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Research Article

Local and semi-local convergence analysis of a multi-step method

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Abstract

This paper is concerned with an m-step $(m \geq 2)$ method of order $m+1$, which was first studied by Ahmad, Tohidi, and Carrasco in [*Numer. Algorithms* **71** (2016) 631–653] to solve nonlinear systems defined on the finite Euclidean space. The method depends on a single real parameter and uses one inverse per a complete step. But the local convergence was shown using Taylor series expansions and by assuming the existence of the seventh derivative; this assumption, however, does not pertain to the method itself. Other constraints include the lack of a priori error estimates and isolation of the solution results. The semi-local convergence has not been considered previously either. In this paper, both local and semi-local convergence analyses are performed using only the operators inherent to the method, namely the operator and its derivative. Moreover, the convergence analysis is carried out in the more general setting of Banach space, employing generalized continuity to control the derivative and to sharpen the error distances. In addition, isolation of the solution results are provided. The proposed technique is quite general and can be employed to extend the applicability of other iterative methods.

Keywords: nonlinear systems; Banach spaces; local convergence; semi-local convergence.

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1. Introduction

A well-studied problem in numerical analysis is that of solving a nonlinear equation or system of nonlinear equations of the form

$$F(x) = 0, \tag{1}$$

where F is a Fréchet differentiable operator from a Banach space B_1 into a Banach space B_2. Let Ω be an open convex subset of B_1. Formulation of problems as an equation like (1) using mathematical modeling [4, 9, 17, 19, 28] arises in multiple disciplines of science and engineering. Obtaining a solution $\mu \in \Omega$ of (1) in analytic form is another significant issue. Non-analytic and complex functions are therefore addressed through a powerful computational approach, namely iterative methods, which approximate the solution μ of (1). To overcome the issues such as slow or non-convergence, divergence, and inefficiency, a substantial body of literature has been devoted to the study of convergence of iterative methods based on algebraic and geometric considerations [19, 28]. Consequently, researchers worldwide have continued to develop higher order iterative methods [3, 5, 10, 11, 13, 15, 20–25].

In this study, we analyze the convergence of a multi-step method, which is defined for $a \in \mathbb{R}$, $a \neq 0$, $x_0 \in \Omega$, $m = 3, 4, \dots,$ and each for $n = 0, 1, 2, \dots,$ by

$$\begin{aligned} y_1 &= F'(x_0)^{-1}F(x_0), \\ x_1 &= x_0 - (1 + a - a^2)y_1, \\ y_2 &= F'(x_0)^{-1}F\left(x_0 - \frac{1}{a}y_1\right), \\ x_2 &= x_1 - a^2 y_2, \\ \text{and for} \quad & j = 1, \dots, m-2, \\ y_{j+2} &= F'(x_0)^{-1}F(x_{j+1}) \\ x_{j+2} &= x_{j+1} - y_{j+2}. \end{aligned} \tag{2}$$

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The local convergence order $(m+1)$ of the method (2) is shown in [1] for $B_1 = B_2 = \mathbb{R}^k$ using Taylor series expansions and by assuming the existence of $F^{(7)}$; this assumption, however, does not pertain to the method itself. To illustrate the idea, let us consider a simple example for $\Omega = \left[\frac{-3}{2}, \frac{3}{2}\right]$ and $B_1 = B_2 = \mathbb{R}$. Define the function $q : \Omega \to \mathbb{R}$ by

$$q(t) = \begin{cases} c_0 t^7 \log t + ct^8 + c_1 t^9, & t \neq 0, \\ 0, & t = 0, \end{cases}$$

where $c_0 \neq 0$ and $c + c_1 = 0$. Then, the number $\mu = 1 \in \Omega$ is a solution of the equation $q(t) = 0$. But the seventh derivative of the function q is not bounded on Ω, since this function is not continuous at $t = 0 \in \Omega$. Moreover, the method converges, say if $x_0 = 0.95$ and $a = \frac{1}{2}$. Thus, the sufficient conditions in [1] can be weakened. There are other restrictions under the Taylor series approach: neither a priori error estimates on $\|\mu - x_n\|$ are available, nor the results concerning isolation of solutions are available. That is why, in the present paper, the new local convergence is shown using conditions only on the operators inherent to the method, namely the operator F and its derivative F'. Moreover, the more important and challenging semi-local convergence, not previously studied, is addressed through the use of majorizing sequences. Both types of analyses rely on generalized continuity used to control F' and are developed in the general framework of Banach spaces. This methodology is sufficiently flexible to extend the applicability of other iterative methods along the same lines [2–31].

The rest of the paper is structured as follows. Section 2 and 3 present the local and semi-local convergence analysis of the method (2), respectively. Two numerical examples are provided in Section 4, and concluding remarks are given in Section 5.

2. Local convergence

In this section, local convergence of (2) for solving (1) is established. Let $\mathcal{U}(x, r)$ and $\mathcal{U}[x, r]$ denote the open and closed balls, respectively, with center x and radius r. Set $\Lambda = [0, +\infty)$. The hypotheses for the local convergence analysis are given as follows.

(C_1) There exists a continuous and nondecreasing function $\Psi_0 : \Lambda \to \Lambda$, (CNDF, in short) such that $\Psi_0(t) - 1$ possesses a smallest positive root (SPR), which is denoted by b.

(C_2) There exists a CNDF $\Psi : \Lambda_1 \to \Lambda$, where $\Lambda_1 = [0, b)$. Define real functions on Λ_1 by

$$A_1(t) = \frac{\int_0^1 \Psi((1-\theta)t)d\theta + |a(1-a)|(1 + \int_0^1 \Psi_0(\theta t)d\theta)}{1 - \Psi_0(t)},$$

$$\overline{\Psi}(t) = \begin{cases} \Psi((1 + A_1(t))t) \\ \quad \text{or} \\ \Psi_0(t) + \Psi_0(A_1(t)t), \end{cases}$$

$$h(t) = \frac{\left(\int_0^1 \Psi((1-\theta)t)d\theta + |1 - \frac{1}{a}|(1 + \int_0^1 \Psi_0(\theta t)d\theta)\right)t}{1 - \Psi_0(t)},$$

$$h_0(t) = \frac{h(t)}{t}, \quad h_1(t) = \frac{1}{2}(h(t) + A_1(t)t), \quad \overline{h_1}(t) = \frac{h_1(t)}{t},$$

$$h_2(t) = h(t) + A_1(t)t, \quad \overline{h_2}(t) = \frac{h_2(t)}{t},$$

$$A_2(t) = \left(\frac{\int_0^1 \Psi((1-\theta)A_1(t)t)d\theta}{1 - \Psi_0(t)} + \frac{\overline{\Psi}(t)(1 + \int_0^1 \Psi_0(\theta A_1(t)t)d\theta)}{(1 - \Psi_0(t))(1 - \Psi_0(A_1(t)t))}\right)A_1(t) +$$

$$\frac{(1 + h_1(t))\overline{h_2}(t) + |1 - a^2|(1 + \int_0^1 \Psi_0(\theta h(t))d\theta)\overline{h}(t)}{1 - \Psi_0(t)},$$

$$\overline{P}_{j+1}(t) = \begin{cases} \Psi((1 + A_{j+1}(t))t) \\ \quad \text{or} \\ \Psi_0(t) + \Psi_0(A_{j+1}(t)t), \end{cases}$$

$$A_{j+2}(t) = \left(\frac{\int_0^1 \Psi((1-\theta)A_{j+1}(t)t)d\theta}{1 - \Psi_0(A_{j+1}(t)t)} + \frac{\overline{P}_{j+1}(1 + \int_0^1 \Psi_0(\theta A_{j+1}(t)t)d\theta)}{(1-\Psi_0(t))(1-\Psi_0(A_{j+1}(t)t))} \right) A_{j+1}(t).$$

(C_3) Every equation

$$A_j(t) - 1 = 0, \text{ with } j = 1, 2, \ldots,$$

has an SPR denoted by b_j. Let

$$r = \min\{b_j\}, \; j = 1, 2, \ldots. \tag{3}$$

(C_4) There exists an invertible linear operator G and a solution $\mu \in \Omega$ of the equation $F(x) = 0$ such that for each $u \in \Omega$

$$\|G^{-1}(F'(u) - F'(\mu))\| \quad \leq \quad \Psi_0(\|u - \mu\|).$$

(C_5)

$$\|G^{-1}(F'(u_2) - F'(u_1))\| \leq \Psi(\|u_2 - u_1\|)$$

for all $v_1, v_2 \in \Omega_0 := \Omega \cap \mathcal{U}(\mu, b)$.

(C_6)

$$\mathcal{U}[\mu, r] \subset \Omega.$$

Theorem 2.1. *Assuming that the conditions* (C_1)–(C_6) *hold true, then the sequence* $\{x_k\}$ *converges to* μ*, provided that* $x_0 \in \mathcal{U}(\mu, r) - \{\mu\}$.

Proof. Using condition (C_4) and the definition of r, we obtain

$$\|G^{-1}(F'(x_0) - F'(\mu))\| \leq \Psi_0(\|x_0 - \mu\|) < 1.$$

Then, $F'(x_0)^{-1}$ exists. Employing the Banach lemma on invertible operators [3–5, 19], we have

$$\|F'(x_0)^{-1}G\| \leq \frac{1}{1 - \Psi_0(\|x_0 - \mu\|)}.$$

The motivation for introducing the real functions in (C_2) is followed by induction and a series of estimates given in subsequent results

$$x_1 - \mu = x_0 - \mu - F^{'}(x_0)^{-1}F(x_0) - a(1-a)F^{'}(x_0)^{-1}F(x_0),$$

$$\begin{aligned} \|x_1 - \mu\| &\leq \left(\frac{\int_0^1 \Psi((1-\theta)\|x_0 - \mu\|)d\theta + |a(1-a)|(1 + \int_0^1 \Psi_0(\theta\|x_0 - \mu\|)d\theta)}{1 - \Psi_0(\|x_0 - \mu\|)} \right) \|x_0 - \mu\| \\ &\leq A_1(\|x_0 - \mu\|)\|x_0 - \mu\| \leq \|x_0 - \mu\| < r. \end{aligned}$$

Thus, $x_1 \in \mathcal{U}(\mu, r)$. Similarly, we can write in turn

$$\begin{aligned} x_2 - \mu &= x_1 - \mu - F^{'}(x_1)^{-1}F(x_1) + F^{'}(x_1)^{-1}F(x_1) - a^2 F^{'}(x_0)^{-1}F\left(x_0 - \frac{1}{a}y_1\right) \\ &= x_1 - \mu - F^{'}(x_1)^{-1}F(x_1) + (F^{'}(x_1)^{-1} - F(x_0)^{-1})F(x_1) + F^{'}(x_0)^{-1}(F(x_1) \\ &\quad - F\left(x_0 - \frac{1}{a}y_1\right)\right) + (1-a^2)F^{'}(x_0)^{-1}F\left(x_0 - \frac{1}{a}y_1\right). \end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\|x_2-\mu\| \le &\left(\frac{\int_0^1 \Psi((1-\theta)\|x_1-\mu\|)d\theta}{1-\Psi_0(\|x_0-\mu\|)} + \frac{\overline{\Psi}_1(1+\int_0^1 \Psi_0(\theta\|x_1-\mu\|)d\theta)}{(1-\Psi_0(\|x_0-\mu\|))(1-\Psi_0(\|x_1-\mu\|))}\right)\|x_1-\mu\|+ \\ &\frac{(1+h_1(\|x_0-\mu\|))h(\|x_0-\mu\|)}{1-\Psi_0(\|x_0-\mu\|)} + \frac{|1-a^2|(1+\int_0^1 \Psi_0(\theta h(\|x_0-\mu\|))d\theta)h(\|x_0-\mu\|)}{1-\Psi_0(\|x_0-\mu\|)} \\ &\le A_2(\|x_0-\mu\|)\|x_0-\mu\| \ \le \|x_0-\mu\|.\end{aligned}$$

Here, we used the estimates

$$F(x_1)-F\left(x_0-\frac{1}{a}y_1\right) = \int_0^1 F'\left(x_0-\frac{1}{a}y_1+\theta(x_1-x_0+\frac{1}{a}y_1)\right)d\theta\left(x_1-x_0+\frac{1}{a}y_1\right)$$

and

$$x_1-x_0+\frac{1}{a}F'(x_0)^{-1}F(x_0) = x_1-\mu-(x_0-\mu-F'(x_0)^{-1}F(x_0))+\left(\frac{1}{a}-1\right)F'(x_0)^{-1}F(x_0).$$

Hence,

$$\begin{aligned}&\|x_1-x_0+\frac{1}{a}F'(x_0)^{-1}F(x_0)\| \le A_1(\|x_0-\mu\|)\|x_0-\mu\| \\ &+\frac{\int_0^1 \Psi((1-\theta)\|x_0-\mu\|)d\theta\|x_0-\mu\| + |\frac{1}{a}-1|(1+\int_0^1 \Psi_0(\theta\|x_0-\mu\|)d\theta)\|x_0-\mu\|}{1-\Psi_0(\|x_0-\mu\|)} \\ &= h_2(\|x_0-\mu\|),\end{aligned}$$

$$x_0-\mu-\frac{1}{a}y_1+\theta(x_1-x_0+\frac{1}{a}y_1) = (1-\theta)(x_0-\mu)+\theta(x_1-\mu)-\frac{1}{a}(1-\theta)y_1,$$

$$\begin{aligned}&\|x_0-\mu-\frac{1}{a}y_1+\theta(x_1-x_0+\frac{1}{a}y_1)\| \\ &\le(1-\theta)\frac{\left(\int_0^1 \Psi((1-\theta)\|x_0-\mu\|)d\theta+|1-\frac{1}{a}|(1+\int_0^1 \Psi_0(\theta\|x_0-\mu\|)d\theta)\right)\|x_0-\mu\|}{1-\Psi_0(\|x_0-\mu\|)} \\ &\quad+\theta A_1(\|x_0-\mu\|)\|x_0-\mu\| = h_1(\|x_0-\mu\|)\end{aligned}$$

and

$$\begin{aligned}\|x_0-\frac{1}{a}y_1-\mu\| &= \|x_0-\mu-F'(x_0)^{-1}F(x_0)+(1-\frac{1}{a})F'(x_0)^{-1}F(x_0)\| \\ &\le \frac{\left(\int_0^1 \Psi((1-\theta)\|x_0-\mu\|)d\theta+|1-\frac{1}{a}|(1+\int_0^1 \Psi_0(\theta\|x_0-\mu\|)d\theta)\right)\|x_0-\mu\|}{1-\Psi_0(\|x_0-\mu\|)} \\ &= h(\|x_0-\mu\|).\end{aligned}$$

Similarly, by method (2), we have

$$x_{j+2}-\mu = x_{j+1}-\mu-F'(x_{j+1})^{-1}F(x_{j+1})+(F'(x_{j+1})^{-1}F'(x_0)^{-1})F(x_{j+1}),$$

$$\begin{aligned}\|x_{j+2}-\mu\| \le &\left[\frac{\int_0^1 \Psi((1-\theta)\|x_{j+1}-\mu\|)d\theta}{1-\Psi_0(\|x_{j+1}-\mu\|)} + \frac{\overline{P}_{j+1}(1+\int_0^1 \Psi_0(\theta\|x_{j+1}-\mu\|)d\theta)}{(1-\Psi_0(\|x_0-\mu\|)(1-\Psi_0(\|x_{j+1}-\mu\|)))}\right]\|x_{j+1}-\mu\| \\ &\le A_{j+2}(\|x_{j+1}-\mu\|)\|x_{j+1}-\mu\| \le \|x_{j+1}-\mu\|.\end{aligned}$$

Therefore, $x_i \in \mathcal{U}(\mu, r)$ for all $i = 1, 2, \ldots, k$ and

$$\|x_k - \mu\| \leq d\|x_0 - \mu\| < r,$$

where $d = A_k(\|x_0 - \mu\|) \in [0, 1)$. Consequently, using the induction, we have

$$\|x_{i+k} - \mu\| \leq d^{n+1}\|x_0 - \mu\|$$

which implies that $\{x_k\} \subset \mathcal{U}(\mu, r)$ and $\lim_{k\to+\infty} x_k = \mu$. □

The next result determines the uniqueness of the solution region.

Proposition 2.1. *Assume that the condition (C_4) is valid on the ball $\mathcal{U}(\mu, t)$ for some $t > 0$, and there exists some $t_1 \geq t$ such that*

$$\int_0^1 \Psi_0(\theta t_1)d\theta < 1.$$

Let $\Omega_1 = \Omega \cap \mathcal{U}[\mu, t_1]$. Then, the equation $F(x) = 0$ has μ as the only solution in the region Ω_1.

Proof. Assume that there exists a solution $\mu^* \in \Omega_1$ of the equation (1) such that $\mu^* \neq \mu$. Define the linear operator Γ by

$$\Gamma = \int_0^1 F'(\mu + \theta(\mu^* - \mu))d\theta.$$

It follows that

$$\begin{aligned}\|G^{-1}(\Gamma - G)\| &\leq \int_0^1 \Psi_0(\theta\|\mu^* - \mu\|)d\theta \\ &\leq \int_0^1 \Psi_0(\theta t_1)d\theta < 1.\end{aligned}$$

Hence, the operator Γ is invertible. Then, from the identity

$$\mu^* - \mu = \Gamma^{-1}(F(\mu^*) - F(\mu)) = \Gamma^{-1}(0) = 0,$$

it follows that $\mu^* = \mu$, which is a contradiction. □

Remark 2.1. **(i).** *The point b can be replaced with r in condition (C_6).*

(ii). *The selections that can be made for the operator G are: I or $F'(\mu)$, where μ is a simple solution of the equation (1). As such an assumption is not made in Theorem 2.1, the method (2) can be utilized to find roots of multiplicity greater than one of the equation (1). The operator G can have other choices also, provided that conditions (C_4) and (C_5) hold.*

3. Semi-local convergence

The computations and formulas are the same but μ, Ψ_0 and Ψ are replaced with x_0, v_0 and v, respectively.

(S_1) Assume that there exists a CNDF $v_0 : \Lambda \to \Lambda$ such that $v_0(t) - 1$ possesses SPR denoted by s_0.

(S_2) Assume that there exists a CNDF $v : \Lambda_2 \to \Lambda$, where $\Lambda_2 = [0, s_0)$. Define the scalar sequence $\{\alpha_n\}$ such that $\alpha_0 = 0$, $\beta \geq 0$, $\gamma = \frac{\beta}{|a|}$, $\alpha_1 = |1 + a - a^2|\beta$,

$$\alpha_2 = \alpha_1 + a^2\left(\beta + \left(1 + \int_0^1 v_0(\theta\gamma)d\theta\right)\gamma\right), \tag{4}$$

and

$$\alpha_{j+2} = \alpha_{j+1} + \left(1 + \int_0^1 v_0(\theta\alpha_{j+1})d\theta\right)\alpha_{j+1} + \beta.$$

(This sequence is shown to be majorizing for the method (2) in Theorem 3.1. But, let us first provide a convergence condition.)

(S_3) There exists $s \in [0, s_0)$ such that

$$\alpha_n \leq s \quad \forall \quad n = 0, 1, 2 \ldots.$$

This condition, (4) and the induction, show that the sequence $\{\alpha_n\}$ is nondecreasing and is bounded from above by s, and hence, it converges to its unique least upper bound s_1. There is also a relationship between v_0 and v, and the operators on the method (2).

(S_4) There exists an invertible linear operator G and a point $x_0 \in \Omega$ such that for each $u \in \Omega$,

$$\|G^{-1}(F'(u) - G)\| \leq v_0(\|u - x_0\|).$$

Set $\Omega_2 = \Omega \cap \mathcal{U}(x_0, s_0)$. From this condition, it follows that, for $u = x_0$,

$$\|G^{-1}(F'(x_0) - G)\| \leq v_0(0) < 1.$$

Thus, the linear operator $F'(x_0)$ is invertible and we can take $\beta \geq \|F'(x_0)^{-1}F(x_0)\|$.

(S_5) The inequality

$$\|G^{-1}(F'(u_1) - F'(u_2))\| \leq v(\|u_2 - u_1\|)$$

holds for all $u_1, u_2 \in \Omega_2$.

(S_6) $\mathcal{U}(x_0, s_1) \subset \Omega$.

Remark 3.1. *Similar to the local analysis, we can choose $G = I$ or $G = F'(\overline{x})$ for some $\overline{x} \in \Omega$ including x_0 or some other selection as long as (S_1)–(S_6) hold.*

Now, we provide the result concerning the semi-local analysis.

Theorem 3.1. *Assume that the conditions (S_1)–(S_6) hold. Then, there exists a solution $\mu \in \mathcal{U}[x_0, s_1]$ of the equation (1) such that $\lim_{n\to+\infty} x_n = \mu$.*

Proof. The motivational calculations are as follows:

$$x_1 - x_0 = -(1 + a - a^2)F^{'}(x_0)^{-1}F(x_0),$$

$$\|x_1 - x_0\| \leq |1 + a - a^2|\|F^{'}(x_0)^{-1}F(x_0)\| = \alpha_1 - \alpha_0,$$

$$x_2 - x_1 = -a^2F^{'}(x_0)^{-1}F\left(x_0 - \frac{1}{a}y_1\right).$$

But,

$$\begin{aligned} F\left(x_0 - \frac{1}{a}y_1\right) &= F(x_0 - \frac{1}{a}y_1) - F(x_0) + F(x_0) \\ &= \int_0^1 F^{'}(x_0 + \theta(x_0 - \frac{1}{a}y_1 - x_0)d\theta(-\frac{1}{a}y_1) + F(x_0) \end{aligned}$$

and

$$\left\|-\frac{1}{a}y_1\right\| \leq \frac{\beta}{|a|} = \gamma.$$

Hence, we have

$$\|x_2 - x_1\| \leq a^2\left[\beta + \left(1 + \int_0^1 v_0(\theta\gamma)d\theta\right)\gamma\right] = \alpha_2 - \alpha_1$$

and

$$\|x_2 - x_0\| \leq s_1$$

by the triangle inequality.

Similarly, we obtain

$$x_{j+2} - x_{j+1} = -F^{'}(x_0)^{-1}F(x_{j+1})$$

and

$$F(x_{j+1}) = F(x_{j+1}) - F(x_0) + F(x_0) = \int_0^1 F^{'}(x_0 + \theta(x_{j+1} - x_0))d\theta(x_{j+1} - x_0) + F(x_0).$$

Consequently, we have

$$\|x_{j+2} - x_{j+1}\| \leq \left(1 + \int_0^1 v_0(\theta\alpha_{j+1})d\theta\right)\alpha_{j+1} + \beta = \alpha_{j+2} - \alpha_{j+1}$$

and

$$\|x_{j+2} - x_0\| \leq s_1$$

by the triangle inequality. It follows that $\{x_n\}$ is a Cauchy sequence in B_1. Therefore, there exists $\mu \in \mathcal{U}[x_0, s_1]$ such that $\lim_{n\to+\infty} x_n = \mu$. Finally, by using the inequality

$$\|F(x_{j+1})\| \leq \|F'(x_0)\|\|x_{j+2} - x_{j+1}\|,$$

we conclude $F(\mu) = 0$. □

Next, the uniqueness region is provided.

Proposition 3.1. *Assume that the following conditions hold.*

(1) *There exists a solution $\zeta^* \in \mathcal{U}(x_0, \vartheta)$ of the equation $F(x) = 0$ for some $\vartheta > 0$.*

(2) *The condition (S_2) holds on the ball $\mathcal{U}(x_0, \vartheta)$.*

(3) *There exists $\vartheta_1 \geq \vartheta$ such that*

$$\int_0^1 v_0((1-\theta)\vartheta + \theta\vartheta_1)d\theta < 1.$$

Set $\Omega_3 = \Omega \cap \mathcal{U}[x_0, \vartheta_1]$. Then, the equation $F(x) = 0$ is uniquely solvable by ζ^* in the region Ω_3.

Proof. As in Proposition 2.1, define the linear operator $\Gamma_1 = \int_0^1 F'(\zeta^* + \theta(\vartheta^* - \zeta^*))d\theta$ for some $\vartheta^* \in \Omega_3$ with $F(\vartheta^*) = 0$. Then, by conditions (1)–(3) of the proposition, we have

$$\begin{aligned}\|F'(x_0)^{-1}(\Gamma_1 - F'(x_0))\| &\leq \int_0^1 v_0((1-\theta)\|\zeta^* - x_0\| + \theta\|\vartheta^* - x_0\|)d\theta \\ &\leq \int_0^1 v_0((1-\theta)\vartheta) + \theta\vartheta_1)d\theta < 1.\end{aligned}$$

Therefore, we conclude that $\zeta^* = \vartheta^*$. □

4. Examples

Example 4.1. Let $B_1 = B_2 = \mathbb{R}^3$ and $\Omega = \mathcal{U}(0,1)$. Define the mapping $H : \Omega \to \mathbb{R}^3$ for $\delta = (\delta_1, \delta_2, \delta_3)^{tr}$ as

$$H(\delta) = \left(e^{\delta_1} - 1, \frac{1}{2}(e-1)\delta_2^2 + \delta_2, \delta_3\right)^{tr},$$

where "tr" stands for the transpose. It follows by the definition of the operator H that

$$H'(\delta) = \begin{pmatrix} e^{\delta_1} & 0 & 0 \\ 0 & (e-1)\delta_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We note that if $\mu = (0,0,0)^{tr}$, then $H'(\delta) = I$. Choose $G = I$. Then, conditions (C_4) and (C_5) hold if we choose $\Psi_0(t) = (e-1)t$ and $\Psi(t) = e^{\frac{1}{e-1}}t$. Since $b = \frac{1}{e-1}$, the radius r defined in (3) is given by $r = 0.192081$.

Example 4.2. Consider the nonlinear integral equation of mixed Hammerstein-type equations given by

$$x(s) = \int_0^1 G_1(s,t)\left(x(t)^{\frac{3}{2}} + \frac{x(t)^2}{2}\right) dt, \tag{5}$$

where $x(t) \in C[0,1]$ and $G_1(s,t)$ is Green's function defined on the interval $[0,1]\times[0,1]$. Finding the solution $x^* = 0$ of (1) is same as finding the solution of

$$F(x)(s) = x(s) - \int_0^1 G_1(s,t)\left(x(t)^{\frac{3}{2}} + \frac{x(t)^2}{2}\right) dt = 0,$$

where $F : C[0,1] \to C[0,1]$. Therefore, for $a = 1$, $\Psi_0(t) = \Psi(t) = \frac{1}{8}\left(\frac{3}{2}t^{\frac{1}{2}} + t\right)$, and the radius of convergence r is given by $r = 0.891432$.

5. Concluding remarks

The current study considers an existing m-step method ($m \geq 2$) of order $m+1$, depending on one real parameter and requiring one inverse per step, defined on a finite Euclidean space. The earlier local convergence analysis was established by assuming the existence of the seventh derivative (a requirement not inherent to the method) and by employing Taylor series expansions. However, no a priori error estimates, isolation results for the solution, or semi-local convergence analysis were provided previously. In contrast, the present work develops both local and semi-local convergence analyses using only the operator and its first derivative in a Banach space setting. A generalized continuity condition is employed to control the derivative and improve the error distances, while isolation results for the solution are also established. This technique is, therefore, an improvement of the previous one and can be used for other methods [2–31] to extend their applicability.

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