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Research Article

A polynomial identity and some of its new consequences

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Abstract

In this article, we present a new proof of a polynomial identity studied by Guo [*Discrete Math. Lett.* **8** (2022) 41–48]. We also provide new applications of this identity to combinatorial sums involving binomial coefficients and to convolution-type sums with harmonic numbers. Finally, we establish some relations involving Catalan numbers.

Keywords: polynomial identity; binomial coefficient; Fibonacci (Lucas) polynomial; harmonic number; Catalan number.

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1. Introduction

Polynomial identities appear frequently in the combinatorial and number-theoretic literature. Many classical polynomial identities are stated in the works of Riordan [26] and Gould [14]. More recent examples can be found in the articles by Munarini [24], Mansour and Sun [22], Alzer [4], Alzer and Prodinger [6], Dattoli et al. [12], Kellner [18], and Adegoke [1], among others. Applications of polynomial identities to Fibonacci and Lucas numbers may be found in [3, 13].

In 2022, Guo, with reference to Batir [9], rederived the polynomial identity (see [16, Lemma 2.1]) given in the following lemma:

Lemma 1.1. *For $n \in \mathbb{N}$, the following relation holds*

$$\sum_{k=0}^{n} \binom{m+k}{k} x^k = \sum_{k=0}^{n} \binom{m+n+1}{k} x^k (1-x)^{n-k}. \tag{A}$$

Guo [16] used the Chu-Vandermonde convolution in his proof. He also derived several combinatorial identities concerning harmonic numbers and binomial coefficients from Lemma 1.1. Another way to obtain identity (A) is to substitute $x = \frac{t}{1+t}$ in the following known identity (see [14]):

$$\sum_{k=0}^{n} \binom{m-n+k}{k} (1+t)^{n-k} t^k = \sum_{k=0}^{n} \binom{m+1}{k} t^k. \tag{1}$$

This leads to

$$\sum_{k=0}^{n} \binom{m-n+k}{k} x^k = \sum_{k=0}^{n} \binom{m+1}{k} x^k (1-x)^{n-k},$$

and replacing m by $m+n$ yields (A).

Surprisingly, identity (A) also provides a short proof of the Chaundy-Bullard identity [4, 19, 20]:

Theorem 1.1. *If we define the polynomial $P_{m,n}(x)$ by*

$$P_{m,n}(x) = (1-x)^{m+1} \sum_{k=0}^{n} \binom{m+k}{k} x^k,$$

then

$$P_{m,n}(x) + P_{n,m}(1-x) = 1.$$

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Proof. From (A), the sum $P_{m,n}(x) + P_{n,m}(1-x)$ successively rewrites as

$$P_{m,n}(x) + P_{n,m}(1-x) = \sum_{k=0}^{n} \binom{m+n+1}{k} x^k (1-x)^{m+n+1-k} + \sum_{k=0}^{m} \binom{m+n+1}{k} (1-x)^k x^{m+n+1-k}$$

$$= \sum_{k=0}^{n} \binom{m+n+1}{k} x^k (1-x)^{m+n+1-k} + \sum_{j=n+1}^{m+n+1} \binom{m+n+1}{j} x^j (1-x)^{m+n+1-j}$$

$$= \sum_{k=0}^{m+n+1} \binom{m+n+1}{k} x^k (1-x)^{m+n+1-k} = 1.$$

□

The goal of this paper is to present a new proof of identity (A) and to discuss three new consequences of this identity: new identities for Fibonacci (and Lucas) numbers and polynomials; new combinatorial sums involving (central) binomial coefficients and harmonic numbers; and new identities involving Catalan numbers. In particular, the identities involving Catalan numbers complement and generalize those obtained by Alzer and Nagy [5].

2. An alternative proof of identity (A)

We deduce identity (A) from the following result:

Lemma 2.1. *If m, n are non-negative integers and $x \neq 1$, then*

$$\sum_{k=0}^{n} \binom{m+k}{k} x^k + \sum_{k=0}^{m} \binom{m+n+1}{m-k} \frac{x^{n+1+k}}{(1-x)^{k+1}} = \frac{1}{(1-x)^{m+1}}. \tag{B}$$

Proof. For $x \neq 1$, we calculate the mth derivative of

$$f(x) = x^m + x^{m+1} + \cdots + x^{m+n} = \frac{x^m}{1-x} - \frac{x^{m+n+1}}{1-x}.$$

in two ways. On the one hand, we have

$$f^{(m)}(x) = m! \sum_{k=0}^{n} \binom{m+k}{m} x^k,$$

and on the other hand, using $\left[(1-x)^{-1}\right]^{(j)} = j!(1-x)^{-1-j}$ and Leibniz's formula

$$f^{(m)}(x) = m! \left(\sum_{j=0}^{m} \binom{m}{m-j} \frac{x^j}{(1-x)^{j+1}} - \sum_{j=0}^{m} \binom{m+n+1}{m-j} \frac{x^{n+1+j}}{(1-x)^{j+1}} \right).$$

Identity (B) follows since

$$\sum_{j=0}^{m} \binom{m}{m-j} \frac{x^j}{(1-x)^{j+1}} = \frac{1}{1-x} \left(1 + \frac{x}{1-x}\right)^m = \frac{1}{(1-x)^{m+1}}.$$

□

Now, the polynomial identity (A) follows immediately from reworking the second term on the left-hand side of (B):

$$\sum_{j=0}^{m} \binom{m+n+1}{m-j} \frac{x^{n+1+j}}{(1-x)^{j+1}} = (1-x)^n \sum_{k=n+1}^{m+n+1} \binom{m+n+1}{k} \left(\frac{x}{1-x}\right)^k$$

$$= (1-x)^n \left(\left(1 + \frac{x}{1-x}\right)^{m+n+1} - \sum_{k=0}^{n} \binom{m+n+1}{k} \left(\frac{x}{1-x}\right)^k \right)$$

$$= \frac{1}{(1-x)^{m+1}} - \sum_{k=0}^{n} \binom{m+n+1}{k} x^k (1-x)^{n-k},$$

which yields the required conclusion.

Remark 2.1. *Identity (B) was used recently by the first author in the solution of Problem B-1347 in the Fibonacci Quarterly [8]. Also, the first author used (A) in the solution to another problem [7].*

Considering $f^{(r)}(x)$ with $0 \le r \le m$ is natural and leads to the following generalization of identity (B).

Theorem 2.1. *If m, n, r are non-negative integers with $0 \le r \le m$, then*

$$(1-x)^{r+1}x^{m-r}\sum_{j=0}^{n}\binom{m+j}{r}x^j + \sum_{j=0}^{r}\binom{m+n+1}{j}x^{m+n+1-j}(1-x)^j = \sum_{j=0}^{r}\binom{m}{j}x^{m-j}(1-x)^j.$$

3. Consequences involving the Fibonacci numbers and polynomials

In this section, we offer a selection of identities involving the Fibonacci numbers, all deduced from (A). They can be extended to Lucas numbers and, more generally, to numbers G_n satisfying $G_{n+1} = G_n + G_{n-1}$ for every integer n.

Recall that the Fibonacci and Lucas numbers F_n and L_n ($n \in \mathbb{Z}$) are defined by

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \qquad L_n = \alpha^n + \beta^n \qquad \text{(Binet formulas)}$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2 = -\alpha^{-1}$. For more information about these sequences, one can consult the On-Line Encyclopedia of Integer Sequences [27] where they are listed under the IDs A000045 and A000032, respectively.

Theorem 3.1. *Let $m, n,$ and r be integers with $m, n \ge 0$. Then,*

$$\sum_{k=0}^{n}\binom{m+k}{k}(-2)^k F_k = \sum_{k=0}^{n}\binom{m+n+1}{k}(-2)^k F_{3n-2k}, \tag{2}$$

$$\sum_{k=0}^{n}\binom{m+k}{k}(-1)^k F_{3k} = \sum_{k=0}^{n}\binom{m+n+1}{k}(-1)^k 2^{n-k} F_{2n+k}, \tag{3}$$

$$\sum_{k=0}^{n}\binom{m+k}{k}(-1)^{k(r+1)} F_{2kr} = (-1)^{rn}\sum_{k=0}^{n}\binom{m+n+1}{k}(-1)^k L_r^{n-k} F_{r(n+k)}, \tag{4}$$

$$\sum_{k=0}^{n}\binom{m+k}{k}(-1)^{rk} L_r^k F_{rk} = (-1)^{n(r+1)}\sum_{k=0}^{n}\binom{m+n+1}{k}(-1)^k L_r^k F_{r(2n-k)}. \tag{5}$$

Proof. Taking $x = -2\alpha$ in (A) and using the relation $\alpha^3 = 2\alpha + 1$, we obtain

$$\sum_{k=0}^{n}\binom{m+k}{k}(-2)^k\alpha^k = \sum_{k=0}^{n}\binom{m+n+1}{k}(-2)^k\alpha^{3n-2k}.$$

Similarly, we have

$$\sum_{k=0}^{n}\binom{m+k}{k}(-2)^k\beta^k = \sum_{k=0}^{n}\binom{m+n+1}{k}(-2)^k\beta^{3n-2k}.$$

Combining these two results readily leads to the first identity.

Taking $x = -\alpha^3$, we obtain

$$\sum_{k=0}^{n}\binom{m+k}{k}(-1)^k\alpha^{3k} = \sum_{k=0}^{n}\binom{m+n+1}{k}(-1)^k 2^{n-k}\alpha^{2n+k},$$

and a similar result holds with β instead of α. The second identity then follows from the Binet formula. To obtain the third identity, we take $x = -(-1)^r\alpha^{2r}$ in (B), so that $1 - x = (-1)^r\alpha^r L_r$. We obtain

$$\sum_{k=0}^{n}\binom{m+k}{k}(-1)^{k(r+1)}\alpha^{2rk} = (-1)^{rn}\sum_{k=0}^{n}\binom{m+n+1}{k}(-1)^k\alpha^{r(n+k)}L_r^{n-k}.$$

The third identity follows. Finally, we take $x = (-1)^r\alpha^r L_r$ and $x = (-1)^r\beta^r L_r$, and then combine according to the Binet formula. □

Let $F_n(z)$ and $L_n(z)$ denote the Fibonacci and Lucas polynomials, respectively, where $z \in \mathbb{C}$. These polynomials are defined recursively by the equations

$$F_n(z) = zF_{n-1}(z) + F_{n-2}(z), \quad n \geq 2,$$

$$L_n(z) = zL_{n-1}(z) + L_{n-2}(z), \quad n \geq 2,$$

with $F_0(z) = 0$, $F_1(z) = 1$, $L_0(z) = 2$ and $L_1(z) = z$. If $z = 1$, then $F_n(1) = F_n$ and $L_n(1) = L_n$ are the Fibonacci and Lucas numbers, respectively. If $z = 2$, then $F_n(2) = P_n$ and $L_n(2) = Q_n$ become the Pell and Pell-Lucas numbers. The polynomials possess compact forms, also known as the Binet formulas,

$$F_n(z) = \frac{\alpha^n(z) - \beta^n(z)}{\alpha(z) - \beta(z)}, \qquad L_n(z) = \alpha^n(z) + \beta^n(z), \tag{6}$$

with

$$\alpha(z) = \frac{1}{2}\left(z + \sqrt{z^2+4}\right) \quad \text{and} \quad \beta(z) = \frac{1}{2}\left(z - \sqrt{z^2+4}\right).$$

New combinatorial identities involving Fibonacci and Lucas polynomials can also be derived from (A).

Theorem 3.2. *Let m, n, t be integers with $m, n \geq 0$ and $z \in \mathbb{C}$. Then,*

$$\sum_{k=0}^{n} \binom{m+k}{k} F_{2k+t}(z) = \sum_{k=0}^{n} \binom{m+n+k}{k} (-1)^{n-k} z^{n-k} F_{n+k+t}(z) \tag{7}$$

and

$$\sum_{k=0}^{n} \binom{m+k}{k} L_{2k+t}(z) = \sum_{k=0}^{n} \binom{m+n+k}{k} (-1)^{n-k} z^{n-k} L_{n+k+t}(z). \tag{8}$$

Proof. We take $x = \alpha^2(z)$ and $x = \beta^2(z)$, in turn, in (B) and use

$$\alpha^2(z) = 1 + z\alpha(z) \qquad \text{and} \qquad \beta^2(z) = 1 + z\beta(z).$$

The desired formulas then follow from the Binet formulas (6). □

Theorem 3.3. *Let m, n, t be integers with $m, n \geq 0$ and $z \in \mathbb{C}$. Then,*

$$\sum_{k=0}^{n} \binom{m+k}{k} (-1)^k F_{2(2k+t)}(z) = \sum_{k=0}^{n} \binom{m+n+k}{k} (-1)^k (z^2+2)^{n-k} F_{2(n+k+t)}(z) \tag{9}$$

and

$$\sum_{k=0}^{n} \binom{m+k}{k} (-1)^k L_{2(2k+t)}(z) = \sum_{k=0}^{n} \binom{m+n+k}{k} (-1)^k (z^2+2)^{n-k} L_{2(n+k+t)}(z). \tag{10}$$

Proof. The proof is similar, taking $x = -\alpha^4(z)$ (so that $1 - x = (z^2+2)\alpha^2(z)$) and $x = -\beta^4(z)$. □

Theorem 3.4. *Let m, n, t be integers with $m, n \geq 0$ and $z \in \mathbb{C}$. Then,*

$$\sum_{k=0}^{n} \binom{m+k}{k} z^{n-k} F_{3k+t}(z) = \sum_{k=0}^{n} \binom{m+n+k}{k} (-1)^{n-k} (1+z^2)^{n-k} F_{n+2k+t}(z) \tag{11}$$

and

$$\sum_{k=0}^{n} \binom{m+k}{k} z^{n-k} L_{3k+t}(z) = \sum_{k=0}^{n} \binom{m+n+k}{k} (-1)^{n-k} (1+z^2)^{n-k} L_{n+2k+t}(z). \tag{12}$$

Proof. This time, we choose $x = \alpha^3(z)/z$ and $x = \beta^3(z)/z$ and use

$$\frac{\alpha^3(z)}{z} = 1 + \alpha(z)\left(\frac{1+z^2}{z}\right) \qquad \text{and} \qquad \frac{\beta^3(z)}{z} = 1 + \beta(z)\left(\frac{1+z^2}{z}\right).$$

□

Remark 3.1. *The polynomial identities can be easily extended to the broader class of Horadam sequences*

$$(w_n)_{n\in\mathbb{Z}} = (w_n(w_0, w_1; p, q)),$$

which are defined for all integers n and arbitrary complex numbers w_0, w_1, $p \neq 0$ and $q \neq 0$, by the recurrence relation

$$w_n = pw_{n-1} - qw_{n-2}, \quad n \geq 2.$$

4. New applications to combinatorial sums

In this section, we use identity (A) to prove some combinatorial identities involving ratios of binomial coefficients and harmonic numbers H_n. We start with some definitions and preliminary observations.

Harmonic numbers H_z and odd harmonic numbers O_z are defined for $0 \neq z \in \mathbb{C} \setminus \mathbb{Z}^-$ by the recurrence relations

$$H_z = H_{z-1} + \frac{1}{z} \qquad \text{and} \qquad O_z = O_{z-1} + \frac{1}{2z-1},$$

with $H_0 = 0$ and $O_0 = 0$. Harmonic numbers are connected to the digamma function through the fundamental relation

$$H_z = \psi(z+1) + \gamma, \tag{13}$$

where γ is the Euler-Mascheroni constant and $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma (or psi) function. When each z is a positive integer, say n, we have the sequences of harmonic and odd harmonic numbers H_n and O_n, respectively, and the recurrence relations give

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \qquad \text{and} \qquad O_n = \sum_{k=1}^{n} \frac{1}{2k-1}.$$

Obvious relations between harmonic numbers H_n and odd harmonic numbers O_n are given by

$$H_{2n} = \frac{1}{2}H_n + O_n \qquad \text{and} \qquad H_{2n-1} = \frac{1}{2}H_{n-1} + O_n. \tag{14}$$

Additional relations are contained in the next lemma.

Lemma 4.1. *If n is an integer, then*

$$H_{n-1/2} = 2O_n - 2\ln 2, \tag{15}$$

$$H_{n-1/2} - H_{-1/2} = 2O_n, \tag{16}$$

$$H_{n-1/2} - H_{1/2} = 2\left(O_n - 1\right), \tag{17}$$

$$H_{n+1/2} - H_{-1/2} = 2O_{n+1}, \tag{18}$$

$$H_{n+1/2} - H_{1/2} = 2\left(O_{n+1} - 1\right), \tag{19}$$

$$H_{n+1/2} - H_{n-1/2} = \frac{2}{2n+1}. \tag{20}$$

Proof. Using (13) as the definition of the harmonic numbers for all complex n (excluding zero and the negative integers), the results are consequences of a known result for the digamma function at half-integer arguments [29, Eq. (51)], namely,

$$\psi(n+1/2) = -\gamma - 2\ln 2 + 2\sum_{k=1}^{n} \frac{1}{2k-1}.$$

□

Our first result, in this section, is the following theorem.

Theorem 4.1. *Let m and n be integers with $m, n \geq 0$. Let $r, s \in \mathbb{C}$ such that $s \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ and $r \in \mathbb{C} \setminus \{-1, -2, \ldots\}$. Then,*

$$\sum_{k=0}^{n} \frac{\binom{m+k}{k}}{\binom{k+r+s}{k+r}} = s \sum_{k=0}^{n} \frac{\binom{m+n+1}{k}}{(n-k+s)\binom{n+r+s}{k+r}}. \tag{21}$$

In particular, for every $m \geq 1$, we have

$$\sum_{k=0}^{n} \frac{1}{(n-k+m)(n+1-k+m)} = \frac{n+1}{m(n+1+m)}. \tag{22}$$

Proof. We replace x by $1-x$ in (A) and multiply through by $(1-x)^r x^{s-1}$ to obtain

$$\sum_{k=0}^{n}\binom{m+k}{k}x^{s-1}(1-x)^{k+r}=\sum_{k=0}^{n}\binom{m+n+1}{k}x^{n-k+s-1}(1-x)^{k+r}.$$

It just remains to integrate both sides from zero to one using the Beta integral ([15,29])

$$\int_0^1 x^u(1-x)^v dx = B(u+1,v+1)=\frac{1}{(u+1)\binom{u+1+v}{u+1}}$$

and to simplify. The special case follows by setting $r=0$ and $s=m$ in (21) or by noticing that

$$\frac{1}{(n-k+m)(n+1-k+m)}=\frac{1}{n-k+m}-\frac{1}{n+1-k+m}.$$

□

Corollary 4.1. *Let m and n be integers with $m,n\geq 0$. Then, we have*

$$\sum_{k=0}^{n}2^{2k}\frac{\binom{m+k}{k}}{\binom{2k+1}{2k}\binom{2k}{k}}=\sum_{k=0}^{n}\frac{2^{2k}}{2(n-k)+1}\frac{\binom{m+n+1}{k}\binom{n}{k}}{\binom{2n+1}{2k}\binom{2k}{k}}. \tag{23}$$

In particular,

$$\sum_{k=0}^{n}\frac{2^{2k}}{\binom{2k+1}{2k}\binom{2k}{k}}=\sum_{k=0}^{n}\frac{2^{2k}}{2(n-k)+1}\frac{\binom{n+1}{k}\binom{n}{k}}{\binom{2n+1}{2k}\binom{2k}{k}} \tag{24}$$

and

$$\sum_{k=0}^{n}2^{2k}\frac{\binom{n+k}{k}}{\binom{2k+1}{2k}\binom{2k}{k}}=\sum_{k=0}^{n}\frac{2^{2k}}{2(n-k)+1}\frac{\binom{2n+1}{k}\binom{n}{k}}{\binom{2n+1}{2k}\binom{2k}{k}}. \tag{25}$$

Proof. After setting $r=0$ and $s=1/2$ in (21), the desired identity (23) is obtained using the following relation [14] valid for all integers $0\leq v\leq u$:

$$\binom{u+1/2}{v}=2^{-2v}\binom{2u+1}{2v}\frac{\binom{2v}{v}}{\binom{u}{v}}.$$

□

Corollary 4.2. *Let m and n be integers with $m,n\geq 0$. Then, we have*

$$\sum_{k=0}^{n}2^{2k}\frac{\binom{m+k}{k}}{\binom{2k}{k}}=\frac{1}{\binom{2n}{n}}\left(2^{2n}\binom{m+n+1}{n}-\sum_{k=0}^{n-1}\frac{2^{2k}}{2(n-k)-1}\frac{\binom{m+n+1}{k}\binom{2(n-k)}{n-k}}{\binom{n}{k}}\right). \tag{26}$$

In particular,

$$\sum_{k=0}^{n}\frac{2^{2k}}{\binom{2k}{k}}=\frac{1}{\binom{2n}{n}}\left((n+1)2^{2n}-\sum_{k=0}^{n-1}\frac{2^{2k}}{2(n-k)-1}\frac{\binom{n+1}{k}\binom{2(n-k)}{n-k}}{\binom{n}{k}}\right) \tag{27}$$

and

$$\sum_{k=0}^{n}2^{2k}\frac{\binom{n+k}{k}}{\binom{2k}{k}}=\frac{1}{\binom{2n}{n}}\left(2^{2n}\binom{2n+1}{n}-\sum_{k=0}^{n-1}\frac{2^{2k}}{2(n-k)-1}\frac{\binom{2n+1}{k}\binom{2(n-k)}{n-k}}{\binom{n}{k}}\right). \tag{28}$$

Proof. We set $r=0$ and $s=-1/2$ in (21) and use the following relation [14] valid for all integers $0\leq v\leq u$:

$$\binom{u-1/2}{v}=2^{-2v}\frac{\binom{u}{v}\binom{2u}{u}}{\binom{2(u-v)}{u-v}}.$$

□

Sums and series involving reciprocals of central binomial coefficients are famous quantities. They have been studied by many authors [10,11,21,28,30,31]. The sum on the left-hand side of (27) can be evaluated exactly as [2,28,31]

$$\sum_{k=0}^{n}\frac{2^{2k}}{\binom{2k}{k}}=\frac{1}{3}\left((n+1)\frac{2^{2n+1}}{\binom{2n}{n}}+1\right).$$

We note that it can be expressed equivalently in terms of Catalan numbers $C_n=\frac{1}{n+1}\binom{2n}{n}$:

$$\sum_{k=0}^{n}\frac{2^{2k}}{\binom{2k}{k}}=\frac{1}{3}\left(\frac{2^{2n+1}}{C_n}+1\right).$$

In the next theorem, we generalize this result.

Theorem 4.2. *Let m and n be integers with $m,n \geq 0$. Then, we have*

$$\sum_{k=0}^{n} 2^{2k}\frac{\binom{m+k}{k}}{\binom{2k}{k}} = \frac{2}{2m+3}\left((m+n+1)\frac{\binom{m+n}{n}}{\binom{2n}{n}}2^{2n} + \frac{1}{2}\right). \tag{29}$$

Consequently,

$$\sum_{k=0}^{n-1} \frac{2^{2k}}{2(n-k)-1}\frac{\binom{m+n+1}{k}\binom{2(n-k)}{n-k}}{\binom{n}{k}} = 2^{2n}\frac{m+n+1}{(m+1)(2m+3)}\binom{m+n}{n} - \frac{1}{2m+3}\binom{2n}{n}. \tag{30}$$

Proof. Fix $m \geq 0$. We prove (29) by induction on n. For $n=0$ both sides equal 1. Let the statement be true for a fixed $n \geq 1$. Then, using the obvious relations for the binomial coefficients

$$\binom{m+n+1}{n+1} = \frac{m+n+1}{n+1}\binom{m+n}{n} \quad \text{and} \quad \binom{2(n+1)}{n+1} = \frac{2(2n+1)}{n+1}\binom{2n}{n},$$

we obtain

$$\begin{aligned}
\sum_{k=0}^{n+1} 2^{2k}\frac{\binom{m+k}{k}}{\binom{2k}{k}} &= \sum_{k=0}^{n} 2^{2k}\frac{\binom{m+k}{k}}{\binom{2k}{k}} + 2^{2(n+1)}\frac{\binom{m+n+1}{n+1}}{\binom{2(n+1)}{n+1}} \\
&= \frac{2}{2m+3}\left(\frac{(n+1)\binom{m+n+1}{n+1}}{\frac{(n+1)}{2(2n+1)}\binom{2(n+1)}{n+1}}2^{2n} + \frac{2m+3}{2}2^{2(n+1)}\frac{\binom{m+n+1}{n+1}}{\binom{2(n+1)}{n+1}} + \frac{1}{2}\right) \\
&= \frac{2}{2m+3}\left(2^{2n+1}\frac{\binom{m+n+1}{n+1}}{\binom{2(n+1)}{n+1}}(2n+1+2m+3) + \frac{1}{2}\right) \\
&= \frac{2}{2m+3}\left(2^{2n+2}\frac{\binom{m+n+1}{n+1}}{\binom{2(n+1)}{n+1}}(m+n+2) + \frac{1}{2}\right),
\end{aligned}$$

and the proof is complete. The second part follows immediately by combining (29) with (26). □

For $m=0$, we obtain

$$\sum_{k=0}^{n-1} \frac{2^{2k}}{2(n-k)-1}\frac{\binom{n+1}{k}\binom{2(n-k)}{n-k}}{\binom{n}{k}} = \frac{1}{3}\left((n+1)2^{2n} - \binom{2n}{n}\right), \tag{31}$$

and for $m=n$, it holds that

$$\sum_{k=0}^{n-1} \frac{2^{2k}}{2(n-k)-1}\frac{\binom{2n+1}{k}\binom{2(n-k)}{n-k}}{\binom{n}{k}} = \frac{1}{2n+3}\binom{2n}{n}\left(\frac{2n+1}{n+1}2^{2n} - 1\right). \tag{32}$$

Theorem 4.3. *Let m and n be integers with $m,n \geq 0$. Let $r,s \in \mathbb{C}$ such that $s \in \mathbb{C}\setminus\{0,-1,-2,\ldots\}$ and $r \in \mathbb{C}\setminus\{-1,-2,\ldots\}$. Then,*

$$\sum_{k=0}^{n} \frac{\binom{m+k}{k}}{\binom{k+r+s}{s}}\left(H_{k+r} - H_{k+r+s} + H_{n+r+s}\right) = s\sum_{k=0}^{n}\frac{\binom{m+n+1}{k}}{(n-k+s)\binom{n+r+s}{n-k+s}}H_{k+r}. \tag{33}$$

In particular, for every $m \geq 1$, we have

$$\sum_{k=0}^{n} \frac{H_k}{(n-k+m)(n+1-k+m)} = \frac{(n+1)(H_{n+m}+H_{n+1}) - (m+n+1)H_{m+n+1} + mH_m}{m(m+n+1)}. \tag{34}$$

Proof. We first differentiate (21) with respect to r, with the help of

$$\frac{d}{dr}\frac{1}{\binom{k+r+s}{s}} = \frac{\psi(k+r+1) - \psi(k+s+r+1)}{\binom{k+r+s}{s}} = \frac{H_{k+r} - H_{k+s+r}}{\binom{k+r+s}{s}}$$

and

$$\frac{d}{dr}\frac{1}{\binom{n+r+s}{n-k+s}} = \frac{\psi(k+r+1) - \psi(n+s+r+1)}{\binom{n+r+s}{n-k+s}} = \frac{H_{k+r} - H_{n+s+r}}{\binom{n+r+s}{n-k+s}}.$$

Then, using (21), we simplify. The special case follows by setting $r = 0$ and $s = m$, and making use of the known summation

$$\sum_{k=0}^{n} H_{k+m} = (m+n+1)H_{m+n+1} - mH_m - (n+1) \quad (m \geq 0),$$

of which $\sum_{k=0}^{n} H_k = (n+1)(H_{n+1} - 1)$ is a special case. □

We record the summation

$$\sum_{k=0}^{n} \frac{H_k}{(n+1-k)(n+2-k)} = \frac{n}{n+2} H_{n+1}. \tag{35}$$

For the next theorem, the following lemma is needed.

Lemma 4.2. *For every $m \geq 2$,*

$$\sum_{k=0}^{n} \frac{\binom{m+k}{k}}{k+2} = \frac{1}{(m-1)m} \frac{(mn+m-(n+2))\binom{m+n+1}{n+1} + n+2}{n+2}. \tag{36}$$

Proof. Fix $m \geq 2$. We prove the statement by induction on n. For $n = 0$ both sides equal $1/2$. Let the statement be true for a fixed $n \geq 1$. Note that the right-hand side of (36) can be expressed as

$$\frac{1}{(m-1)m}\left((m-1)\binom{m+n+1}{n+1} - \frac{m}{n+2}\binom{m+n+1}{n+1} + 1\right).$$

Then,

$$\begin{aligned}
\sum_{k=0}^{n+1} \frac{\binom{m+k}{k}}{k+2} &= \sum_{k=0}^{n} \frac{\binom{m+k}{k}}{k+2} + \frac{\binom{m+n+1}{n+1}}{n+3} \\
&= \frac{1}{(m-1)m}\left(\left((m-1) - \frac{m}{n+2} + \frac{(m-1)m}{n+3}\right)\binom{m+n+1}{n+1} + 1\right) \\
&= \frac{1}{(m-1)m}\left(\left((m-1) - \frac{m}{n+2} + \frac{(m-1)m}{n+2} - \frac{(m-1)m}{(n+2)(n+3)}\right)\binom{m+n+1}{n+1} + 1\right) \\
&= \frac{1}{(m-1)m}\left(\left((m-1)\left(1 + \frac{m}{n+2}\right) - \frac{m(n+3)}{(n+2)(n+3)} - \frac{(m-1)m}{(n+2)(n+3)}\right)\binom{m+n+1}{n+1} + 1\right) \\
&= \frac{1}{(m-1)m}\left(\left((m-1)\frac{m+n+2}{n+2} - \frac{m}{n+3}\frac{m+n+2}{n+2}\right)\binom{m+n+1}{n+1} + 1\right) \\
&= \frac{1}{(m-1)m}\left((m-1)\binom{m+n+2}{n+2} - \frac{m}{n+3}\binom{m+n+2}{n+2} + 1\right),
\end{aligned}$$

as desired. □

Theorem 4.4. *The following identities hold*

$$\sum_{k=0}^{n} \frac{k+1}{n+1-k} \frac{\binom{m+n+1}{k}}{\binom{n+1}{k}} H_{k+1}$$
$$= \begin{cases} (n+2)\left(H_{n+2}^2 - H_{n+2}^{(2)} - H_{n+2} + 1\right), & \text{if } m = 0; \\ (n+2)\left((n+1)H_{n+2} - H_{n+2}^2 + H_{n+2}^{(2)}\right), & \text{if } m = 1; \\ \left((mn+m-(n+2))\binom{m+n+1}{n+1} + n+2\right)\dfrac{H_{n+2}}{(m-1)m} - (n+2)\displaystyle\sum_{k=0}^{n} \frac{\binom{m+k}{k}}{(k+2)^2}, & \text{if } m \geq 2. \end{cases} \tag{37}$$

Proof. Setting $r = s = 1$ in (33) produces

$$\sum_{k=0}^{n} \frac{k+1}{n+1-k} \frac{\binom{m+n+1}{k}}{\binom{n+1}{k}} H_{k+1} = (n+2)\left(H_{n+2}\sum_{k=0}^{n} \frac{\binom{m+k}{k}}{k+2} - \sum_{k=0}^{n} \frac{\binom{m+k}{k}}{(k+2)^2}\right).$$

The special case for $m = 0$ is easily established. For $m = 1$, we proceed similarly using

$$\sum_{k=0}^{n} \frac{k+1}{k+2} = n + 2 - H_{n+2}$$

and

$$\sum_{k=0}^{n} \frac{k+1}{(k+2)^2} = \sum_{k=0}^{n} \left(\frac{1}{k+2} - \frac{1}{(k+2)^2} \right).$$

For $m \geq 2$, we use Lemma 4.2 to complete the proof. □

The explicit expressions in cases $m = 2$ and $m = 3$ read as

$$\sum_{k=0}^{n} \frac{(k+1)H_{k+1}}{(n+1-k)(n+2-k)(n+3-k)} = \frac{1}{2(n+3)} \left(\frac{1}{2}(n^2+3n+4)H_{n+2} - n - 2 \right) \tag{38}$$

and

$$\sum_{k=0}^{n} \frac{(k+1)H_{k+1}}{(n+1-k)(n+2-k)(n+3-k)(n+4-k)}$$
$$= \frac{H_{n+2}}{6(n+3)(n+4)} \left(\frac{1}{6}(n+3)(n+4)(2n+1) + 2 \right) - \frac{1}{12}\frac{n+2}{n+4}. \tag{39}$$

The harmonic number counterpart of Corollary 23 is given in the next corollary.

Corollary 4.3. *Let m and n be integers with $m, n \geq 0$. Then, we have*

$$\sum_{k=0}^{n} 2^{2k} \frac{\binom{m+k}{k}}{(2k+1)\binom{2k}{k}} (2O_{k+1} - H_k) = \sum_{k=0}^{n} \frac{2^{2k}}{2(n-k)+1} \frac{\binom{m+n+1}{k}\binom{n}{k}}{\binom{2n+1}{2k}\binom{2k}{k}} (2O_{n+1} - H_k) \tag{40}$$

with

$$\sum_{k=0}^{n} \frac{2^{2k}}{(2k+1)\binom{2k}{k}} (2O_{k+1} - H_k) = \sum_{k=0}^{n} \frac{2^{2k}}{2(n-k)+1} \frac{\binom{n+1}{k}\binom{n}{k}}{\binom{2n+1}{2k}\binom{2k}{k}} (2O_{n+1} - H_k) \tag{41}$$

being a special case.

Proof. We set $r = 0$ and $s = 1/2$ in (33) and appeal to (18) to transform the result. □

The counterpart of Corollary 4.2 is stated next.

Corollary 4.4. *Let m and n be integers with $m, n \geq 0$. Then, we have*

$$\sum_{k=0}^{n} 2^{2k} \frac{\binom{m+k}{k}}{\binom{2k}{k}} (2O_k - H_k) = \frac{1}{\binom{2n}{n}} \left(2^{2n} \binom{m+n+1}{n} (2O_n - H_n) \right.$$
$$\left. - \sum_{k=0}^{n-1} \frac{2^{2k}}{2(n-k)-1} \frac{\binom{m+n+1}{k}\binom{2(n-k)}{n-k}}{\binom{n}{k}} (2O_n - H_k) \right) \tag{42}$$

with

$$\sum_{k=0}^{n} \frac{2^{2k}}{\binom{2k}{k}} (2O_k - H_k) = \frac{1}{\binom{2n}{n}} \left(2^{2n}(n+1)(2O_n - H_n) \right.$$
$$\left. - \sum_{k=0}^{n-1} \frac{2^{2k}}{2(n-k)-1} \frac{\binom{n+1}{k}\binom{2(n-k)}{n-k}}{\binom{n}{k}} (2O_n - H_k) \right) \tag{43}$$

being a special case.

Proof. We use (33) with $r = 0$ and $s = -1/2$ and (18). □

Using Theorem 4.2, the above relation can be expressed equivalently as

$$\sum_{k=0}^{n} 2^{2k} \frac{\binom{m+k}{k}}{\binom{2k}{k}} (2O_k - H_k) = \frac{4O_n}{2m+3} \left((m+n+1)2^{2n} \frac{\binom{m+n}{n}}{\binom{2n}{n}} + \frac{1}{2} \right)$$
$$+ \frac{1}{\binom{2n}{n}} \sum_{k=0}^{n} \frac{2^{2k}}{2(n-k)-1} \frac{\binom{m+n+1}{k}\binom{2(n-k)}{n-k}}{\binom{n}{k}} H_k. \tag{44}$$

We also note that $2O_n - H_n = 2(H_{2n} - H_n)$.

5. More convolutional-type relations

In this section, we study some convolutional-type relations involving Catalan numbers C_n and central binomial coefficients. The section is related to the paper [5] of Alzer and Nagy.

Some results about the general sum $S_d(n)=\sum_{k=0}^n 2^{-2k}\binom{2k}{k}k^d$ (where d is a non-negative integer) will be needed. This sum is the object of the following theorem and its corollary.

Theorem 5.1. *For all* $n,d\geq 0$,

$$S_0(n)=2^{-2n}(2n+1)\binom{2n}{n} \tag{45}$$

and

$$(2d+1)S_d(n)=n^d 2^{-2n}(2n+1)\binom{2n}{n}+2\sum_{k=1}^{d-1}\binom{d}{k+1}(-1)^{k+1}S_{d-k}(n). \tag{46}$$

Proof. Let $u_k=2^{-2k}\binom{2k}{k}$. From the relation $u_k=2(k+1)(u_k-u_{k+1})$ (which can be easily checked), we deduce that

$$S_0(n)=\sum_{k=0}^n u_k=2\sum_{k=0}^n (k+1)(u_k-u_{k+1})=2\Big(\sum_{k=0}^n u_k-(n+1)u_{n+1}\Big)=2S_0(n)-(2n+1)u_n.$$

Hence, (45) is obtained.

Suppose that $d\geq 1$. To calculate $S_d(n)$, we use summation by parts again:

$$S_d(n)=\sum_{k=0}^n k^d u_k=2\sum_{k=0}^n k^d(k+1)(u_k-u_{k+1})=2\sum_{k=0}^n a_{k+1}(u_k-u_{k+1})$$

where $a_k=k(k-1)^d$, so that

$$\begin{aligned}S_d(n)&=2\sum_{k=0}^n u_k(a_{k+1}-a_k)-2u_{n+1}a_{n+1}\\&=2\sum_{k=0}^n u_k\left((d+1)k^d-\sum_{j=2}^d\binom{d}{j}(-1)^j k^{d-j+1}\right)-2(n+1)n^d\frac{(2n+1)u_n}{2(n+1)}\\&=(2d+2)S_d(n)-2\sum_{j=2}^d\binom{d}{j}(-1)^jS_{d-j+1}(n)-n^d(2n+1)2^{-2n}\binom{2n}{n}.\end{aligned}$$

Therefore, (46) is readily obtained. □

Remark 5.1. *Identity (45) is already known in the literature. It is stated as Theorem 23 in [25] and as Theorem 4 in [17]. Both proofs are based on integration and apply integral representations for the Catalan numbers. It is also contained in Gould's book [14, Page 14].*

The explicit values of $S_d(n)$ for $d=1,2,3,$ are given in the next corollary.

Corollary 5.1. *We have*

$$\sum_{k=0}^n 2^{-2k}\binom{2k}{k}k=\frac{n}{3}2^{-2n}(2n+1)\binom{2n}{n}, \tag{47}$$

$$\sum_{k=0}^n 2^{-2k}\binom{2k}{k}k^2=\frac{n}{15}2^{-2n}(3n+2)(2n+1)\binom{2n}{n}, \tag{48}$$

and

$$\sum_{k=0}^n 2^{-2k}\binom{2k}{k}k^3=\frac{n}{105}2^{-2n}(15n^2+18n+2)(2n+1)\binom{2n}{n}. \tag{49}$$

Proof. For $d=1$, the sum on the right-hand side of (46) is empty, and hence,

$$3S_1(n)=n(2n+1)2^{-2n}\binom{2n}{n}.$$

From (46), we deduce that

$$5S_2(n) = 2S_1(n) + n^2(2n+1)2^{-2n}\binom{2n}{n} = n(2n+1)2^{-2n}\binom{2n}{n}\left(n+\frac{2}{3}\right).$$

Hence, (48) follows. Similarly, we have

$$7S_3(n) = 6S_2(n) - 2S_1(n) + n^3(2n+1)2^{-2n}\binom{2n}{n}.$$

Now, a simple calculation yields (49). □

Note that, for $d \geq 1$, the sum $S_d(n)$ is of the form $n(2n+1)2^{-2n}\binom{2n}{n}Q_d(n)$ for some polynomial Q_d of degree $d-1$. By (46), the polynomials $Q_d(x)$ $(d \geq 1)$ satisfy $Q_1(x) = \frac{1}{3}$ and the recursion

$$(2d+1)Q_d(x) = x^{d-1} + 2\sum_{k=1}^{d-1}\binom{d}{k+1}(-1)^{k+1}Q_{d-k}(x).$$

Our main result, in this section, is the relation given in the following theorem.

Theorem 5.2. *Let m and n be integers with $m, n \geq 0$. Then, we have*

$$\sum_{k=0}^{n} 2^{2(n-k)}\binom{m+k}{k}\binom{2k}{k} = \sum_{k=0}^{n}\frac{\binom{m+n+1}{k}}{\binom{n}{k}}\binom{2k}{k}\binom{2(n-k)}{n-k}. \tag{50}$$

Proof. We first substitute $\cos^2(x)$ for x in (A) to obtain the relation

$$\sum_{k=0}^{n}\binom{m+k}{k}\cos^{2k}(x) = \sum_{k=0}^{n}\binom{m+n+1}{k}\cos^{2k}(x)\sin^{2(n-k)}(x).$$

Now, we integrate both sides from 0 to π and invoke the known results [1]

$$\int_0^{\pi}\cos^p(x)\,dx = \begin{cases} 2^{-p}\,\pi\binom{p}{p/2}, & \text{if } p \text{ is even;} \\ 0, & \text{if } p \text{ is odd;} \end{cases}$$

and

$$\int_0^{\pi}\cos^p(x)\sin^q(x)\,dx = \begin{cases} 2^{-(p+q)}\,\pi\binom{p}{p/2}\binom{q}{q/2}\binom{(p+q)/2}{p/2}^{-1}, & \text{if } p \text{ is even;} \\ 0, & \text{if } p \text{ is odd.} \end{cases}$$

Rearranging the terms completes the proof. □

Corollary 5.2. *For every $n \geq 0$,*

$$\sum_{k=0}^{n}\binom{2(n-k)}{n-k}C_k = (2n+1)C_n. \tag{51}$$

where C_n is the nth Catalan number.

Proof. Identity (51) is obtained by setting $m = 0$ in (50) and making use of identity (45). □

Remark 5.2. *Identity (51) is not new. It is identity (b) of Lemma 2 in the paper [5], stated there in the following equivalent form:*

$$\sum_{k=0}^{n} B_k C_{n-k} = \frac{1}{2}B_{n+1},$$

where $B_n = \binom{2n}{n}$ is the central binomial coefficient, and proved by completely combinatorial arguments.

Corollary 5.3. *For every $n \geq 0$,*

$$\sum_{k=0}^{n} kC_k\binom{2(n-k)}{n-k} = 2^{2n} - (2n+1)C_n, \tag{52}$$

$$\sum_{k=0}^{n} kC_{n-k}\binom{2k}{k} = (2n+1)\binom{2n}{n} - 2^{2n}. \tag{53}$$

Proof. Identities (52) and (53) follow by combining identity (51) with the following widely known identity (see for instance [23], [5, Lemma 2 (c)] or [1, Theorem 6]):

$$\sum_{k=0}^{n} \binom{2(n-k)}{n-k}\binom{2k}{k} = 2^{2n}.$$

□

Also, the well-known recurrence relation for Catalan numbers immediately follows:

Corollary 5.4. *For every $n \geq 0$,*

$$\sum_{k=0}^{n} C_k C_{n-k} = \frac{2(2n+1)}{n+2} C_n = C_{n+1}.$$

Corollary 5.5. *For every $n \geq 0$,*

$$\sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} \binom{2(n-k)}{n-k}\binom{2k}{k} = \frac{(n+3)(2n+1)}{3(n+2)} C_n \tag{54}$$

and

$$\sum_{k=0}^{n} \frac{k}{(n+1-k)(n+2-k)} \binom{2(n-k)}{n-k}\binom{2k}{k} = \frac{n(2n+1)}{3} C_n. \tag{55}$$

Proof. First, we obtain identity (54) by setting $m = 1$ in (50) and making use of (47). Then, identity (55) is derived by combining (54) with identity (51) and noting that

$$\frac{1}{(n+2-k)(n+1-k)} - \frac{1}{n+1-k} = \frac{k-(n+1)}{(n+2-k)(n+1-k)}.$$

□

Corollary 5.6. *For every $n \geq 0$,*

$$\sum_{k=0}^{n} \frac{k}{(k+1)(k+2)} \binom{2(n-k)}{n-k}\binom{2k}{k} = \frac{n(2n+1)}{3(n+2)} C_n \tag{56}$$

and

$$\sum_{k=0}^{n} \frac{1}{k+2} \binom{2(n-k)}{n-k}\binom{2k}{k} = \frac{(2n+1)(2n+3)}{3(n+2)} C_n. \tag{57}$$

Remark 5.3. *Subtracting (54) from (57) gives*

$$\sum_{k=1}^{n} \frac{1}{k+2} \binom{2(n-k)}{n-k} \left(\binom{2k}{k} - \frac{1}{k+1}\binom{2k}{k} \right) = \frac{n(2n+1)}{3(n+2)} C_n.$$

This is Theorem 4 of Alzer and Nagy [5], stated there equivalently as

$$\sum_{k=0}^{n-1} \frac{1}{n-k+2} B_k (B_{n-k} - C_{n-k}) = \frac{n}{6(n+2)} B_{n+1}, \tag{58}$$

where $B_n = \binom{2n}{n}$ is the central binomial coefficient. By using the alternating versions, and the odd and even variants of Alzer and Nagy, more connections could be stated.

Corollary 5.7. *For every $n \geq 0$,*

$$\sum_{k=0}^{n} \frac{1}{(k+1)(k+2)(k+3)} \binom{2(n-k)}{n-k}\binom{2k}{k} = \frac{3n^2+17n+30}{30(n+2)(n+3)} (2n+1) C_n. \tag{59}$$

We also have

$$\sum_{k=0}^{n} \frac{1}{(k+1)(k+3)} \binom{2(n-k)}{n-k}\binom{2k}{k} = \frac{(7n+15)(n+4)}{30(n+2)(n+3)} (2n+1) C_n \tag{60}$$

and

$$\sum_{k=0}^{n} \frac{k+2}{(k+1)(k+3)} \binom{2(n-k)}{n-k}\binom{2k}{k} = \frac{23n^2+107n+120}{30(n+2)(n+3)} (2n+1) C_n. \tag{61}$$

Proof. To obtain identity (59), we set $m = 2$ in Theorem 5.2, simplify, and appeal to identities (48)–(45). The other two identities follow by linear combinations of (59) with (54), respectively (57), while using

$$1 - \frac{1}{k+3} = \frac{k+2}{k+3} \qquad \text{and} \quad 1 + \frac{1}{(k+1)(k+3)} = \frac{(k+2)^2}{(k+1)(k+3)}.$$

□

Corollary 5.8. *For every $n \geq 0$,*

$$\sum_{k=0}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)} \binom{2(n-k)}{n-k} \binom{2k}{k} = \frac{(n+5)(5n^2+23n+42)}{210(n+2)(n+3)(n+4)} (2n+1)C_n. \tag{62}$$

Proof. The proof is similar to the previous one. This time, we set $m = 3$ in Theorem 5.2 and use identities (49)–(45). □

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