $c, d \in K(A)$ and $a \in A$, $cd \leq j(a)$ implies $d \leq j(c \to a)$.*

Proof. By Lemma 3.7, we have $(i) \Leftrightarrow (ii)$. Also, by Lemma 3.8(ii), we have $(ii) \Leftrightarrow (iii)$. Finally, it is obvious that $(iii) \Leftrightarrow (iv)$. □

Lemma 3.11. *Let $u : A_1 \to A_2$ be a surjective coherent quantale morphism. If A_1 is hyperarchimedean, then A_2 is hyperarchimedean.*

Proof. By Lemma 3.2, we have $K(A_2) = u(K(A_1))$. In order to show that A_2 is hyperarchimedean, consider an element $d \in K(A_2)$. Hence, $d = u(c)$ for some $c \in K(A_1)$. Since A_1 is hyperarchimedean, one can find an integer $n \geq 1$ such that $c^n \in B(A_1)$. Therefore, $d^n = (u(c))^n = u(c^n) \in B(A_2)$. Consequently, A_2 is hyperarchimedean. □

4. Anderson localization in coherent quantales

Let A be a quantale. A subset S of $K(A)$ is said to be multiplicatively closed if $1 \in K(A)$ and S is closed under multiplication ($x, y \in S$ implies $xy \in S$). Let us fix a coherent quantale A and a multiplicatively closed subset S of A. Following [1, 18, 21], for any $a \in A$, we denote $a_S = \bigvee\{c \in K(A) : cs \leq a \text{ for some } s \in S\}$. According to [1], a_S is called the localization of a at S. Let us set $A_S = \{a_S : a \in A\}$. In this section, we provide proofs of certain results from [1], derived through applications of the following basic lemma.

Lemma 4.1 (Basic Lemma). *For all $a \in A$ and $d \in K(A)$, $d \leq a_S$ if and only if $ds \leq a$ for some $s \in S$.*

Proof. Suppose that $d \leq a_S$. Then, we have $d \leq \bigvee\{c \in K(A) : cs \leq a \text{ for some } s \in S\}$. Since d is a compact element of A, there exist an integer $n \geq 1$, $c_1, \ldots, c_n \in K(A)$ and $s_1, \ldots, s_n \in S$ such that $d \leq \bigvee_{i=1}^n c_i$ and $c_i s_i \leq a$, for any $i = 1, \ldots, n$. We set $s = s_1 \ldots s_n$ and $c = \bigvee_{i=1}^n c_i$. Then $s \in S$, $c \in K(A)$ and the following hold:

$$ds \leq cs = \bigvee_{i=1}^{n} c_i s \leq \bigvee_{i=1}^{n} c_i s_i \leq a.$$

The converse implication follows from the definition of a_S. □

Now, we provide a proof of the following known result (see [1]) using Lemma 4.1.

Lemma 4.2. *For all $a, b \in A$ and $d \in K(A)$, the following statements hold:*

(i). *If $a \leq b$, then $a_S \leq b_S$.*

(ii). $a \leq a_S$.

(iii). $(a_S)_S = a_S$.

(iv). $(a \wedge b)_S = a_S \wedge b_S$.

(v). $a_S = 1$ *if and only if* $s \leq a$ *for some* $s \in S$.

(vi). $a \leq b_S$ *if and only if* $a_S \leq b_S$.

(vii). $d_S = 0_S$ *if and only if* $ds = 0$ *for some* $s \in S$.

Proof. (i). By using the definition of a_S.

(ii). Let c be a compact element of A such that $c \leq a$. Thus, $c1 \leq a$ and $1 \in S$. Hence, using Lemma 4.1, we obtain $c \leq a_S$. Since A is a coherent quantale, it follows that $a \leq a_S$.

(iii). By (ii), we have $a_S \leq (a_S)_S$. Assume that $c \in K(A)$ and $c \leq (a_S)_S$. Hence, $cs \leq a_S$ for some $s \in S$ (cf. Lemma 4.1). Thus, $cs \in K(A)$. So, $cs \leq a_S$ implies $cst \leq a$ for some $t \in S$ (cf. Lemma 4.1). Thus, $st \in S$, and hence, from $cst \leq a$, we obtain $c \leq a_S$. Therefore, $(a_S)_S \leq a_S$. Consequently, we conclude that $(a_S)_S = a_S$.

(iv). Using (i), we obtain $(a \wedge b)_S \leq a_S \wedge b_S$. In order to establish the converse inequality $a_S \wedge b_S \leq (a \wedge b)_S$, consider a compact element c of A such that $c \leq a_S \wedge b_S$. By using Lemma 4.1, we obtain $cs \leq a$ and $ct \leq b$ for some $s, t \in S$. Then, $st \in S$ and $cst \leq a \wedge b$. Using again Lemma 4.1, we obtain $c \leq (a \wedge b)_S$, and hence, $a_S \wedge b_S \leq (a \wedge b)_S$.

(v). According to Lemma 4.1, $a_S = 1$ if and only if $1 \leq a_S$, that is, if and only if $s = 1s \leq a$ for some $s \in S$.

(vi). If $a \leq b_S$ then $a_S \leq (b_S)_S = b_S$ (cf. (i) and (iii)). Conversely, from (i), it follows that $a_S \leq b_S$ implies $a \leq a_S \leq b_S$.

(vii). Firstly, we observe that $0_S \leq d_S$ (by (i)). By hypothesis, d is a compact element of A. Hence, using (vi) and Lemma 4.1, the following equivalences hold: $d_S = 0_S$ if and only if $d_S \leq 0_S$ if and only if $d \leq 0_S$ if and only if $ds = 0$ for some $s \in S$. □

If $p \in Spec(A)$ then $S = \{c \in K(A) : c \not\leq p\}$ is a multiplicatively closed subset of A. In this case, we denote $A_p = A_S$ and $a_p = a_S$ for any $a \in A$. Using Lemma 4.1, one can prove that $x_p = \bigvee\{c \in K(A) : c \to x \not\leq p\}$ (see Remark 1 of [15]), where, x_p is called the localization of x at p. Applying Lemma 4.2(v) to this particular case, one obtains the following equivalence: $a_p = 1$ if and only if $a \not\leq p$.

Now let us consider the map $j_S : A \to A$ defined by $j_S(a) = a_S$ for any $a \in A$.

Proposition 4.1. *The map j_S is a nucleus on the quantale A.*

Proof. In accordance with Lemma 4.2, it is clear that j_S is a closure operator on A. Let a and b be two elements of A. We have to prove the inequality $j_S(a)j_S(b) \leq j_S(ab)$. By the definition of j_S, the following hold:

$$j_S(a) = \bigvee\{c \in K(A) : cs \leq a \text{ for some } s \in S\} \text{ and } j_S(b) = \bigvee\{d \in K(A) : dt \leq b \text{ for some } t \in S\}.$$

By virtue of the distributivity law, we obtain

$$j_S(a)j_S(b) = \bigvee\{cd : c, d \in K(A), cs \leq a, dt \leq b \text{ for some } s, t \in S\}.$$

Therefore, taking into account the fact that S is closed under multiplication, the following inequality holds:

$$j_S(a)j_S(b) \leq \bigvee\{e \in K(A), eu \leq ab \text{ for some } u \in S\}.$$

Hence, $j_S(a)j_S(b) \leq j_S(ab)$, and therefore, the map j_S is a nucleus on the quantale A. □

Remark 4.1. *By virtue of Proposition 4.1 and Lemmas 3.1 and 3.9, A_S is a quantale with respect to the following operations:*

(i). $\bigvee_{j_S}\{(a_i)_S : i \in I\} = (\bigvee_{i\in I} a_i)_S$;

(ii). $a_S \wedge b_S = (a \wedge b)_S$;

(iii). $a_S \cdot_{j_S} b_S = (ab)_S$.

The above equalities show that $j_S : A \to A_S$ is a quantale morphism.

According to [1], A_S is called the localization quantale of A at S. In the rest of this paper, we use the following notations:

$$\bigvee_S\{(a_i)_S : i \in I\} = \bigvee_{j_S}\{(a_i)_S : i \in I\} \text{ and } a_S \cdot_S b_S = a_S \cdot_{j_S} b_S.$$

Let $p \in Spec(A)$, $S = \{c \in K(A) : c \not\leq p\}$ and $A_p = A_S$ (= the localization quantale of A at p). In this case, the quantale morphism $j_S : A \to A_S$ is denoted by $j_p : A \to A_p$. Then, $j_p(a) = a_p$ for any $a \in A$. We also use the following notations:

$$a_p \cdot_p b_p = a_S \cdot_S b_S = (ab)_p \text{ and } \bigvee_p\{(a_i)_p : i \in I\} = \bigvee_S\{(a_i)_S : i \in I\} = (\bigvee_{i\in I} a_i)_p.$$

Corollary 4.1. *The map j_S is a flat quantale morphism.*

Proof. Assume that $c, d \in K(A)$ and $a \in A$. By virtue of Lemma 4.1, the following properties are equivalent:

- $cd \leq j_S(a)$;
- there exists $t \in S$ such that $cdt \leq a$;
- there exists $t \in S$ such that $dt \leq c \to a$;
- $d \leq j_S(c \to a)$.

Therefore, from Lemma 3.10 it follows that j_S is a flat quantale morphism. □

Lemma 4.3. *The map j_S preserves the compact elements.*

Proof. Let c be a compact element of A. We have to show that $c_S \in K(A_S)$. Assume that $T \subseteq A$ and $c_S \leq \bigvee_S\{t_S : t \in T\}$. Then, $c_S \leq (\bigvee T)_S$. By Lemma 4.1 we have $cs \leq a$ for some $s \in S$. Since $cs \in K(A)$ we have $cs \leq \bigvee T_0$ for some finite subset T_0 of T. Using again Lemma 4.1, we obtain $c \leq (\bigvee T_0)_S = \bigvee_S\{t_S : t \in T_0\}$. Hence, c_S is a compact element of A_S. □

Corollary 4.2 (see [1])**.** $K(A_S) = \{c_S : c \in K(A)\}$.

Proof. We apply Lemmas 3.2 and 4.3 to the quantale morphism $j_S : A \to A_S$. □

Corollary 4.3. *A_S is a coherent quantale and $j_S : A \to A_S$ is a coherent quantale morphism.*

Now, we provide a proof of the following known result due to [1] by using Lemma 4.1.

Theorem 4.1. $Spec(A_S) = \{p \in Spec(A) : t \in S \Rightarrow t \not\leq p\}$.

Proof. Let p be an m-prime element of A such that $t \not\leq p$ for any $t \in S$. We want to prove that $p_S \leq p$. Let c be a compact element of A such that $c \leq p_S$. Then, there exists $t \in S$ such that $ct \leq p$ (by Lemma 4.1). Thus, $t \not\leq p$, and hence, $c \leq p$ (because $p \in Spec(A)$). It follows that $p_S \leq p$. Hence, $p_S = p$ (by Lemma 4.2(ii)).

Let c and d be two compact elements of A such that $c_S \cdot_S d_S \leq p$. Then, $(cd)_S \leq p$. Thus, $cd \leq p$ (by Lemma 4.2(ii)). So, $c \leq p$ or $d \leq p$. Thus, $c_S \leq p_S = p$ or $d_S \leq p_S = p$. It follows that $p \in Spec(A_S)$. Hence, we obtain the inclusion

$$\{p \in Spec(A) : t \in S \Rightarrow t \not\leq p\} \subseteq Spec(A_S).$$

Assume now that $p \in Spec(A_S)$. Then, $p = p_S$ (by Lemma 4.2(iii)). It is clear that $p \neq 1$. Let c and d be two compact elements of A such that $cd \leq p$. Then, $c_S \cdot_S d_S = (cd)_S \leq p_S = p$. So, $c_S \leq p$ or $d_S \leq p$ (because $p \in Spec(A_S)$). Hence, $c \leq p$ or $d \leq p$ (by Lemma 4.2(ii)). Consequently, $p \in Spec(A)$. Suppose that there exists $t \in S$ such that $t \leq p$. By using Lemma 4.2(v), we obtain $p_S = 1$, which contradicts the fact that $p = p_S$ is an m-prime element of A_S. So, $t \not\leq p$ for any $t \in S$. Therefore, we have proved the inclusion

$$Spec(A_S) \subseteq \{p \in Spec(A) : t \in S \Rightarrow t \not\leq p\}.$$

In conclusion, we have obtained the equality $Spec(A_S) = \{p \in Spec(A) : t \in S \Rightarrow t \not\leq p\}$. □

From Theorem 4.1, it follows that $Spec(A_S) \subseteq Spec(A)$. Hence, we have the following corollary of this theorem.

Corollary 4.4. *If A_S is semiprime, then A is semiprime.*

Proof. Using the inclusion $Spec(A_S) \subseteq Spec(A)$, we have $\rho_A(0) = \bigwedge Spec(A) \leq \bigwedge A_S = 0$. □

Lemma 4.4. *$(j_S)_*(z) = z$ for any $z \in A_S$.*

Proof. Assume that $z \in A_S$. Then, $z = a_S$ for some $a \in A$. Using Lemma 4.2(vi) and the adjointness property of j_S and $(j_S)_*$, for any $c \in K(A)$, we have $c \leq (j_S)_*(a_S)$ if and only if $j_S(c) \leq a_S$ if and only if $c_S \leq a_S$ if and only if $c \leq a_S$. Therefore, $(j_S)_*(a_S) = a_S$. □

Proposition 4.2. *For any $a \in A$, $\rho_A(a_S) = (\rho_A(a))_S = (\rho_A(a_S))_S$.*

Proof. Assume that $c \in K(A)$. By Lemmas 4.1 and 4.2(ii), the following properties are equivalent:

- $c \leq \rho_A(a_S)$;
- there exists an integer $n \geq 1$ such that $c^n \leq a_S$;
- there exist an integer $n \geq 1$ and $s \in S$ such that $c^n s \leq a$;
- there exist an integer $n \geq 1$ and $s \in S$ such that $c^n s^n \leq a$;
- there exists $s \in S$ such that $cs \leq \rho_A(a)$;
- $c \leq (\rho_A(a))_S$.

By virtue of these equivalences, we have $\rho_A(a_S) = (\rho_A(a))_S$.

In order to show that $(\rho_A(a_S))_S \leq \rho_A(a_S)$, consider a compact element c of A such that $c \leq (\rho_A(a_S))_S$. By Lemma 4.1, there exists $s \in S$ such that $cs \leq \rho_A(a_S)$. Hence, $c^n s^n \leq a_S$ for some integer $n \geq 1$ (cf. Lemma 2.3(ii)). Also, Lemma 4.1 gives $c^n s^n t \leq a$ for some $t \in S$. Since $s^n t \in S$, by Lemma 4.1, we have $c^n \leq a_S$. Hence, using Lemma 2.3(ii), we obtain $c \leq \rho_A(a_S)$, and therefore, $(\rho_A(a_S))_S \leq \rho_A(a_S)$. By Lemma 4.2(ii), we have $\rho_A(a_S) \leq (\rho_A(a_S))_S$, and consequently, $\rho_A(a_S) = (\rho_A(a_S))_S$. □

Proposition 4.3. *For any $a \in A$, $\rho_{A_S}(a_S) = (\rho_A(a))_S$.*

Proof. First, we note that $\rho_{A_S}(a_S)$ and $(\rho_A(a))_S$ are elements of A_S. On the other hand, any compact element of A_S has the form c_S for some $c \in K(A)$.

Let c be a compact element of A. By Lemmas 2.3(ii) and 4.1, the following properties are equivalent:

- $c_S \leq \rho_{A_S}(a_S)$;
- $(c^n)_S \leq a_S$ for some integer $n \geq 1$;
- $c^n \leq a_S$ for some integer $n \geq 1$;
- $c^n s \leq a$ for some integer $n \geq 1$ and $s \in S$;
- $c^n t^n \leq a$ for some integer $n \geq 1$ and $t \in S$;
- $ct \leq \rho_A(a)$ for some $t \in S$;
- $c \leq (\rho_A(a))_S$;
- $c_S \leq (\rho_A(a))_S$. □

5. Weak fraction-dense quantales

In this section, we define and study the weak fraction-dense and fraction-dense quantales, two notions that constitute generalizations of the fraction-dense frames. In order to define the weak fraction-dense quantales, we need the notion of the weak annihilator. The weak annihilators in quantales were defined in [14] as an abstraction of the weak annihilators in commutative algebra (see [22]). Let A be a quantale. Recall from [14] that the weak annihilator of an element $a \in A$ is defined by $a^{\perp_w} = a^{\perp_{A,w}} = a \to \rho(0)$. The following lemma summarizes some basic properties of weak annihilators.

Lemma 5.1. *Let A be a coherent quantale. For all elements $a, b \in A$ the following hold:*

(i). *If $a \leq b$ then $b^{\perp_w} \leq a^{\perp_w}$.*

(ii). $a \leq a^{\perp_w \perp_w}$. $a^{\perp_w \perp_w \perp_w} = a^{\perp_w}$.

(iii). $(\rho(a))^{\perp_w} = a^{\perp_w}$.

(iv). *If $a \in R(A)$ then $a^{\perp_{R(A)}} = a^{\perp_w}$.*

(v). $a^{\perp_w} = (\rho(a))^{\perp_{R(A)}}$.

(vi). $(a \vee a^{\perp_w})^{\perp_w} = \rho(0)$.

Proof. For the proofs of (i)–(iv), see [14]. Also, (v) follows from (iii) and (iv). In what follows, we prove (vi). We note that $(a \vee a^{\perp_w})^{\perp_w} = a^{\perp_w} \wedge a^{\perp_w \perp_w}$. Hence, we need to show that $a^{\perp_w} \wedge a^{\perp_w \perp_w} = \rho(0)$. Let c be a compact element of A such that $c \leq a^{\perp_w} \wedge a^{\perp_w \perp_w}$. So, $c \leq a \to \rho(0)$ and $c \leq (a \to \rho(0)) \to \rho(0)$. This last inequality implies $c(a \to \rho(0)) \leq \rho(0)$, and therefore $c \leq a \to \rho(0) \leq c \to \rho(0)$. Thus, $c^2 \leq \rho(0)$, and hence, $c^{2n} = 0$ for some integer $n \geq 1$ (cf. Lemma 2.3(ii)). Another application of the mentioned lemma gives $c \leq \rho(0)$. It follows that $(a \vee a^{\perp_w})^{\perp_w} \leq \rho(0)$. The converse inequality is obvious. Now, the desired equality $(a \vee a^{\perp_w})^{\perp_w} = \rho(0)$ follows. □

The following proposition highlights how the operation $(\cdot)^{\perp_w}$ behaves with respect to localization.

Proposition 5.1. *Let S be a multiplicatively closed subset of a coherent quantale A. For any $a \in A$ we have $(a^{\perp_w})_S = (a_S)^{\perp_w}$.*

Proof. Recall from Corollary 4.2 that any compact element of A_S has the form c_S, where $c \in K(A)$. By Lemmas 2.3(ii), 4.1 and 4.2(vi), for any $c \in K(A)$, the following equivalences hold:

$c_S \leq (a^{\perp_w})_S$ if and only if $c \leq (a^{\perp_w})_S$,
if and only if $cs \leq a^{\perp_w}$ for some $s \in S$,
if and only if $cs \leq a \to \rho(0)$ for some $s \in S$,
if and only if $csa \leq \rho(0)$ for some $s \in S$,
if and only if $c^n s^n a^n = 0$ for some integer $n \geq 1$ and $s \in S$,
if and only if $c^n t a^n = 0$ for some integer $n \geq 1$ and $t \in S$,
if and only if $(c^n a^n)_S = 0_S$ for some integer $n \geq 1$,
if and only if $(c^n)_S \cdot_S (a^n)_S = 0_S$ for some integer $n \geq 1$,
if and only if $c_S \cdot_S a_S \leq \rho_S(0_S)$,
if and only if $c_S \leq a_S \to \rho_S(0_S)$,
if and only if $c_S \leq (a_S)^{\perp_w}$.

By Corollary 4.3, A_S is a coherent quantale. Therefore, by using the previous equivalences, we have $(a^{\perp_w})_S = (a_S)^{\perp_w}$. □

Corollary 5.1. *Let S be a multiplicatively closed subset of a semiprime coherent quantale A. Then, for any $a \in A$,*

$$(a^{\perp_A})_S = (a_S)^{\perp_{A_S}}.$$

The definitions given below generalize, within the framework of quantale theory, the notion of a fraction-dense quantale introduced in Definition 3.4 of [3].

Definition 5.1. *A quantale A is said to be fraction-dense if for any $a \in A$ there exists $c \in K(A)$ such that $a^{\perp_A} = c^{\perp_A}$.*

Definition 5.2. *A quantale A is said to be weak fraction-dense if for any $a \in A$ there exists $c \in K(A)$ such that $a^{\perp_w} = c^{\perp_w}$.*

We remark that any fraction-dense quantale is also a weak fraction-dense. If A is semiprime, then A is a weak fraction-dense quantale if and only if A is fraction-dense. In particular, any weak fraction-dense coherent frame is fraction-dense.

The following result establishes a strong connection between weak fraction-dense coherent quantales and fraction-dense coherent frames.

Theorem 5.1. *If A is a coherent quantale, then the following statements are equivalent:*

(i). *A is a weak fraction-dense quantale.*

(ii). *$R(A)$ is a fraction-dense frame.*

Proof. **(i) ⇒ (ii).** Assume that A is a weak fraction-dense quantale. Let a be an element of $R(A)$. Then, $a = \rho(x)$ for some $x \in A$. Since A is weak fraction-dense, there exists $c \in K(A)$ such that $x^{\perp_w} = c^{\perp_w}$. According to Lemma 8 of [5], $\rho(c)$ is a compact element of $R(A)$. Taking into account the properties (iii) and (iv) of Lemma 5.1, we have the following equivalences:

$$a^{\perp_{R(A)}} = (\rho(x))^{\perp_{R(A)}} = (\rho(x))^{\perp_w} = x^{\perp_w} = (\rho(c))^{\perp_w} = (\rho(c))^{\perp_{R(A)}}.$$

Therefore, $R(A)$ is a fraction-dense frame.

(ii) ⇒ (i). Suppose that $x \in A$. Then, $\rho(x) \in R(A)$. Since $R(A)$ is a fraction-dense frame, there exists $e \in K(R(A))$ such that $(\rho(x))^{\perp_{R(A)}} = e^{\perp_{R(A)}}$. In accordance with Lemma 8 of [5], we have $e = \rho(c)$ for some $c \in K(A)$. Therefore, $(\rho(x))^{\perp_{R(A)}} = (\rho(c))^{\perp_{R(A)}}$. Applying again the properties (iii) and (iv) of Lemma 5.1, we obtain the following equivalences:

$$x^{\perp_w} = (\rho(x))^{\perp_w} = (\rho(x))^{\perp_{R(A)}} = (\rho(c))^{\perp_{R(A)}} = (\rho(c))^{\perp_w} = c^{\perp_w}.$$

Hence, A is weak fraction-dense. □

Corollary 5.2. *If A is a semiprime coherent quantale, then the following statements are equivalent:*

(i). *A is a fraction-dense quantale.*

(ii). *$R(A)$ is a fraction-dense frame.*

Lemma 5.2. *If A is a coherent quantale, then the following statements are equivalent:*

(i). *For any $a \in A$, there exists $c \in K(A)$ such that $a^{\perp_w} = c^{\perp_w \perp_w}$.*

(ii). *For any $x \in R(A)$, there exists $y \in K(R(A))$ such that $x^{\perp_{R(A)}} = y^{\perp_{R(A)} \perp_{R(A)}}$.*

Proof. **(i) ⇒ (ii).** Assume that $x \in R(A)$. Then, $x = \rho(a)$ for some $a \in A$. By hypothesis (i), there exists $c \in K(A)$ such that $a^{\perp_w} = c^{\perp_w \perp_w}$. We note that $\rho(c) \in K(A)$. By virtue of the properties (iii) and (iv) of Lemma 5.1, we obtain the following equalities:

$$x^{\perp_{R(A)}} = (\rho(a))^{\perp_{R(A)}} = a^{\perp_w} = c^{\perp_w \perp_w} = ((\rho(c))^{\perp_{R(A)}})^{\perp_w} = (\rho(c))^{\perp_{R(A)} \perp_{R(A)}}.$$

(ii) ⇒ (i). By using the arguments similar to the ones provided in the proof of (i) ⇒ (ii), we obtain the required conclusion. □

Proposition 5.2. *If A is a coherent quantale, then the following statements are equivalent:*

(i). *A is a weak fraction-dense quantale.*

(ii). *For any $a \in A$, there exists $c \in K(A)$ such that $a^{\perp_w} = c^{\perp_w \perp_w}$.*

Proof. By Proposition 3.6 of [3], Theorem 5.1 and Lemma 5.2, the following properties are equivalent:

- A is a weak fraction-dense quantale.
- $R(A)$ is a fraction-dense frame.
- For any $x \in R(A)$ there exists $y \in K(R(A))$ such that $x^{\perp_{R(A)}} = y^{\perp_{R(A)}\perp_{R(A)}}$.
- For any $a \in A$ there exists $c \in K(A)$ such that $a^{\perp_w} = c^{\perp_w \perp_w}$. □

It follows that a semiprime coherent quantale A is fraction-dense if and only if for any $a \in A$ there exists $c \in K(A)$ such that $a^{\perp_A} = c^{\perp_A \perp_A}$.

Theorem 5.2. *If A is a coherent quantale, then the following statements are equivalent:*

(i). *The quantale A is a weak fraction-dense quantale.*

(ii). *The topological space $Min(A)$ is a Boolean space.*

Proof. We know from [5] that the topological spaces $Min(A)$ and $Min(R(A))$ are homeomorphic. Therefore, by Corollary 3.7 of [3] and Theorem 5.1, the following properties are equivalent:

- A is a weak fraction-dense quantale.
- $R(A)$ is a fraction-dense frame.
- $Min(R(A))$ is a Boolean space.
- $Min(A)$ is a Boolean space. □

The proofs of the two previous results highlight the way in which one can use Theorem 5.1 in order to translate some properties from fraction-dense coherent frames to weak fraction-dense coherent quantales.

Let L be a coherent frame and $S(L) = \{c \in K(L) : c^{\perp_L} = 0\}$. Then, $S(L)$ is a multiplicatively closed subset of L, and hence, one can consider the localization frame of L at $S(L)$: $Q(L) = L_{S(L)}$. We remark here that $Q(L)$ is exactly the construction qL from Section 3 of [3]. Now, we extend this concept to the quantale theory.

Let us fix a coherent quantale A. For the rest of this section, we set

$$S = S(A) = \{c \in K(A) : c^{\perp_A} = 0\}. \quad (1)$$

Lemma 5.3. *The set S defined via (1) is a multiplicatively closed subset of the coherent quantale A.*

Proof. First, we remark that $1 \in S$ (because $1^{\perp_A} = 0$). Assume that $c, d \in K(A)$ such that $c^{\perp_A} = d^{\perp_A} = 0$. Let e be a compact element of A such that $e \leq (cd)^{\perp_A}$. Then, $cde = 0$. Thus, $de \leq c^{\perp_A}$, and so, $de = 0$. Hence, $e \leq d^{\perp_A}$, and therefore, $e = 0$. We conclude that $(cd)^{\perp_A} = 0$, and consequently, $cd \in S$. □

We denote by $Q(A)$ the localization quantale A_S with respect to the multiplicatively closed set S. If L is a coherent frame, then $Q(L)$ is exactly the localization frame defined in Section 3 of [3]. In what follows, we denote by $j_A : A \to Q(A)$ the canonical quantale morphism $j_S : A \to A_S$, where $S = \{c \in K(A) : c^{\perp_A} = 0\}$.

Lemma 5.4. *If $c \in K(A)$ and $a \in A$, then the following hold:*

(i). *$c_S = 1$ if and only if $c^{\perp_A} = 0$.*

(ii). *If $j_A(a) = 0_S$, then $a = 0$ (j_A is dense).*

Proof. (i). According to Lemma 4.2(v), $c_S = 1$ if and only if $s \leq c$ for some $s \in S$ if and only if $s \leq c$ for some $s \in K(A)$ such that $s^{\perp_A} = 0$ if and only if $c^{\perp_A} = 0$.

(ii). Assume that $a_S = 0_S$. Let c be a compact element A such that $c \leq a_S = 0_S$. Then, there exists $s \in S$ such that $cs = 0$. Thus, $c \leq s^{\perp_A} = 0$, and hence, $c = 0$. It follows that $a = 0$. □

Proposition 5.3. *For any $a \in A$, it holds that $(a^{\perp_A})_S = a^{\perp_A}$.*

Proof. Let c be a compact element of A such that $c \leq (a^{\perp_A})_S$. By Lemma 4.1, there exists $s \in S$ such that $cs \leq a^{\perp_A}$. So, $csa = 0$. Then, $ca \leq s^{\perp_A} = 0$, and hence, $ca = 0$. Therefore, $c \leq a^{\perp_A}$, and consequently, the inequality $(a^{\perp_A})_S \leq a^{\perp_A}$ holds. The converse inequality $a^{\perp_A} \leq (a^{\perp_A})_S$ follows from Lemma 4.2(ii). □

Recall from Section 3 of [3] that an algebraic frame L is strongly projectable if any polar of L is complemented. We extend this definition to quantale theory: a quantale A is said to be strongly projectable if each polar is a complemented element of A ($Pol(A) \subseteq B(A)$). We note that the quantale A is strongly projectable if and only if for any $a \in A$, we have $a^{\perp_A} \vee a^{\perp_A \perp_A} = 1$.

The following result is a quantale version of Theorem 3.10 of [3].

Theorem 5.3. *If A is a semiprime coherent quantale, then the following statements are equivalent:*

(i). *A is a fraction-dense quantale.*

(ii). *$Q(A)$ is a strongly projectable quantale.*

Proof. (i) ⇒ (ii). Assume that A is a fraction-dense quantale. In order to prove that $Q(A)$ is strongly projectable, we need to show that for any $a \in A$, the following equality holds:

$$(a_S)^{\perp_{A_S}} \vee_S (a_S)^{\perp_{A_S} \perp_{A_S}} = 1.$$

Let a be an element of A. Since A is fraction-dense, we have $a^{\perp_A} = c^{\perp_A \perp_A}$ and $a^{\perp_A \perp_A} = c^{\perp_A} = d^{\perp_A \perp_A}$ for some $c, d \in K(A)$ (cf. Proposition 5.2). We note that $d^{\perp_A} = d^{\perp_A \perp_A \perp_A} = a^{\perp_A \perp_A \perp_A} = a^{\perp_A}$. Hence,

$$(c \vee d)^{\perp_A} = c^{\perp_A} \wedge d^{\perp_A} = a^{\perp_A \perp_A} \wedge a^{\perp_A} = 0.$$

Therefore, by Lemma 5.4(i), we have $(c \vee d)_S = 1$. On the other hand, by applying Corollary 5.1, we obtain

$$(a_S)^{\perp_{A_S}} \vee_S (a_S)^{\perp_{A_S} \perp_{A_S}} = (a^{\perp_A})_S \vee_S (a^{\perp_A \perp_A})_S = (a^{\perp_A} \vee a^{\perp_A \perp_A})_S = (c^{\perp_A \perp_A} \vee d^{\perp_A \perp_A})_S.$$

Since $c \leq c^{\perp_A \perp_A}$ and $d \leq d^{\perp_A \perp_A}$, we have

$$1 = (c \vee d)_S \leq (c^{\perp_A \perp_A} \vee d^{\perp_A \perp_A})_S = (a_S)^{\perp_{A_S}} \vee_S (a_S)^{\perp_{A_S} \perp_{A_S}}.$$

Therefore, $Q(A)$ is strongly projectable.

(ii) ⇒ (i). Assume that $Q(A)$ is strongly projectable. Let a be an arbitrary element of A. By Proposition 5.3, we have $(a^{\perp_A})_S = a^{\perp_A}$, and hence, $a^{\perp_A} \in Q(A)$. Since $Q(A)$ is strongly projectable, $a^{\perp_A} \in B(Q(A))$. From Lemma 15 of [5], we have $B(Q(A)) \subseteq K(Q(A))$, and so, $a^{\perp_A} \in K(Q(A))$. According to Corollary 4.2, there exists $c \in K(A)$ such that $a^{\perp_A} = c_S$. By virtue of Proposition 5.3 and Corollary 5.1, the following equalities hold:

$$c^{\perp_A} = (c^{\perp_A})_S = (c_S)^{\perp_{A_S}} = (a^{\perp_A})^{\perp_{A_S}} = ((a^{\perp_A})_S)^{\perp_{A_S}} = (a^{\perp_A \perp_A})_S = a^{\perp_A \perp_A} = a^{\perp_A \perp_A}.$$

Therefore, from Proposition 5.2, it follows that A is a fraction-dense quantale. □

Lemma 5.5 (see [11]). *Let A be a semiprime coherent quantale. If $p \in Spec(A)$, then $p \in Min(A)$ if and only if for every $c \in K(A)$, $c \leq p$ implies $c^{\perp_A} \not\leq p$.*

Proposition 5.4. *If A is a semiprime coherent quantale, then $Min(A) = Min(Q(A))$.*

Proof. First, we prove that $Min(A) \subseteq Min(Q(A))$. Assume that $p \in Min(A)$. We want to show that $p \in Spec(Q(A))$. According to Theorem 4.1, we need to prove that

$$t \in K(A),\ t^{\perp_A} = 0 \Rightarrow t \not\leq p.$$

Let t be a compact element of A such that $t \leq p$. By Lemma 5.5, we have $t^{\perp_A} \not\leq p$, and hence, $t^{\perp_A} \neq 0$. The desired implication is proven. Therefore, $p \in Spec(Q(A))$. Since $Spec(Q(A)) \subseteq Spec(A)$ and $p \in Min(A)$, we have that $p \in Min(Q(A))$.

In order to show that $Min(Q(A)) \subseteq Min(A)$, consider a minimal m-prime element p of $Q(A)$. Then, $p \in Spec(A)$. We have to prove that $p \in Min(A)$. Let c be a compact element of A such that $c \leq p$. By Lemma 4.2(vi), we have $c_S \leq p$, and hence, $(c_S)^{\perp_{A_S}} \not\leq p$ (cf. Lemma 5.5). In accordance with Corollary 5.1, we have $(c^{\perp_A})_S = (c_S)^{\perp_{A_S}}$, and so, $(c^{\perp_A})_S \not\leq p$. By virtue of Proposition 5.3, we obtain $c^{\perp_A} \not\leq p$. Using again Lemma 5.5, we have $p \in Min(A)$. □

Corollary 5.3. *If A is a coherent quantale, then A is semiprime if and only if $Q(A)$ is semiprime.*

Proof. Assume that A is semiprime, i.e., $\rho_A(0) = 0$. By Proposition 5.4, we have $Min(A) = Min(Q(A))$. Hence,

$$\rho_{Q(A)} = \bigwedge Min(Q(A)) = \bigwedge Min(A) = \rho_A(0) = 0.$$

Therefore, $Q(A)$ is semiprime. Conversely, by virtue of Corollary 4.4, if $Q(A) = A_{S(A)}$ is semiprime, then A is semiprime. □

A frame morphism $u : L \to M$ is said to be skeletal (see [3]) if for all $x, y \in L$, $x^{\perp_L} = y^{\perp_L}$ implies $u(x)^{\perp_M} = u(y)^{\perp_M}$. Let $u : A \to B$ be a quantale morphism. We say that u is w-skeletal (respectively, skeletal) if $x, y \in A$, $x^{\perp_{A,w}} = y^{\perp_{A,w}}$ (respectively, $x^{\perp_A} = y^{\perp_A}$) implies $u(x)^{\perp_{B,w}} = u(y)^{\perp_{B,w}}$ (respectively, $u(x)^{\perp_B} = u(y)^{\perp_B}$).

Proposition 5.5. *If $u : A \to B$ is a coherent quantale morphism, then the following statements are equivalent:*

(i). *$u : A \to B$ is a w-skeletal quantale morphism.*

(ii). *$u^\rho : R(A) \to R(B)$ is a skeletal frame morphism.*

Proof. (i) ⇒ (ii). Assume that $u : A \to B$ is a w-skeletal quantale morphism. Let x and y be two elements of $R(A)$ such that $x^{\perp_{R(A)}} = y^{\perp_{R(A)}}$. By Lemma 5.1(iv), we have $x^{\perp_{A,w}} = x^{\perp_{R(A)}} = y^{\perp_{R(A)}} = y^{\perp_{A,w}}$. Therefore, $u(x)^{\perp_{B,w}} = u(y)^{\perp_{B,w}}$. We note that $u^\rho(x) = \rho_B(u(x))$ and $u^\rho(y) = \rho_B(u(y))$. Consequently, using the definition of u^ρ and the properties (iii) and (iv) of Lemma 5.1, we obtain the following equalities:

$$(u^\rho(x))^{\perp_{R(B)}} = (u^\rho(x))^{\perp_{B,w}} = (\rho_B(u(x)))^{\perp_{B,w}} = ((u(x))^{\perp_{B,w}} = ((u(y))^{\perp_{B,w}} = (u^\rho(y))^{\perp_{R(B)}}.$$

Hence, we conclude that $u^\rho : R(A) \to R(B)$ is a skeletal frame morphism.

(ii) ⇒ (i). The proof of this implication is similar to that of (i) ⇒ (ii). □

Now, we use Proposition 5.5 in order to obtain a characterization of w-skeleted coherent quantale morphisms.

Proposition 5.6. *If $u : A \to B$ is a coherent quantale morphism, then the following statements are equivalent:*

(i). *u is a w-skeletal quantale morphism.*

(ii). *For any $a \in A$, $a^{\perp_{A,w}} = \rho_A(0)$ implies $u(a)^{\perp_{B,w}} = \rho_B(0)$.*

Proof. We know from [3] that a coherent frame morphism is skeletal if and only if it sends dense elements to dense elements. Therefore, using Proposition 5.5, the following properties are equivalent:

(a) u is w-skeletal.

(b) u^ρ is a skeletal frame morphism.

(c) For any $x \in R(A)$, $x^{\perp_{R(A)}} = \rho_A(0)$ implies $(u^\rho(x))^{\perp_{R(B)}} = \rho_B(0)$.

Hence, to conclude the proof, it suffices to verify the equivalence between statements (ii) and (c).

First, we prove the implication (ii) ⇒ (c). Let x be an element of $R(A)$ such that $x^{\perp_{R(A)}} = \rho_A(0)$. One can find an element $a \in A$ such that $x = \rho_A(a)$. Hence, $(\rho_A(a))^{\perp_{R(A)}} = \rho_A(0)$. Using Lemma 5.1(v), we have $a^{\perp_{A,w}} = \rho_A(0)$. According to (ii), it holds $u(a)^{\perp_{B,w}} = \rho_B(0)$. From Lemma 5.1(v) and Corollary 5.1, it follows that

$$u(a)^{\perp_{B,w}} = (\rho_B(u(a)))^{\perp_{R(B)}} = (u^\rho(\rho_A(a)))^{\perp_{R(B)}} = (u^\rho(x))^{\perp_{R(B)}}.$$

Therefore, $(u^\rho(x))^{\perp_{R(B)}} = \rho_B(0)$, and consequently, the property (c) holds.

The implication (c) ⇒ (ii) follows from the arguments similar to the ones provided in the proof of (ii) ⇒ (c). □

Lemma 5.6. *If a is an element of a coherent quantale A, then the following hold:*

(i). $R([a)_A) = [\rho_A(a))_{R(A)}$.

(ii). $((u_a^A)^\rho)(x) = x \dot\vee \rho_A(a)$.

(iii). $(u_a^A)^\rho = u_{\rho_A(a)}^{R(A)}$.

Proof. (i). By Lemma 3.5(iv), we have

$$R([a)_A) = \{x \in [a)_A : \rho[a)_A(x) = x\} = \{x \in A : a \leq x, \rho_A(x) = x\} = \{x \in R(A) : \rho_A(a) \leq x\} = [\rho_A(a))_{R(A)}.$$

(ii). Let x be an element of $R(A)$. Using Lemma 3.5(iv) and the definition of $(u_a^A)^\rho$, we obtain

$$(u_a^A)^\rho(x) = \rho_{[a)_A}(u_a^A(x)) = \rho_A(x \vee a) = \rho_A(x) \dot\vee \rho_A(a) = x \dot\vee \rho_A(a).$$

(iii). This part follows from (i) and (ii). □

Proposition 5.7. *Let A be a coherent quantale and $a \in A$. Then, the quantale morphism $u_a^A : A \to [a)_A$ is w-skeletal if and only if $a^{\perp_{A,w}\perp_{A,w}} = \rho_A(0)$.*

Proof. By Lemma 4.10 of [6], if x is an element of a coherent frame L, then the coherent frame morphism $u_x^A : L \to [x)_L$ is skeletal if and only if $x = x^{\perp_L \perp_L}$. Therefore, by Proposition 5.5 and Lemma 5.6, the following properties are equivalent:

(i) $u_a^A : A \to [a)_A$ is w-skeletal.

(ii) $(u_a^A)^\rho : R(A) \to R([a)_A)$ is a skeletal frame morphism.

(iii) $\rho_A(a) = (\rho_A(a))^{\perp_{R(A)}\perp_{R(A)}}$.

We note that $(\rho_A(a))^{\perp_{R(A)}} \in R(A)$. Using Lemma 5.1, we have

$$a^{\perp_{R(A)}\perp_{R(A)}} = ((\rho_A(a))^{\perp_{R(A)}})^{\perp_{A,w}} = a^{\perp_{A,w}\perp_{A,w}}.$$

Therefore, by virtue of the equivalence $(i) \Leftrightarrow (iii)$, we conclude that u_a^A is w-skeletal if and only if $a^{\perp_{A,w}\perp_{A,w}} = \rho_A(0)$. □

Remark 5.1. *Let A be a coherent quantale. According to Proposition 4.5 of [14], the set $Pol_w(A) = \{a^{\perp_w} : a \in A\}$ has a canonical structure of the complete Boolean algebra (in fact, $Pol_w(A) = Pol(R(A))$). Hence, Proposition 5.7 can be reformulated in the following way: $Pol_w(A) = \{a \in A : u_a^A$ is w $-skeletal\}$. Particularly, $u_{\rho_A(0)}^A : A \to [\rho_A(0))_A$ is w-skeletal.*

Proposition 5.8. *Let $u : A \to B$ be a surjective w-skeletal coherent quantale morphism. If A is weak fraction-dense, then B is weak fraction-dense.*

Proof. Recall from Lemma 3.3 that $u^\rho \circ \rho_A = \rho_B \circ u$. Hence, u^ρ is surjective (because ρ_A, ρ_B and u are surjective maps). In accordance with Proposition 5.5, $u^\rho : R(A) \to R(B)$ is a skeletal frame morphism (cf. Lemma 5.1). From Theorem 5.1, it follows that the frame $R(A)$ is fraction dense. Applying Proposition 4.1 of [3] to the coherent frame morphism u^ρ, we conclude that the frame $R(B)$ is fraction-dense. Another application of Theorem 5.1 gives that B is weak fraction-dense. □

Corollary 5.4. *If A is a weak fraction-dense coherent quantale, then $[a)_A$ is weak fraction-dense for any $a \in Pol_w(A)$.*

We say that a quantale morphism $u : A \to B$ is dense if for every $a \in A$, $u(a) = 0$ implies $a = 0$.

Lemma 5.7 (see [14]). *If $u : A \to B$ is a dense coherent quantale morphism, then $u^\rho : R(A) \to R(B)$ is a dense coherent frame morphism.*

Proposition 5.9. *If $u : A \to B$ is a surjective dense coherent quantale morphism, then the following statements are equivalent:*

(i). *A is a weak fraction-dense quantale.*

(ii). *B is a weak fraction-dense quantale.*

Proof. By Theorem 5.1 and Lemma 5.7, $u^\rho : R(A) \to R(B)$ is a dense coherent frame morphism. We know that u^ρ is surjective (see the proof of Proposition 5.8). Therefore, using Proposition 4.3 of [3] and Theorem 5.1, we obtain the equivalence of the following properties:

- A is a weak fraction-dense quantale.
- $R(A)$ is a fraction-dense frame.
- $R(B)$ is a fraction-dense frame.
- B is a weak fraction-dense quantale. □

Corollary 5.5. *If A is a coherent quantale, then the following statements are equivalent:*

(i). *A is weak fraction-dense.*

(ii). *$Q(A)$ is weak fraction-dense.*

Proof. By Lemma 5.4(i), $j_A : A \to Q(A)$ is a surjective dense coherent quantale morphism. Now, the equivalence (i) $\Leftrightarrow$ (ii) follows from Proposition 5.9. □

6. Characterizing some classes of quantales by means of $Q(A)$

Recall from Definition 3.5 of [3] that a coherent frame L is complemented if for each element $c \in K(L)$ there exists $d \in K(L)$ such that $c \wedge d = 0$ and $(c \vee d)^{\perp_A} = 0$. This definition can be extended to quantale theory: a coherent quantale A is complemented if for each element $c \in K(A)$ there exists $d \in K(A)$ such that $cd = 0$ and $(c \vee d)^{\perp_A} = 0$. We note that the condition defining the notion of complemented quantale appeared in the characterization of coherent quantales for which the minimal m-prime spectrum is compact (see Theorem 8.8 of [11]).

Proposition 6.1. *If A is a semiprime coherent quantale, then the following statements are equivalent:*

(i). *A is a complemented quantale.*

(ii). *$R(A)$ is a complemented frame.*

Proof. **(i) ⇒ (ii)**. Recall that $K(R(A)) = \{\rho_A(c) : c \in K(A)\}$ (cf. [5]). Assume that $x \in K(R(A))$. Then, $x = \rho_A(c)$ for some $c \in K(A)$. Using the hypothesis that A is a complemented quantale, there exists $d \in K(A)$ such that $cd = 0$ and $(c \vee d)^{\perp_A} = 0$. We set $y = \rho_A(d)$. Hence, $y \in K(R(A))$. By virtue of Lemma 5.1(iv), we have

$$x \wedge y = \rho_A(c) \wedge \rho_A(d) = \rho_A(cd) = \rho_A(0) = 0$$

and

$$(x \dot{\vee} y)^{\perp_{R(A)}} = (\rho_A(c) \dot{\vee} \rho_A(d))^{\perp_{R(A)}} = (\rho_A(c \vee d))^{\perp_{R(A)}} = (c \vee d)^{\perp_A} = 0.$$

Therefore, $R(A)$ is a complemented frame.

(ii) ⇒ (i). Assume that $c \in K(A)$. Then, $\rho_A(c) \in K(R(A))$. According to hypothesis (ii), there exists $y \in K(R(A))$ such that $\rho_A(c) \wedge y = 0$ and $(\rho_A(c) \dot{\vee} y)^{\perp_{R(A)}} = 0$. One can find an element $d \in K(A)$ such that $y = \rho_A(d)$. Therefore, by using Lemma 5.1(iv) again, we have

$$\rho(cd) = \rho_A(c) \wedge \rho_A(d) = \rho_A(c) \wedge y = 0$$

and

$$(c \vee d)^{\perp_A} = (\rho_A(c \vee d))^{\perp_{R(A)}} = (\rho_A(c) \dot{\vee} \rho_A(d))^{\perp_{R(A)}} = (\rho_A(c) \dot{\vee} y)^{\perp_{R(A)}} = 0.$$

Since A is semiprime, $\rho(cd) = 0$ implies $cd = 0$, and therefore, A is a complemented quantale. □

Lemma 6.1. *If A is a semiprime coherent quantale, then $c^{\perp_A} = (c^n)^{\perp_A}$ for all $c \in K(A)$ and $n \geq 1$.*

Proof. Since $c^n \leq c$, we have $c^{\perp_A} \leq (c^n)^{\perp_A}$. In order to show that $(c^n)^{\perp_A} \leq c^{\perp_A}$, consider a compact element d of A such that $d \leq (c^n)^{\perp_A}$. Then, $dc^n = 0$. Thus, $d^n c^n = 0$, and hence, $dc = 0$ (because A is semiprime). Therefore, $d \leq c^{\perp_A}$, and consequently, it follows that $(c^n)^{\perp_A} \leq c^{\perp_A}$. □

Proposition 6.2. *If A is a semiprime coherent quantale, then the following statements are equivalent:*

(i). *A is hyperarchimedean;*

(ii). *$K(A) = B(A)$;*

(iii). *$K(A) \subseteq B(A)$.*

Proof. **(i) ⇒ (ii)**. We know that $B(A) \subseteq K(A)$ (by Lemma 15 of [5]). In order to show that $K(A) \subseteq B(A)$, assume that $c \in K(A)$. Since A is hyperarchimedean, $c^n \in B(A)$ for some integer $n \geq 1$. Then, $c^n \vee (c^n)^{\perp_A} = 1$, and hence, $c \vee (c^n)^{\perp_A} = 1$. Using Lemma 6.1, we obtain $c \vee c^{\perp_A} = 1$, and therefore, $c \in B(A)$.

(ii) ⇒ (i). This implication obviously holds.

(ii) ⇔ (iii). This implication follows from the fact that $B(A) \subseteq K(A)$. □

Recall from [12] that the coherent quantale A is a PP-quantale if for any $c \in K(A)$, we have $c^{\perp_A} \in B(A)$. In the previous section, the notion of the strongly projectable quantale was introduced. Since each strongly projectable quantale is a PP-quantale, the latter can also be named a projectable quantale.

Corollary 6.1. *Any semiprime hyperarchimedean quantale A is a PP-quantale.*

Proof. Let c be a compact element of A. By Proposition 6.2 we have $c \in K(A)$, and hence, $c^{\perp_A} \in B(A)$. □

Lemma 6.2. *Any PP-quantale A is complemented.*

Proof. We know that any PP-quantale is semiprime (see Lemma 8.1 of [12]). Assume that $c \in K(A)$. Then, $c^{\perp_A} \in B(A)$. Thus, we have $c^{\perp_A} \in K(A)$. We set $d = c^{\perp_A}$. Since $c \leq c^{\perp_A \perp_A}$, we obtain $cd \leq c^{\perp_A \perp_A} c^{\perp_A} = 0$, and hence, $cd = 0$. On the other hand, $(c \vee d)^{\perp_A} = (d \vee d^{\perp_A})^{\perp_A} = d^{\perp_A} \wedge d^{\perp_A \perp_A} = 0$. Therefore, A is complemented. □

Theorem 6.1. *If A is a semiprime coherent quantale, then the following statements are equivalent:*

(i). *A is a complemented quantale.*

(ii). *$Min(A)$ is a compact space.*

(iii). *$Min(A)$ is a Boolean space.*

(iv). *$Q(A)$ is a hyperarchimedean quantale.*

(v). *$Q(A)$ is a PP-quantale.*

(vi). *$Q(A)$ is a complemented quantale.*

Proof. **(i) ⇔ (ii) ⇔ (iii)**. These equivalences follow from Theorem 8.8 of [11].

(i)⇒(iv). Recall that $Q(A) = A_S$, where $S = \{s \in K(A) : s^{\perp_A} = 0\}$. Let x be a compact element of $Q(A)$. Then, $x = c_S$ for some $c \in K(A)$ (cf. Corollary 4.2). Since A is complemented, there exists $d \in K(A)$ such that $cd = 0$ and $(c \vee d)^{\perp_A} = 0$. By Lemma 10 of [13], $cd = 0$ implies $c \wedge d = 0$. Using again Corollary 4.2, we have $d_S \in K(Q(A))$. Taking into account Proposition 5.1, from $(c \vee d)^{\perp_A} = 0$, we obtain $(c \vee d)_S = 1$. In accordance with the form of operations of $Q(A)$, the following equalities hold:

$$x \wedge d_S = c_S \wedge d_S = (c \wedge d)_S = 0_S$$

and

$$x \vee_S d_S = c_S \vee_S d_S = (c \vee d)_S = 1.$$

Thus, $x \in B(Q(A))$, and hence, $K(Q(A)) \subseteq B(Q(A))$. From Proposition 6.2, it follows that $Q(A)$ is a hyperarchimedean quantale.

(iv) ⇒ (v). This implication follows from Corollary 6.1.

(v) ⇒ (vi). This implication follows from Lemma 6.2.

(vi) ⇒ (i). Suppose that $c \in K(A)$. Then, $c_S \in K(A_S)$ (by Corollary 4.2). By the hypothesis that $Q(A)$ is a complemented quantale, there exists $d \in K(A)$ such $c_S d_S = 0_S$ and $(c_S \vee_S d_S)^{\perp_{A_S}} = 0_S$. In accordance with the form of operations of $Q(A)$ and Corollary 5.1, the following equalities hold:

$$(cd)_S = c_S d_S = 0_S$$

and

$$((c \vee d)^{\perp_A})_S = ((c \vee d)_S)^{\perp_{A_S}} = (c_S \vee_S d_S)^{\perp_{A_S}} = 0_S.$$

By Lemma 5.4(ii), $(cd)_S = 0_S$ and $((c \vee d)^{\perp_A})_S = 0_S$ imply that $cd = 0$ and $(c \vee d)^{\perp_A} = 0$. Therefore, A is a complemented quantale. □

Theorem 6.2. *If A is a coherent quantale, then the following statements are equivalent:*

(i). *$Q(A)$ is a semiprime hyperarchimedean quantale.*

(ii). *A is a semiprime quantale and $Min(A)$ is a compact space.*

Proof. **(i) ⇒ (ii)**. Suppose that $Q(A)$ is a semiprime hyperarchimedean quantale. From Corollary 4.4, it follows that A is a semiprime quantale. Now, using Proposition 5.4, we have $Min(A) = Min(Q(A))$. But, $Q(A)$ is hyperarchimedean, and hence, $Min(Q(A)) = Spec(Q(A))$ (cf. Proposition 6.2 of [11]). Thus, $Min(A) = Spec(Q(A))$. We know that the m-prime spectrum of any coherent quantale is compact (see e.g. [9], in a more general case). Therefore, $Min(A)$ is compact.

(ii) ⇒ (i). Suppose that A is a semiprime quantale and $Min(A)$ is a compact space. Then, due to Corollary 5.3, $Q(A)$ is a semiprime coherent quantale. Applying Proposition 5.4, we obtain the equality $Min(A) = Min(Q(A))$. Hence, $Min(Q(A))$ is a compact space. Therefore, in accordance with Theorem 6.1, we conclude that $Q(A)$ is a hyperarchimedean quantale. □

Corollary 6.2. *If A is a semiprime coherent quantale such that $Min(A)$ is a finite set, then $Q(A)$ is a semiprime hyperarchimedean quantale.*

According to [11, 12], an element a of the quantale A is pure if for any $c \in K(A)$, $c \leq a$ implies $a \vee c^{\perp_A} = 1$. Recall from [11] that the quantale A is a PF-quantale if for any $c \in K(A)$, $c^{\perp_A}$ is a pure element of A.

Lemma 6.3 (see [11]). *Any PF-quantale is semiprime.*

Proposition 6.3 (see [12]). *If A is a semiprime coherent quantale, then the following statements are equivalent:*

(i). *A is a PP-quantale.*

(ii). *A is a PF-quantale and $Min(A)$ is a compact space.*

Proposition 6.4 (see [12]). *If A is a coherent quantale, then the following statements are equivalent:*

(i). *A is a PF-quantale.*

(ii). *For all $c, d \in K(A)$, $cd = 0$ implies $c^{\perp_A} \vee d^{\perp_A} = 1$.*

(iii). *For all $c, d \in K(A)$, $c^{\perp_A} \vee d^{\perp_A} = (cd)^{\perp_A}$.*

Lemma 6.4. *Let p be an m-prime element of a coherent quantale A. For any $c \in K(A)$, $c_p = 0_p$ if and only if $c^{\perp_A} \not\leq p$.*

Proof. The required conclusion follows from Lemma 4.1 (or Lemma 10 of [15]). □

A coherent quantale A is said to be a quantale domain (or lattice domain, in the terminology of [18]) if $0 \in Spec(A)$. Let $u : A_1 \to A_2$ be a quantale morphism. We say that the complemented elements can be lifted along u if for any $y \in B(A_1)$ there exists $x \in B(A_1)$ such that $y = u(x)$. Recall that $Q(A) = A_S$, where $S = \{c \in K(A) : c^{\perp_A} = 0\}$ and $j_A : A \to Q(A)$ is the quantale morphism $j_S : A \to A_S$. The motivation for establishing the following result comes from Theorem 2.1 of [26] and Proposition 1 of [8].

Theorem 6.3. *If A is a coherent quantale, then the following statements are equivalent:*

(i). *A is a PP-quantale.*

(ii). *A is a PF-quantale and $Q(A)$ is a semiprime hyperarchimedean quantale.*

(iii). *The quantale $Q(A)$ is a semiprime hyperarchimedean quantale and, for any multiplicatively closed subset T of A, the complemented elements can be lifted along $j_T : A \to A_T$.*

(iv). *The quantale $Q(A)$ is a semiprime hyperarchimedean quantale and the complemented elements can be lifted along $j_A : A \to Q(A)$.*

(v). *$Q(A)$ is a semiprime hyperarchimedean quantale and A_m is a quantale domain for any $m \in Max(A)$.*

Proof. (i) ⇒ (ii). Assume that A is a PP-quantale. Then, by Proposition 6.3, A is a PF-quantale, and hence, it is a semiprime quantale (by Lemma 6.3). Next, we prove that $Q(A)$ is a semiprime hyperarchimedean quantale. According to Proposition 6.2, it suffices to verify that $K(Q(A)) \subseteq B(Q(A))$. Suppose that $x \in K(Q(A))$. Then, $x = c_S$ for some $c \in K(A)$ (cf. Corollary 4.2). Since A is a PP-quantale, we have $c^{\perp_A} \in B(A)$. We know that $B(A) \subseteq K(A)$. Hence, $c \vee c^{\perp_A} \in K(A)$. On the other hand, $(c \vee c^{\perp_A})^{\perp_A} = c^{\perp_A} \wedge c^{\perp_A \perp_A} = 0$. Thus, $c \vee c^{\perp_A} \in S$. By Lemma 5.4(i), we obtain $(c \vee c^{\perp_A})_S = 1$. By Corollary 5.1, we obtain

$$c_S \dot{\vee} (c_S)^{\perp_{A_S}} = c_S \dot{\vee} (c^{\perp_A})_S = (c \vee c^{\perp_A})_S = 1.$$

It follows that $x = c_S \in B(A_S)$, and therefore, $K(Q(A)) \subseteq B(Q(A))$.

(ii) ⇒ (i). In accordance with Lemma 6.3, A is a semiprime quantale. Therefore, by Proposition 5.4, it holds that $Min(A) = Min(Q(A))$. Since $Q(A)$ is a hyperarchimedean quantale, it follows that $Min(Q(A)) = Spec(Q(A))$ (by Proposition 6.2 of [11]). We know that $Spec(Q(A))$ is a compact space. Therefore, $Min(Q(A))$ is a compact space. Hence, $Min(A)$ is a compact space, and therefore, the PF-quantale A is a PP-quantale (cf. Proposition 6.3).

(i) ⇒ (iii). According to the equivalence of the properties (i) and (ii), $Q(A)$ is a semiprime hyperarchimedean quantale. Let T be an arbitrary multiplicatively closed subset of A and $j_S : A \to A_T$ be the canonical quantale morphism associated with T defined by $j_T(a) = a_T$ for any $a \in A$. We want to show that the complemented elements can be lifted along j_T.

Assume that a is an element of A such that $a_T \in B(A_T)$. We know that $B(A_T) \subseteq K(A_T)$. Therefore, $a_T \in K(A_T)$, and hence, there exists $c \in K(A)$ such that $a_T = c_T$. Consequently, due to Corollary 5.1, the following equalities hold:

$$c_T = (c_T)^{\perp_{A_T}\perp_{A_T}} = (c^{\perp_A\perp_A})_T = j_T(c^{\perp_A\perp_A}).$$

Since A is a PP-quantale, from $c \in K(A)$, we infer that $c^{\perp_A} \in B(A)$. Hence, $c^{\perp_A\perp_A} \in B(A)$. Therefore, the complemented elements can be lifted along j_T.

(iii) $\Rightarrow$ (iv). This implication is obvious.

(iv) $\Rightarrow$ (i). We need to prove that A is a PP-quantale. Let c be a compact element of A. Then, $c_S \in K(A_S)$ (cf. Corollary 4.2). Since $Q(A) = A_S$ is hyperarchimedean, we have $(c^n)_S \in B(A_S)$ for some integer $n \geq 1$. Applying the hypothesis that the complemented elements can be lifted along $j_A : A \to Q(A)$, one can find an element $e \in B(A)$ such that $e_S = (c^n)_S$. Hence, $(e_S)^{\perp_{A_S}} = ((c^n)_S)^{\perp_{A_S}}$. From Corollary 4.4, it follows that A is a semiprime quantale, and hence, $(c^n)^{\perp_A} = c^{\perp_A}$ (by Lemma 6.1). In accordance with Proposition 5.3 and Corollary 5.1, the following equalities hold:

$$e^{\perp_A} = (e^{\perp_A})_S = (e_S)^{\perp_{A_S}} = ((c^n)_S)^{\perp_{A_S}} = ((c^n)^{\perp_A})_S = (c^{\perp_A})_S = c^{\perp_A}.$$

By using these equalities, we obtain $c^{\perp_A} \in B(A)$. Therefore, A is a PP-quantale.

(ii) $\Rightarrow$ (v). Let m be a maximal element of A. Recall from Corollary 4.2 that $K(A_m) = \{c_m : c \in K(A)\}$. In order to show that $0_m \in Spec(A_m)$, consider two compact elements c and d of A such that $c_m \cdot_m d_m = 0_m$. According to the definition of multiplication $\cdot_m$ of A_m, we obtain $(cd)_m = 0_m$. Hence, $(cd)^{\perp_A} \not\leq m$. (cf. Lemma 6.4). Since A is a PF-quantale, we can apply Proposition 6.4, and hence, $c^{\perp_A} \vee d^{\perp_A} \not\leq m$. Therefore, we obtain $c^{\perp_A} \not\leq m$ or $d^{\perp_A} \not\leq m$. Another application of Lemma 6.4 provides $c_m = 0_m$ or $d_m = 0_m$. It follows that $0_m \in Spec(A_m)$, and hence, A_m is a quantale domain.

(v) $\Rightarrow$ (ii). It suffices to prove that A is a PF-quantale. Assume that c and d are two compact elements of A such that $cd = 0$. We want to show that $c^{\perp_A} \vee d^{\perp_A} = 1$. According to Lemma 24 of [15], it suffices to prove that $(c^{\perp_A} \vee d^{\perp_A})_m = 1$, for any $m \in Max(A)$. Let m be a maximal element of A. By Corollary 5.1, we have

$$(c^{\perp_A} \vee d^{\perp_A})_m = (c^{\perp_A})_m \vee_m (d^{\perp_A})_m = (c_m)^{\perp_{A_m}} \vee (d_m)^{\perp_{A_m}}.$$

From $cd = 0$, we obtain $c_m \cdot d_m = (cd)_m = 0_m$. Hence, $c_m = 0_m$ or $d_m = 0_m$ (because $0_m \in Spec(A_m)$). Thus, $(c_m)^{\perp_{A_m}} = 1$ or $(c_m)^{\perp_{A_m}} = 1$. In both cases, we obtain $(c^{\perp_A} \vee d^{\perp_A})_m = 1$. Therefore, $c^{\perp_A} \vee d^{\perp_A} = 1$. Consequently, by Proposition 6.4, A is a PF-quantale. □

7. The functor $Q(\cdot)$

Let $u : A \to B$ be a coherent quantale morphism. Let us consider the following multiplicatively closed sets:

$$S(A) = \{c \in K(A) : c^{\perp_A} = 0\} \text{ and } S(B) = \{d \in K(B) : d^{\perp_B} = 0\}.$$

Recall that $Q(A) = A_{S(A)}$ and $Q(B) = B_{S(B)}$. We say that u preserves the dense compact elements if $s \in S(A)$ implies $u(s) \in S(B)$.

Lemma 7.1. *If u preserves the dense compact elements, then, for all $c, d \in K(A)$, the following hold:*

(i). $c_{S(A)} \leq d_{S(A)}$ *implies* $u(c)_{S(B)} \leq u(d)_{S(B)}$;

(ii). $c_{S(A)} = d_{S(A)}$ *implies* $u(c)_{S(B)} = u(d)_{S(B)}$.

Proof. **(i)**. Suppose that $c_{S(A)} \leq d_{S(A)}$. Then, $c \leq d_{S(A)}$ (by Lemma 4.2(vi)). From Lemma 4.1, it follows that $cs \leq d$ for some $s \in S(A)$. Thus, we have $u(c)u(s) \leq u(d)$, $u(s) \in S(B)$ and $u(c), u(d) \in K(B)$. Another application of Lemma 4.1 provides the inequality $u(c) \leq u(d)_{S(B)}$. Therefore, we obtain $u(c)_{S(B)} \leq u(d)_{S(B)}$ (cf. Lemma 4.2(vi)).

(i). This part follows from (i). □

Recall from Corollary 4.2 that any compact element of $Q(A)$ has the form $c_{S(A)}$ for some $c \in K(A)$. By Lemma 7.1, one can define a map $\tilde{u} : K(Q(A)) \to K(Q(B))$ by setting $\tilde{u}(c_{S(A)}) = u(c)_{S(B)}$ for each $c \in K(A)$.

Lemma 7.2. *The map $\tilde{u}$ preserves multiplication, finite joins and the top element.*

Proof. Assume that $c, d \in K(A)$. According to the form of the multiplication in $Q(A)$ and $Q(B)$, we have

$$\tilde{u}(c_{S(A)} \cdot_{S(A)} d_{S(A)}) = \tilde{u}((cd)_{S(A)}) = (u(cd))_{S(B)} = (u(c)u(d))_{S(B)} = \tilde{u}(c)_{S(B)} \cdot_{S(B)} \tilde{u}(d)_{S(B)}.$$

These equalities show that $\tilde{u}$ preserves the multiplication. In a similar way, one can prove that $\tilde{u}$ preserves finite joins and the top element. □

Theorem 7.1. *If $u : A \to B$ is a coherent quantale morphism that preserves the dense compact elements, then there exists a unique coherent quantale morphism $Q(u) : Q(A) \to Q(B)$ such that the following diagram is commutative:*

$$\begin{array}{ccc} A & \xrightarrow{\quad u \quad} & B \\ {\scriptstyle j_A}\downarrow & & \downarrow{\scriptstyle j_B} \\ Q(A) & \xrightarrow{\quad Q(u) \quad} & Q(B) \end{array}$$

Proof. The existence and uniqueness of $Q(u)$ is obtained by using Lemmas 3.4 and 7.2. □

Proposition 7.1. *Let $u : A \to B$ be a coherent quantale morphism for which there exists a coherent quantale morphism $v : Q(A) \to Q(B)$ such that $j_B \circ u = v \circ j_A$. Then, u preserves the dense compact elements.*

Proof. Let c be a compact element of A such that $c^{\perp_A} = 0$. By Lemma 5.4(i), we have $j_A(c) = c_{S(A)} = 1$. Therefore, using the hypothesis $j_B \circ u = v \circ j_A$, we obtain

$$u(c)_{S(B)} = j_B(u(c)) = v(j_A(c)) = v(1) = 1.$$

Another application of Lemma 5.4(i) shows that $u(c)^{\perp_B} = 0$. Therefore, the quantale morphism u preserves the dense compact elements. □

Lemma 7.3. *If A is a coherent quantale, then $S(Q(A)) = \{1\}$.*

Proof. Let $S = S(A)$. Then, $Q(A) = A_S$. Recall from Corollary 4.2 that $K(Q(A)) = \{c_S : c \in K(A)\}$. By Corollary 5.1 and Lemma 5.4, for every $c \in K(A)$, the following equivalences hold:

$$(c_S)^{\perp_{A_S}} = 0_S \Leftrightarrow (c^{\perp_A})_S = 0_S \Leftrightarrow c^{\perp} = 0 \Leftrightarrow c_S = 1.$$

Therefore, $S(Q(A)) = \{c_S : c \in K(A), (c_S)^{\perp_{A_S}} = 0_S\} = \{1\}$. □

Remark 7.1. *Let $u : A \to B$ be a coherent quantale morphism that preserves the dense compact elements. In accordance with Lemma 7.3, the quantale morphism $Q(u) : Q(A) \to Q(B)$ preserves the dense compact elements.*

Definition 7.1. *A coherent quantale A is said to be a Q-quantale if $Q(A) = A$.*

We remark here that the notion of Q-quantale generalizes the notion of q-frame, defined in [3].

Lemma 7.4. *A coherent quantale A is a Q-quantale if and only if $S(A) = \{1\}$.*

Proof. Assume that $A = Q(A)$. Let $x \in S(Q(A))$. Then, $x = c_{S(A)}$ for some $c \in K(A)$ such that $(c_{S(A)})^{\perp_{S(A)}} = 0_{S(A)}$. By using Corollary 5.1, we obtain $(c^{\perp_A})_{S(A)} = 0_{S(A)}$, and hence, $c^{\perp_A} = 0$ (cf. Lemma 5.4(ii)). Therefore, by Lemma 5.4(i), we have $c_{S(A)} = 1$. Consequently, $S(Q(A)) = \{1\}$, and hence, $S(A) = \{1\}$.

Conversely, suppose that $S(A) = \{1\}$. Let a be an element of A. We want to prove that $a_{S(A)} = a$. By Lemma 4.2(ii), we have $a \leq a_{S(A)}$. In order to prove that $a_{S(A)} \leq a$, consider a compact element c such that $c \leq a_{S(A)}$. Then, $c = c1 \leq a$ (cf. Lemma 4.1). It follows that $a_{S(A)} \leq a$, and hence, $a = a_{S(A)}$. Therefore, we conclude that $A = Q(A)$. □

By Lemmas 7.3 and 7.4, if A is a coherent quantale, then $Q(A)$ is a Q-quantale.

Theorem 7.2. *If A is a semiprime coherent quantale, then the following statements are equivalent:*

(i). *A is hyperarchimedean;*

(ii). *$A = Q(A)$.*

Proof. (i) ⇒ (ii). Assume that A is hyperarchimedean. Then, $K(A) = B(A)$ (by Proposition 6.2). Thus, $S(A) \subseteq B(A)$, and hence, for each $s \in S(A)$ we have $s = s \vee s^{\perp_A} = 1$. So, $S(A) = \{1\}$. Consequently, we obtain $A = Q(A)$ (by Lemma 7.4).

(ii) ⇒ (i). Suppose that $A = Q(A)$. Then, $S(A) = \{1\}$. For any $c \in K(A)$, we have $(c \vee c^{\perp_A})^{\perp_A} = 0$, and so, $c \vee c^{\perp_A} \in S(A)$. Thus, $c \vee c^{\perp_A} = 1$, and so, $c \in B(A)$. Therefore, $K(A) \subseteq B(A)$, and hence, A is hyperarchimedean (by Proposition 6.2). □

Denote by $CohQuant^*$ the category of coherent quantales and the coherent morphisms that preserve the dense compact elements, and by $Q - CohQuant^*$ the full subcategory of $CohQuant^*$ whose objects are the Q-quantales.

Remark 7.2. *Consider two morphisms $u : A \to B$ and $v : B \to C$ of $CohQuant^*$. Using Theorem 7.1, one can prove that $Q(v \circ u) = Q(v) \circ Q(u)$. Therefore, the assignments $A \mapsto Q(A)$ and $A \to B \mapsto Q(A) \to Q(B)$ define a covariant functor $Q : CohQuant^* \to Q - CohQuant^*$.*

Proposition 7.2. *If $u : A \to B$ is a coherent quantale morphism that preserves dense compact elements, then*

$$(Q(u))_*(b_{S(B)}) = u_*(b_{S(B)}) \text{ for any } b \in B.$$

Proof. Let c be a compact element of A. By Theorem 7.1, we have $Q(u)(c_{S(A)}) = u(c)_{S(B)}$. Therefore, using the adjointness property for $Q(u)$ and $(Q(u))_*$, we obtain the following equivalences: $c_{S(A)} \leq (Q(u))_*(b_{S(B)})$ if and only if $Q(u)(c_{S(A)}) \leq b_{S(B)}$,
if and only if $u(c)_{S(B)} \leq b_{S(B)}$,
if and only if $u(c) \leq b_{S(B)}$,
if and only if $c \leq u_*(b_{S(B)})$.
Since $Q(A)$ is a coherent quantale, from the above equivalences, we obtain $(Q(u))_*(b_{S(B)}) = u_*(b_{S(B)})$. □

Lemma 7.5. *Let $u : A \to B$ be a coherent quantale morphism that preserves the dense compact elements. If u is surjective, then $Q(u)$ is surjective.*

Proof. By Theorem 7.1, we have $Q(u) \circ j_A = j_B \circ u$. Since j_A and j_B are surjective maps, it follows that $Q(u)$ is surjective. □

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