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Research Article

On the nX-complementary generations of a Chevalley group

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Abstract

A finite non-abelian group G is said to be nX-complementary generated if, for any arbitrary non-identity element $x \in G$, there exists an element $y \in nX$ such that $G = \langle x, y \rangle$, where nX is a non-trivial conjugacy class of elements of order n. In this paper, we classify all the non-trivial conjugacy classes of the Chevalley group $G_2(4)$, determining whether they are complementary generators of $G_2(4)$ or not. We prove that the group $G_2(4)$ is nX-complementary generated if and only if $n \geq 5$ and $nX \notin \{5A, 5B\}$.

Keywords: conjugacy classes; nX-complementary generation; structure constant; Chevalley group.

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1. Introduction

The generation of finite groups is a well-studied topic in group theory and has a rich history. A finite group G can be generated in many different ways, such as probabilistic generation, $\frac{3}{2}$-generation, (p, q, r)-generation, ranks of non-trivial classes of G, nX-complementary generation, and several other methods.

A finite group G is said to be (l, m, n)-*generated* if it is generated by two elements x and y, such that the orders of x, y, and xy are l, m and n, respectively [20]. Here $[x] = lX$, $[y] = mY$ and $[z] = nZ$, where $[x]$ is the conjugacy class of lX in G containing elements of order l. The same applies to $[y]$ and $[z]$. In this case, G is also a quotient group of the triangular group $T(l, m, n)$ and, by definition of the triangular group, G is also $(\sigma(l), \sigma(m), \sigma(n))$-generated group for any $\sigma \in S_3$. Therefore, we may assume that $l \leq m \leq n$. A triple (p, q, r) is called (p, q, r)-generation of a group G if G is (p, q, r)-generated. In this case, p, q and r are prime numbers, which divide the order of the group. For a non-trivial conjugacy class nX of a finite non-abelian group G, we say that G is nX-*complementary generated* if for any non-identity element $x \in G$, there exists an element $y \in nX$ such that $G = \langle x, y \rangle$. We say that y is a complementary. The label nX is taken according to the Atlas [12]. The motivation for studying this type of generation comes from a conjecture of Brenner, Guralnick, and Wiegold [9], which states that every finite simple group can be generated by an arbitrary non-trivial element together with another suitable element.

In a series of papers [2, 13, 16–21], the nX-complementary generations of the sporadic simple groups Th, Co_1, J_1, J_2, J_3, HS, McL, Co_3, Co_2 and F_{22} have been investigated. The present authors determined all (p, q, r)-generations of the Chevalley group $G_2(3)$ (see [7]) and, in [6], they showed that the group $G_2(3)$ is nX-complementary generated if and only if $n \geq 6$ and $nX \notin \{6A, 6B\}$. In this paper, we find all nX-complementary generations of the Chevalley group $G_2(4)$, and prove that the group $G_2(4)$ is nX-complementary generated if and only if $n \geq 5$ and $nX \notin \{5A, 5B\}$. We follow the methods used in the papers [3–5].

Note that, in general, if G is a $(2, 2, n)$-generated group, then G is a dihedral group and therefore G is not simple. Also, by [10], if G is a non-abelian (l, m, n)-generated group then either $G \cong A_5$ or $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. Thus, for our purpose of establishing the nX-complementary generations of $G = G_2(4)$, the only cases we need to consider are when $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$.

The following proposition gives a criterion for G to be nX-complementary generated or not.

Proposition 1.1 (see [14]). *A finite non-abelian group G is nX-complementary generated if and only if for each conjugacy class pY of G, where p is prime, there exists a conjugacy class $t_{pY}Z$, depending on pY, such that G is $(pY, nX, t_{pY}Z)$-generated. Moreover, if G is a finite simple group then G is not $2X$-complementary generated for any conjugacy class of involutions.*

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The results obtained in the present paper for the nX-complementary generations of $G_2(4)$ are summarized in the following theorem.

Theorem 1.1. *The group $G_2(4)$ is nX-complementary generated if and only if $n \geq 5$ and $nX \notin \{5A, 5B\}$.*

2. Preliminaries

Let G be a finite group. For $k \geq 3$, let $C_1, C_2, \ldots, C_k$ (not necessarily distinct) be conjugacy classes of G with $g_1, g_2, \ldots, g_k$ being representatives for these classes, respectively.

For a fixed representative $g_k \in C_k$ and for $g_i \in C_i$, $1 \leq i \leq k-1$, denote by $\Delta_G = \Delta_G(C_1, C_2, \ldots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \ldots, g_{k-1}) \in C_1 \times C_2 \times \ldots \times C_{k-1}$ such that $g_1 g_2 \ldots g_{k-1} = g_k$. This number is known as *class algebra constant* or *structure constant*. With $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \ldots, \chi_r\}$, the number Δ_G is easily calculated from the character table of G using the following equation:

$$\Delta_G(C_1, C_2, \ldots, C_k) = \frac{\prod_{i=1}^{k-1} |C_i|}{|G|} \sum_{i=1}^{r} \frac{\chi_i(g_1)\chi_i(g_2)\ldots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{(\chi_i(1_G))^{k-2}}. \tag{1}$$

Also, for a fixed $g_k \in C_k$, we denote by $\Delta^*_G(C_1, C_2, \ldots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \ldots, g_{k-1})$ satisfying

$$g_1 g_2 \ldots g_{k-1} = g_k \quad \text{and} \quad G = \langle g_1, g_2, \ldots, g_{k-1} \rangle. \tag{2}$$

Definition 2.1. *If $\Delta^*_G(C_1, C_2, \ldots, C_k) > 0$, then the group G is said to be $(C_1, C_2, \ldots, C_k)$-generated.*

Let H be any subgroup of G containing the fixed element $g_k \in C_k$. Let $\Sigma_H(C_1, C_2, \ldots, C_k)$ be the total number of distinct tuples $(h_1, h_2, \ldots, h_{k-1})$ which are in $C_1 \times C_2 \times \ldots \times C_{k-1}$ such that

$$h_1 h_2 \ldots h_{k-1} = g_k \quad \text{and} \quad \langle h_1, h_2, \ldots, h_{k-1} \rangle \leq H. \tag{3}$$

The value of $\Sigma_H(C_1, C_2, \ldots, C_k)$ can be obtained as a sum of the structure constants $\Delta_H(c_1, c_2, \ldots, c_k)$ of H-conjugacy classes $c_1, c_2, \ldots, c_k$ such that $c_i \subseteq H \cap C_i$.

Finally, for conjugacy classes $c_1, c_2, \ldots, c_k$ of a proper subgroup H of G and a fixed $g_k \in c_k$, let $\Sigma^*_H(c_1, c_2, \ldots, c_k)$ represent the number of tuples $(h_1, h_2, \ldots, h_{k-1}) \in c_1 \times c_2 \times \ldots \times c_{k-1}$ such that $h_1 h_2 \ldots h_{k-1} = g_k$ and $\langle h_1, h_2, \ldots, h_{k-1} \rangle = H$.

When it is clear from the context which conjugacy classes of H are considered, we will use the notations $\Sigma(H)$ and $\Sigma^*(H)$ to denote $\Sigma_H(c_1, c_2, \ldots, c_k)$ and $\Sigma^*_H(c_1, c_2, \ldots, c_k)$, respectively.

Theorem 2.1. *Let G be a finite group and H be a subgroup of G containing a fixed element g such that*

$$\gcd\left(o(g), [N_G(H){:}H]\right) = 1.$$

Then, the number $h(g, H)$ of conjugates of H containing g is $\chi_H(g)$, where $\chi_H(g)$ is the permutation character of G with action on the conjugates of H. In particular,

$$h(g, H) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_{N_G(H)}(x_i)|}, \tag{4}$$

where $x_1, x_2, \ldots, x_m$ are representatives of the $N_G(H)$-conjugacy classes fused to the G-class of g.

Proof. See [17] and [15, Theorem 2.1]. □

The number $h(g, H)$, in (4), is useful in establishing a lower bound for $\Delta^*_G(C_1, C_2, \ldots, C_k)$, namely

$$\Delta^*_G(C_1, C_2, \ldots, C_k) \geq \Theta_G(C_1, C_2, \ldots, C_k),$$

where

$$\Theta_G(C_1, C_2, \ldots, C_k) = \Delta_G(C_1, C_2, \ldots, C_k) - \sum h(g_k, H)\Sigma^*_H(C_1, C_2, \ldots, C_k), \tag{5}$$

g_k is a representative of the class C_k and the sum is taken over all the representatives H of G-conjugacy classes of maximal subgroups of G containing elements of all the classes $C_1, C_2, \ldots, C_k$.

Theorem 2.2 (see [4]). *Let G be a $(2X, sY, tZ)$-generated simple group, then G is $(sY, sY, (tZ)^2)$-generated.*

Lemma 2.1 (e.g., see [1] or [11]). *Let G be a finite centerless group. If $\Delta_G(C_1, C_2, \ldots, C_k) < |C_G(g_k)|$ and $g_k \in C_k$, then $\Delta^*_G(C_1, C_2, \ldots, C_k) = 0$ and therefore, G is not $(C_1, C_2, \ldots, C_k)$-generated.*

Lemma 2.2 (see [14]). *If G is sY-complementary generated and $(rX)^n = sY$, then G is rX-complementary generated.*

Proof. Let rX and sY be non-trivial conjugacy classes of G such that $(rX)^n = sY$ for some integer n. If G is not rX-complementary generated, then there exists an element x of prime order such that $\langle x, y\rangle < G$ for all $y \in rX$. Since $x, y^n \in \langle x, y\rangle$, it follows that $\langle x, y^n\rangle \leq \langle x, y\rangle < G$ for all $y^n \in sY$. □

Theorem 2.2 and Lemma 2.2 are often useful in determining whether G is nX-complementary generated, whereas Lemma 2.1 is sometimes useful in establishing non-generation results for finite groups.

The following result is due to Scott (see [11] and [22]).

Theorem 2.3 (Scott's Theorem). *Let $g_1, g_2, \ldots, g_s$ be elements generating a group G such that $g_1 g_2 \ldots g_s = 1_G$ and $\mathbb{V}$ be an irreducible module for G with $\dim \mathbb{V} = n \geq 2$. Let $C_{\mathbb{V}}(g_i)$ denote the fixed point space of $\langle g_i\rangle$ on $\mathbb{V}$ and let d_i be the codimension of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$. Then $\sum_{i=1}^{s} d_i \geq 2n$.*

With χ being the ordinary irreducible character afforded by the irreducible module $\mathbb{V}$ and $\mathbf{1}_{\langle g_i\rangle}$ being the trivial character of the cyclic group $\langle g_i\rangle$, the codimension d_i of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$ can be computed using the following formula (see [14]):

$$d_i = \dim(\mathbb{V}) - \dim(C_{\mathbb{V}}(g_i)) = \dim(\mathbb{V}) - \left\langle \chi\downarrow^{G}_{\langle g_i\rangle}, \mathbf{1}_{\langle g_i\rangle} \right\rangle$$
$$= \chi(1_G) - \frac{1}{|\langle g_i\rangle|} \sum_{j=0}^{o(g_i)-1} \chi(g_i^j). \quad (6)$$

3. Results on the nX-complementary generations of $G_2(4)$

In this section, we apply the results discussed in Section 2 to the group $G_2(4)$. Throughout this section, we assume that $G = G_2(4)$. We determine the non-trivial conjugacy classes nX such that G is nX-complementary generated.

The group G is a simple group of order $251596800 = 2^{12} \times 3^3 \times 5^2 \times 7 \times 13$. According to the Atlas [12], the group G has exactly 32 conjugacy classes of its elements and 8 conjugacy classes of its maximal subgroups. Representatives of these classes of maximal subgroups are given in Table 3.1. This table and Table 3.2 are prepared using information given in [12].

Table 3.1: Maximal subgroups of $G_2(4)$.

Maximal Subgroup	Order
$J_2 = M_1$	$604800 = 2^7 \times 3^3 \times 5^2 \times 7$
$2^{2+8}{:}(A_5 \times 3) = M_2$	$184320 = 2^{12} \times 3^2 \times 5$
$2^{4+6}{:}(A_5 \times 3) = M_3$	$184320 = 2^{12} \times 3^2 \times 5$
$U_3(4){:}2 = M_4$	$124800 = 2^7 \times 3 \times 5^2 \times 13$
$3^{\cdot}L_3(4){:}2_3 = M_5$	$120960 = 2^7 \times 3^3 \times 5 \times 7$
$U_3(3){:}2 = M_6$	$12096 = 2^6 \times 3^3 \times 7$
$A_5 \times A_5 = M_7$	$3600 = 2^4 \times 3^2 \times 5^2$
$L_2(13) = M_8$	$1092 = 2^2 \times 3 \times 7 \times 13$

Using Equation (4), we calculated the values of $h(g, M_i)$, where g is a representative of a non-trivial conjugacy class of G and over all the maximal subgroups M_i of G. We list these values in Table 3.2.

If $2X$ is a class of involutions of non-abelian finite simple group H then H is not $2X$-complementary generated; if it is, then H would be $(2Y, 2X, t_{2Y}Z)$-generated, but this would mean that H is a dihedral group, which would contradict the fact that H is a simple group. Thus, in the process of investigating whether a group H is nX-complementary generated or not, we consider classes nX of H with $n \geq 3$.

Table 3.2: The values $h(g, M_i),\ 1 \le i \le 8$, for non-identity classes and maximal subgroups of $G_2(4)$.

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
$2A$	32	21	85	96	160	320	256	0
$2B$	16	25	21	32	32	80	256	320
$3A$	56	105	42	0	1	280	336	0
$3B$	5	12	15	6	10	10	21	30
$4A$	16	13	5	0	16	64	0	0
$4B$	0	1	5	0	16	16	0	0
$4C$	0	5	5	16	0	0	0	0
$5A$	1	0	5	6	10	0	13	0
$5B$	1	0	5	6	10	0	13	0
$5C$	6	5	0	1	0	0	13	0
$5D$	6	5	0	1	0	0	13	0
$6A$	8	9	10	0	1	8	16	0
$6B$	1	4	3	2	2	2	1	2
$7A$	3	0	0	0	1	3	0	9
$8A$	4	1	1	0	4	8	0	0
$8B$	0	1	1	4	0	0	0	0
$10A$	2	1	0	1	0	0	1	0
$10B$	2	1	0	1	0	0	1	0
$10C$	1	0	1	2	2	0	1	0
$10D$	1	0	1	2	2	0	1	0
$12A$	4	1	2	0	1	4	0	0
$12B$	0	1	2	0	1	4	0	0
$12C$	0	1	2	0	1	4	0	0
$13A$	0	0	0	1	0	0	0	1
$13B$	0	0	0	1	0	0	0	1
$15A$	1	0	2	0	1	0	1	0
$15B$	1	0	2	0	1	0	1	0
$15C$	0	2	0	1	0	0	1	0
$15D$	0	2	0	1	0	0	1	0
$21A$	0	0	0	0	1	0	0	0
$21B$	0	0	0	0	1	0	0	0

Proposition 3.1. *The group $G = G_2(4)$ is not $3A$-complementary generated.*

Proof. For the case $(2A, 3A, tZ)$, we need only to check the conjugacy classes of $G_2(4)$ with elements of orders greater than or equal to 7 because of the condition $\frac{1}{l}+\frac{1}{m}+\frac{1}{n} < 1$. From Proposition 7 of [8], we know that $G_2(4)$ is not $(2A, 3A, 7A)$-generated. Computations with the GAP system [23] yield $\Delta_{G_2(4)}(2A, 3A, tZ) = 0$ for $tZ \in S = \{8X, 10Y, 12B, 12C, 13X, 15Y, 21X\}$, where $X \in \{A, B\}$ and $Y \in \{A, B, C, D\}$. Thus, $\Delta^*_{G_2(4)}(2A, 3A, tZ) = 0$ for all $tZ \in S$. For the remaining class $12A$, we obtain $\Delta_{G_2(4)}(2A, 3A, 12A) = 3 < 48 = |C_G(g)|,\ g \in 12A$, and by Lemma 2.1, we must have $\Delta^*_{G_2(4)}(2A, 3A, 12A) = 0$. Since we have $\Delta^*_{G_2(4)}(2A, 3A, tZ) = 0$ for every conjugacy classes tZ, it follows that $G_2(4)$ is not $(2A, 3A, tZ)$-generated for every tZ. Hence, $G_2(4)$ is not $3A$-complementary generated. □

The group $G_2(4)$ has 78-dimensional complex irreducible module $\mathbb{V}$. For any conjugacy class nX, let

$$d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX))$$

denote the codimension of the fixed space (in $\mathbb{V}$) of a representative of nX. Using Equation (6) together with the power maps associated with the character table of $G_2(4)$ given in the Atlas [12], we compute the values of d_{nX} for some non-trivial classes nX of G, with respect to $\mathbb{V}$. We list these values in Table 3.3.

Table 3.3: $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX)),\ nX$ is a non-trivial class of G and $\dim(\mathbb{V}) = 78$.

nX	$7A$	$8A$	$8B$	$10A$	$10B$	$10C$	$10D$	$12A$	$12B$	$12C$	$15A$	$15B$
d_{nX}	66	66	66	68	68	70	70	66	68	68	70	70

The values of the codimension of the fixed space, given in Table 3.3, help us in determining the non-generation in the proof of the next proposition.

Proposition 3.2. *The group $G = G_2(4)$ is not $3B$-complementary generated.*

Proof. The group $G_2(4)$ cannot be generated by the triples $(2A, 3B, nX)$ for $n \in \{2, 3, 4, 5, 6\}$ since each of these triples violate the condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$.

Let $S = \{7A, 8A, 8B, 10A, 10B, 10C, 10D, 12A, 12B, 12C, 15A, 15B\}$. Here, we use Scott's Theorem to show that $G_2(4)$ is not $(2A, 3B, nX)$-generated for any $nX \in S$. Using Equation (6), we have $d_{2A} = 32$ and $d_{3B} = 52$. If $G_2(4)$ is $(2A, 3B, nX)$-generated group for any non-trivial class nX of $G_2(4)$ in S, then we must have $d_{2A} + d_{3B} + d_{nX} \geq 2 \times 78$. However, it is clear from Table 3.3 that $d_{2A} + d_{3B} + d_{nX} < 156$ for every nX in S. Thus, $G_2(4)$ is not $(2A, 3B, nX)$-generated group for any $nX \in S$.

By Proposition 8 of [8], $G_2(4)$ is not generated by the triple $(2A, 3B, 13X)$ for $X \in \{A, B\}$. Next, we handle the final cases: $(2A, 3B, 15X)$ with $X \in \{C, D\}$ and $(2A, 3B, 21Y)$ with $Y \in \{A, B\}$. By Table 3.2, only the maximal subgroups M_2, M_4 and M_7 have a nonempty intersection with the triple $(2A, 3B, 15X)$ for $X \in \{C, D\}$. Computations with the GAP system yield $\Delta_{G_2(4)}(2A, 3B, 15X) = 15$, $\Sigma(M_4) = 15$ and $\Sigma(M_2) = 0 = \Sigma(M_7)$. Further computations reveal that $\Sigma(M_{41}) = 15$, where $M_{41} \cong PSU(3, 4)$ is a maximal subgroup of M_4 with a nonempty intersection with the triple $(2A, 3B, 15X)$. For maximal subgroups of M_{41}, we have $\Sigma(M_{411}) = 0 = \Sigma(M_{412})$ ($M_{411} \cong (2^{2 \cdot} 2^4){:}15$ and $M_{412} \cong 5 \times A_5$), whereas $M_{413} \cong 5^2{:}S_3$ and $M_{414} \cong 13{:}3$ have no classes of elements of order 15. Thus, $\Sigma^*(M_{41}) = \Sigma(M_{41}) = 15$ and $\Sigma^*(M_4) = \Sigma(M_4) - 1 \cdot \Sigma^*(M_{41}) = 0$. Now,

$$\Delta^*_{G_2(4)}(2A, 3B, 15X) = \Delta_{G_2(4)}(2A, 3B, 15X) - 1 \cdot \Sigma^*(M_4) - 1 \cdot \Sigma^*(M_{41}) = 15 - 0 - 15 = 0.$$

Since $\Delta^*_{G_2(4)}(2A, 3B, 15X) = 0$, it follows from Lemma 2.1 that $G_2(4)$ is not generated by the triple $(2A, 3A, 15X)$ for any $X \in \{C, D\}$.

Finally, for $(x, y) \in 2A \times 3B$ and a fixed $g \in 21X$, with $X \in \{A, B\}$, the computations with the GAP system reveal that

$$\left|\left\{(x, y) \in 2A \times 3B : \ xy = g\right\}\right| = 21;$$

i.e., $\Delta_{G_2(4)}(2A, 3B, 21X) = 21$. Hence, all the 21 pairs generate groups isomorphic to the group $3 \times PSL(3, 2)$, which is clearly of order 504. Thus, none of the 21 triples generates the entire group $G_2(4)$. Consequently, we deduce that $G_2(4)$ is not $(2A, 3B, 21X)$-generated for $X \in \{A, B\}$. □

Proposition 3.3. *The group $G = G_2(4)$ is not $4A$-complementary generated.*

Proof. For the case $(2A, 4A, nX)$, we need only to check the conjugacy classes of $G_2(4)$ with elements of orders greater than or equal to 5 because of the condition $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. Direct computations yield $\Delta_G(2A, 4A, nX) = 0$ for all non-trivial classes nX, $n \geq 5$, of $G_2(4)$ except for $nX \in \{5X, 7A, 8X, 12A, 15X, 21X\}$ with $X \in \{A, B\}$. For these remaining cases, we have

$$\begin{aligned}
\Delta_G(2A, 4A, 5X) &= 25 < 300 = |C_G(g)|, \ g \in 5X, \ \ X \in \{A, B\},\\
\Delta_G(2A, 4A, 7A) &= 7 < 21 = |C_G(g)|, \ g \in 7A,\\
\Delta_G(2A, 4A, 8A) &= 4 < 32 = |C_G(g)|, \ g \in 8A,\\
\Delta_G(2A, 4A, 8B) &= 8 < 32 = |C_G(g)|, \ g \in 8B,\\
\Delta_G(2A, 4A, 12A) &= 16 < 48 = |C_G(g)|, \ g \in 12A,\\
\Delta_G(2A, 4A, 15X) &= 5 < 15 = |C_G(g)|, \ g \in 15X, \ \ X \in \{A, B\},\\
\Delta_G(2A, 4A, 21X) &= 7 < 21 = |C_G(g)|, \ g \in 21X, \ \ X \in \{A, B\}.
\end{aligned}$$

Hence, using Lemma 2.1, we deduce that $G_2(4)$ is not $(2A, 4A, nX)$-generated for any conjugacy class nX of $G_2(4)$. It follows that $G_2(4)$ is not a $4A$-complementary generated group. □

Proposition 3.4. *The group $G = G_2(4)$ is not $4B$-complementary generated.*

Proof. The group $G_2(4)$ cannot be generated by any triple $(2A, 4B, nX)$ for $n < 5$ because such a triple violates the condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. Direct computations yield $\Delta_G(2A, 4B, nX) = 0$ for every $nX \in \{5X, 6A, 8Y, 10Y, 12A, 13Y, 15X\}$, where $X \in \{C, D\}$ and $Y \in \{A, B\}$.

For the remaining cases, we have

$$\begin{aligned}
\Delta_G(2A,4B,5X) &= 50 < 300 = |C_G(g)|,\ g \in 5X, X \in \{A,B\},\\
\Delta_G(2A,4B,6B) &= 6 < 12 = |C_G(g)|,\ g \in 6B,\\
\Delta_G(2A,4B,7A) &= 14 < 21 = |C_G(g)|,\ g \in 7A,\\
\Delta_G(2A,4B,10X) &= 5 < 20 = |C_G(g)|,\ g \in 10X, X \in \{C,D\},\\
\Delta_G(2A,4B,12X) &= 16 < 48 = |C_G(g)|,\ g \in 12X, X \in \{B,C\},\\
\Delta_G(2A,4B,15X) &= 10 < 15 = |C_G(g)|,\ g \in 15X, X \in \{A,B\},\\
\Delta_G(2A,4B,21X) &= 14 < 21 = |C_G(g)|,\ g \in 21X, X \in \{A,B\}.
\end{aligned}$$

Hence, using Lemma 2.1, we deduce that $G_2(4)$ is not a $(2A,4B,nX)$-generated group for any conjugacy class nX of $G_2(4)$. Consequently, it follows that $G_2(4)$ is not $4B$-complementary generated. □

Proposition 3.5. *The group $G = G_2(4)$ is not $4C$-complementary generated.*

Proof. Again, due to the condition $\frac{1}{l}+\frac{1}{m}+\frac{1}{n} < 1$, we accomplish the result by dealing with the case $(2A,4C,nX)$ for $n \geq 5$. Direct computations with the GAP system yield $\Delta_G(2A,4C,nX) = 0$ for the classes $nX \in S = \{5Y,6A,7A,12Z,15W,21W\}$, where $Y \in \{A,B,C,D\}$, $Z \in \{A,B,C\}$ and $W \in \{A,B\}$. For the classes $nX \in R = \{6B,8A,8B,10C,10D\}$, we obtain

$$\begin{aligned}
\Delta_G(2A,4C,6B) &= 6 < 12 = |C_G(g)|,\ g \in 6B,\\
\Delta_G(2A,4C,8A) &= 4 < 32 = |C_G(g)|,\ g \in 8A,\\
\Delta_G(2A,4C,8B) &= 8 < 32 = |C_G(g)|,\ g \in 8B,\\
\Delta_G(2A,4C,10X) &= 10 < 20 = |C_G(g)|,\ g \in 10X,\ X \in \{C,D\}.
\end{aligned}$$

Consequently, we have $\Delta^*(2A,4C,nX) = 0$ for $nX \in R$. Now, by applying Lemma 2.1, we deduce that $G_2(4)$ is not a $(2A,4C,nX)$-generated group for any conjugacy class $nX \in S \cup R$. For the remaining classes, we have three cases.

Case $(2A,4C,10X)$ with $X \in \{A,B\}$: Only the maximal subgroups M_2 and M_4 have a nonempty intersection with the triple $(2A,4C,10X)$ for $X \in \{A,B\}$. We obtain $\Delta_G(2A,4C,10X) = 20$, $\Sigma(M_2) = 0$ and $\Sigma(M_4) = 20$. Now,

$$\Delta^*_G(2A,4C,10X) = \Delta_G(2A,4C,10X) - 1 \cdot \Sigma(M_2) - 1 \cdot \Sigma(M_4) = 20 - 0 - 20 = 0.$$

Case $(2A,4C,13X)$ with $X \in \{A,B\}$: Only the maximal subgroup M_4 has a nonempty intersection with the triple $(2A,4C,13X)$ for $X \in \{A,B\}$. We obtain $\Delta_G(2A,4C,13X) = 13$ and $\Sigma(M_4) = 13$. But,

$$\Delta^*_G(2A,4C,13X) = \Delta_G(2A,4C,13X) - 1 \cdot \Sigma(M_4) = 13 - 13 = 0.$$

Case $(2A,4C,15X)$ with $X \in \{C,D\}$: Only the maximal subgroups M_2 and M_4 have a nonempty intersection with the triple $(2A,4C,15X)$ for $X \in \{C,D\}$. We obtain $\Delta_G(2A,4C,15X) = 15$, $\Sigma(M_2) = 0$ and $\Sigma(M_4) = 15$. However,

$$\Delta^*_G(2A,4C,15X) = \Delta_G(2A,4C,15X) - 2 \cdot \Sigma(M_2) - 1 \cdot \Sigma(M_4) = 15 - 0 - 15 = 0.$$

In all three cases, we have $\Delta^*(G) = 0$. Hence, $G_2(4)$ is not $(2A,4C,nX)$-generated for $nX \in \{10X,13X,15Y\}$, $X \in \{A,B\}$ and $Y \in \{C,D\}$. Therefore, based on the above discussion, $G_2(4)$ is not $4C$-complementary generated. □

Proposition 3.6. *The group $G = G_2(4)$ is not $5X$-complementary generated for $X \in \{A,B\}$.*

Proof. Since the proofs for the cases $X = A$ and $X = B$ are very similar to each other, we give the proof only for the case $X = A$. Due to the condition $p \leq q \leq r$, we only consider classes nX with $n \geq 5$ to show that G is not $(2A,5A,nX)$-generated. Direct computations with the GAP system yield $\Delta_G(2A,5A,nX) = 0$ for the classes $nX \in S = \{6A,8A,10B,10C\}$. For the classes $nX \in R = \{10A,12A,12B,12C,15C\}$, we obtain

$$\begin{aligned}
\Delta_G(2A,5A,10A) &= 4 < 20 = |C_G(g)|,\ g \in 10A,\\
\Delta_G(2A,5A,12X) &= 16 < 48 = |C_G(g)|,\ g \in 12X,\ X \in \{A,B,C\},\\
\Delta_G(2A,5A,15C) &= 3 < 15 = |C_G(g)|,\ g \in 15C.
\end{aligned}$$

Now, by applying Lemma 2.1, we deduce that $G_2(4)$ is not a $(2A,5A,nX)$-generated group for any conjugacy class $nX \in S \cup R$. Propositions 9, 12 and 13 of [8] show that $G_2(4)$ is not $(2A,5A,nX)$-generated for $nX \in V = \{5X,7A,13A,13B\}$ with $X \in \{A,B,C,D\}$.

For the remaining nX classes, we let $U = \{6B, 8B, 10D, 15A, 15B, 15D, 21A, 21B\}$. For $nX \in U$, we find that

- $\Delta_G(2A, 5A, 6B) = 12$ and these 12 pairs generate groups isomorphic to $(2^4{:}A_5){:}2$ of order 1920,
- $\Delta_G(2A, 5A, 8B) = 32$ and these 32 pairs generate groups isomorphic to $2^{4\cdot}(2^4{:}D_{10})$ of order 2560,
- $\Delta_G(2A, 5A, 10D) = 20$ and these 20 pairs generate groups isomorphic to $(2^4{:}A_5){:}2$ of order 1920,
- $\Delta_G(2A, 5A, 15A) = 15$ and these 15 pairs generate groups isomorphic to $SL(3,4)$ of order 60480,
- $\Delta_G(2A, 5A, 15B) = 15$ and these 15 pairs generate groups isomorphic to $3^{\cdot}A_6$ of order 1080,
- $\Delta_G(2A, 5A, 15D) = 15$ and these 15 pairs generate groups isomorphic to $PSU(3,4)$ of order 62400,
- $\Delta_G(2A, 5A, 21Y) = 21$ and these 21 pairs generate groups isomorphic to $SL(3,4)$ of order 60480 for $Y \in \{A, B\}$.

We observe that none of the generated groups has order equal to the order of the group $G_2(4)$. Hence, we conclude that $G_2(4)$ cannot be generated by the triple $(2A, 5A, nX)$ for any $nX \in U$. Since we have shown that $G_2(4)$ is not $(2A, 5A, nX)$-generated for all non-trivial conjugacy classes nX of $G_2(4)$, it follows that $G_2(4)$ is not $5A$-complementary generated. □

For the purpose of applying Proposition 1.1, let $T = \{2A, 2B, 3A, 3B, 5A, 5B, 5C, 5D, 7A, 13A, 13B\}$ be a set of all non-trivial conjugacy classes of $G_2(4)$ of elements of prime orders. The set T is used in the proofs of Propositions 3.7 to 3.14 as well as Propositions 3.16 and 3.17.

Proposition 3.7. *The group $G = G_2(4)$ is $5Y$-complementary generated for $Y \in \{C, D\}$.*

Proof. Let $Y \in \{C, D\}$. We achieve the result by showing that G is $(pX, 5Y, tZ)$-generated for all conjugacy classes $pX \in T$ and some non-trivial conjugacy class tZ of G. We observe from Table 3.2 that for every $pX \in T$, all the maximal subgroups of G have an empty intersection with the triple $(pX, 5Y, 21A)$. Thus, $\Delta^*_G(pX, 5Y, 21A) = \Delta_G(pX, 5Y, 21A)$. Now, direct computations with the GAP system yield

- $\Delta_G(2A, 5Y, 21A) = 21$,
- $\Delta_G(2B, 5Y, 21A) = 84$,
- $\Delta_G(3A, 5Y, 21A) = 0$,
- $\Delta_G(3B, 5Y, 21A) = 4977$,
- $\Delta_G(5X, 5Y, 21A) = 3129$, $X \in \{A, B\}$,
- $\Delta_G(5X, 5Y, 21A) = 2037$, $X \in \{C, D\}$,
- $\Delta_G(7A, 5Y, 21A) = 35385$,
- $\Delta_G(13X, 5Y, 21A) = 64449$ $X \in \{A, B\}$.

Since all of the structure constants calculated above are greater than zero except when $pX = 3A$, it follows that the inequality $\Delta^*_G(pX, 5Y, 21A) > 0$ holds for every $pX \in T$ except $pX = 3A$. Though $\Delta_G(3A, 5Y, 21A) = 0$, we find that $\Delta_G(3A, 5Y, 13A) = 13$. Again, from Table 3.2, we see that no maximal subgroup of G has a nonempty intersection with the triple $(3A, 5Y, 13A)$. It follows that $\Delta^*_G(3A, 5Y, 13A) = \Delta_G(3A, 5Y, 13A) = 13$, showing generation by this triple. Since G is $(3A, 5Y, 13A)$-generated and $(pX, 5Y, 21A)$-generated for all $pX \in T \setminus \{3A\}$, we conclude that G is $5Y$-complementary generated for $Y \in \{C, D\}$. □

Proposition 3.8. *The group $G = G_2(4)$ is $6A$-complementary generated.*

Proof. We show that G is $(pX, 6A, 13A)$-generated for all conjugacy classes $pX \in T$. We observe from Table 3.2 that no maximal subgroup of G has a nonempty intersection with the triple $(pX, 6A, 21A)$ for any $pX \in T$. Therefore, we have $\Delta^*_G(pX, 6A, 13A) = \Delta_G(pX, 6A, 13A)$. Direct computations with the GAP system show that

- $\Delta_G(2A, 6A, 13A) = 13$,
- $\Delta_G(2B, 6A, 13A) = 312$,
- $\Delta_G(3A, 6A, 13A) = 13$,

- $\Delta_G(3B, 6A, 13A) = 6968$,
- $\Delta_G(5X, 6A, 13A) = 4368$, $X \in \{A, B, C, D\}$,
- $\Delta_G(7A, 6A, 13A) = 61984$,
- $\Delta_G(13X, 6A, 13A) = 102336$ $X \in \{A, B\}$.

We observe that all the structure constants are positive. It follows that the inequality $\Delta^*_G(pX, 6A, 21A) > 0$ holds, and hence, G is $(pX, 6A, 13A)$-generated for every $pX \in T$. Therefore, G is $6A$-complementary generated. □

Proposition 3.9. *The group $G = G_2(4)$ is $6B$-complementary generated.*

Proof. We show that G is $(pX, 6B, 21A)$-generated for all conjugacy classes $pX \in T$. We observe from Table 3.2 that only the maximal subgroup M_5 has a nonempty intersection with the triple $(pX, 6B, 21A)$ for all $pX \in T$ except when $pX \in \{5C, 5D, 13A, 13B\}$ and the corresponding value of h is 1.

For the case $(2A, 6B, 21A)$, we have $\Delta_G(2A, 6B, 21A) = 336$ and $\Sigma^*(M_5) = \Sigma(M_5) = 0$. Therefore,

$$\Delta^*_G(2A, 6B, 21A) = \Delta_G(2A, 6B, 21A) - 1 \cdot \Sigma^*(M_5) = 336 - 0 = 336,$$

showing that G is $(2A, 6B, 21A)$-generated.

For the case $(2B, 6B, 21A)$, we have $\Delta_G(2B, 6B, 21A) = 5715$ and $\Sigma(M_5) = 357$. Therefore,

$$\Delta^*_G(2B, 6B, 21A) \geq \Delta_G(2B, 6B, 21A) - 1 \cdot \Sigma(M_5) = 5715 - 1(357) = 5358,$$

and thus, G is $(2B, 6B, 21A)$-generated.

For the case $(3A, 6B, 21A)$, we have $\Delta_G(3A, 6B, 21A) = 420$ and $\Sigma^*(M_5) = \Sigma(M_5) = 0$. Therefore,

$$\Delta^*_G(3A, 6B, 21A) = \Delta_G(3A, 6B, 21A) - 1 \cdot \Sigma^*(M_5) = 420 - 0 = 420,$$

showing that G is $(3A, 6B, 21A)$-generated.

For the case $(3B, 6B, 21A)$, we have $\Delta_G(3B, 6B, 21A) = 114240$ and $\Sigma^*(M_5) = \Sigma(M_5) = 0$. Therefore,

$$\Delta^*_G(3B, 6B, 21A) = \Delta_G(3B, 6B, 21A) - 1 \cdot \Sigma^*(M_5) = 114240 - 0 = 114240,$$

and thus, G is $(3B, 6B, 21A)$-generated.

For the case $(5X, 6B, 21A)$, we have $\Delta_G(5X, 6B, 21A) = 69888$ and $\Sigma^*(M_5) = \Sigma(M_5) = 0$. Therefore,

$$\Delta^*_G(5X, 6B, 21A) = \Delta_G(5X, 6B, 21A) - 1 \cdot \Sigma^*(M_5) = 69888 - 0 = 69888,$$

showing that G is $(5X, 6B, 21A)$-generated for $X \in \{A, B\}$.

For the case $(7A, 6B, 21A)$, we have $\Delta_G(7A, 6B, 21A) = 1032192$ and $\Sigma^*(M_5) = \Sigma(M_5) = 0$. Therefore,

$$\Delta^*_G(7A, 6B, 21A) = \Delta_G(7A, 6B, 21A) - 1 \cdot \Sigma^*(M_5) = 1032192 - 0 = 1032192,$$

and thus, G is $(7A, 6B, 21A)$-generated.

For the remaining cases $5C, 5D, 13A, 13B$ in T, we note the lack of the required fusion. So, the generation arises because $\Delta^*_G(5X, 6B, 21A) = \Delta_G(5X, 6B, 21A) = 75264$ for $X \in \{C, D\}$ and $\Delta^*_G(13X, 6B, 21A) = \Delta_G(13X, 6B, 21A) = 1612800$ for $X \in \{A, B\}$. Since G is $(pX, 6B, 21A)$-generated for all the conjugacy classes $pX \in T$, it follows from Proposition 1.1 that $G_2(4)$ is $6B$-complementary generated. □

Proposition 3.10. *The group $G = G_2(4)$ is $7A$-complementary generated.*

Proof. It is shown in Propositions 15, 30, 41 and 43 of [8] that the group G is $(pX, 7A, 13A)$-generated for all $pX \in T$ except for $pX \in \{13A, 13B\}$. We complete the proof by showing that G is $(13X, 7A, 21A)$-generated for $X \in \{A, B\}$. We note from Table 3.2 that no maximal subgroup is involved in the calculations of $\Delta_G(13X, 7A, 21A)$. Hence,

$$\Delta^*_G(13X, 7A, 21A) = \Delta_G(13X, 7A, 21A) = 927969,$$

which yields the required result. □

Proposition 3.11. *The group $G = G_2(4)$ is $8A$-complementary generated.*

Proof. Here, we show that G is $(pX, 8A, 13A)$-generated for every $pX \in T$. Using the GAP system, we obtain the following structure constants:

- $\Delta_G(2A, 8A, 13A) = 104$,
- $\Delta_G(2B, 8A, 13A) = 1872$,
- $\Delta_G(3A, 8A, 13A) = 104$,
- $\Delta_G(3B, 8A, 13A) = 42432$,
- $\Delta_G(5X, 8A, 13A) = 26624$, $X \in \{A, B, C, D\}$,
- $\Delta_G(7A, 8A, 13A) = 374400$,
- $\Delta_G(13A, 8A, 13A) = 612352$,
- $\Delta_G(13B, 8A, 13A) = 610688$.

We observe that $\Delta_G(pX, 8A, 13A) > 0$ for every $pX \in T$. From Table 3.2, we conclude that no maximal subgroup of G contains elements from $8A$ and $13A$. Therefore, maximal subgroups of G make no contribution in the calculations of $\Delta^*_G(pX, 8A, 13A)$. Hence, $\Delta^*_G(pX, 8A, 13A) = \Delta_G(pX, 8A, 13A) > 0$. Therefore, G is $(pX, 8A, 13A)$-generated for all the conjugacy classes $pX \in T$. Therefore, it follows from Proposition 1.1 that G is $8A$-complementary generated. □

Proposition 3.12. *The group $G = G_2(4)$ is $8B$-complementary generated.*

Proof. We show that G is $(pX, 8B, 21A)$-generated for every $pX \in T$. Again, from Table 3.2, we note that no maximal subgroup makes any contribution in the calculations of $\Delta^*_G(pX, 8B, 21A)$ for every $pX \in T$. Consequently, we have $\Delta^*_G(pX, 8B, 21A) = \Delta_G(pX, 8B, 21A)$. Using the GAP system, we have

- $\Delta_G(2A, 8B, 21A) = 168$,
- $\Delta_G(2B, 8B, 21A) = 1512$,
- $\Delta_G(3A, 8B, 21A) = 126$,
- $\Delta_G(3B, 8B, 21A) = 43680$,
- $\Delta_G(5X, 8B, 21A) = 26624$, $X \in \{A, B\}$,
- $\Delta_G(5X, 8B, 21A) = 26208$, $X \in \{C, D\}$,
- $\Delta_G(7A, 8B, 21A) = 373632$,
- $\Delta_G(13X, 8B, 21A) = 604800$, $X \in \{A, B\}$.

We note that $\Delta^*_G(pX, 8B, 21A) > 0$ for every $pX \in T$. Consequently, G is $8B$-complementary generated. □

Proposition 3.13. *The group $G = G_2(4)$ is $10Y$-complementary generated for $Y \in \{A, B\}$.*

Proof. We prove the result by establishing generation of G by the triples $(pX, 10Y, 21A)$ for all $pX \in T$ and $Y \in \{A, B\}$. As in the proof of Proposition 3.12, we notice from Table 3.2 that no maximal subgroup of G makes any contribution in the calculations of $\Delta^*_G(pX, 10Y, 21A)$ for all $pX \in T$ and $Y \in \{A, B\}$. Hence, $\Delta^*_G(pX, 10Y, 21A) = \Delta_G(pX, 10Y, 21A)$. Direct computations with the GAP system yield the following:

- $\Delta_G(2A, 10Y, 21A) = 168$,
- $\Delta_G(2B, 10Y, 21A) = 3276$,
- $\Delta_G(3A, 10Y, 21A) = 105$,
- $\Delta_G(3B, 10Y, 21A) = 70896$,
- $\Delta_G(5X, 10Y, 21A) = 40320$, $X \in \{A, B\}$,

- $\Delta_G(5X, 10Y, 21A) = 37044$, $X \in \{C, D\}$,
- $\Delta_G(7A, 10Y, 21A) = 573216$,
- $\Delta_G(13X, 10Y, 21A) = 967680$, $X \in \{A, B\}$.

Hence, $\Delta_G(pX, 10Y, 21A) > 0$ for all $pX \in T$ and $Y \in \{A, B\}$. Consequently, $\Delta^*_G(pX, 10Y, 21A) > 0$. Therefore, G is $(pX, 10Y, 21A)$-generated for all the conjugacy classes $pX \in T$ and $Y \in \{A, B\}$. It follows that G is $10Y$-complementary generated for $Y \in \{A, B\}$. □

Remark 3.1. *Proposition 3.13 can also be proved by applying Lemma 2.2 as follows. From Proposition 3.7, we know that G is $5X$-complementary generated for $X \in \{C, D\}$. Since $(10A)^2 = 5C$ and $(10B)^2 = 5D$, it then follows from Lemma 2.2 that G is $10Y$-complementary generated for $Y \in \{A, B\}$.*

Proposition 3.14. *The group $G = G_2(4)$ is $10Y$-complementary generated for $Y \in \{C, D\}$.*

Proof. As in the proof of Proposition 3.13, we prove the result by establishing the generation of G by the triple $(pX, 10Y, 21A)$ for all $pX \in T$ and $Y \in \{C, D\}$. Let $R = \{5C, 5D, 13A, 13B\}$. We observe from Table 3.2 that M_5 is the only maximal subgroup of G that makes a contribution in the calculations of $\Delta^*_G(pX, 10Y, 21A)$ for any $pX \in T \setminus R$ and $Y \in \{C, D\}$, whereas no maximal subgroup makes any such contribution for $pX \in R$.

For the case $(2A, 10Y, 21A)$, we have $\Delta_G(2A, 10Y, 21A) = 168$ and $\Sigma^*(M_5) = \Sigma(M_5) = 0$. Therefore

$$\Delta^*_G(2A, 10Y, 21A) = \Delta_G(2A, 10Y, 21A) - 1 \cdot \Sigma^*(M_5) = 168 - 0 = 168,$$

showing that G is $(2A, 10Y, 21A)$-generated.

For the case $(2B, 10Y, 21A)$, we obtain $\Delta_G(2B, 10Y, 21A) = 3528$ and $\Sigma(M_5) = 189$. Therefore,

$$\Delta^*_G(2B, 10Y, 21A) \geq \Delta_G(2B, 10Y, 21A) - 1 \cdot \Sigma(M_5) = 3528 - 189 = 3339,$$

and thus, G is $(2B, 10Y, 21A)$-generated.

For the case $(3A, 10Y, 21A)$, we have $\Delta_G(3A, 10Y, 21A) = 168$ and $\Sigma^*(M_5) = \Sigma(M_5) = 0$. Therefore,

$$\Delta^*_G(3A, 10Y, 21A) = \Delta_G(3A, 10Y, 21A) - 1 \cdot \Sigma^*(M_5) = 168 - 0 = 168,$$

showing that G is $(3A, 10Y, 21A)$-generated.

For the case $(3B, 10Y, 21A)$, we obtain $\Delta_G(3B, 10Y, 21A) = 69888$ and $\Sigma^*(M_5) = \Sigma(M_5) = 0$. Therefore,

$$\Delta^*_G(3B, 10Y, 21A) = \Delta_G(3B, 10Y, 21A) - 1 \cdot \Sigma^*(M_5) = 69888 - 0 = 69888,$$

showing that G is $(3B, 10Y, 21A)$-generated.

For the case $(5X, 10Y, 21A)$, we get $\Delta_G(5X, 10Y, 21A) = 40320$ and $\Sigma^*(M_5) = \Sigma(M_5) = 0$ for $X \in \{A, B\}$. Therefore,

$$\Delta^*_G(5X, 10Y, 21A) = \Delta_G(5X, 10Y, 21A) - 1 \cdot \Sigma^*(M_5) = 40320 - 0 = 40320,$$

showing that G is $(5X, 10Y, 21A)$-generated for $X \in \{A, B\}$.

For the case $(5X, 10Y, 21A)$, we have $\Delta_G(5X, 10Y, 21A) = 40320$ for $X \in \{C, D\}$. In this case, the subgroup M_5 does not intersect the triple $(5X, 10Y, 21A)$. Hence,

$$\Delta^*_G(5X, 10Y, 21A) = \Delta_G(5X, 10Y, 21A) = 40320,$$

and thus, G is $(5X, 10Y, 21A)$-generated for $X \in \{C, D\}$.

For the case $(7A, 10Y, 21A)$, we have $\Delta_G(7A, 10Y, 21A) = 591360$ and $\Sigma^*(M_5) = \Sigma(M_5) = 0$. Therefore,

$$\Delta^*_G(7A, 10Y, 21A) = \Delta_G(7A, 10Y, 21A) - 1 \cdot \Sigma^*(M_5) = 591360 - 0 = 591360,$$

and thus, G is $(7A, 10Y, 21A)$-generated.

Finally, we deal with the case $(13X, 10Y, 21A)$ for $X \in \{A, B\}$. In this case, the subgroup M_5 does not intersect the triple $(13X, 10Y, 21A)$ and hence $\Delta^*_G(13X, 10Y, 21A) = \Delta_G(13X, 10Y, 21A) = 967680$. Thus, G is $(13X, 10Y, 21A)$-generated for $X \in \{A, B\}$.

Combining the conclusions of all cases, we deduce that G is $(pX, 10Y, 21A)$-generated for all $pX \in T$ and $Y \in \{C, D\}$. Consequently, it follows from Proposition 1.1 that G is $10Y$-complementary generated for $Y \in \{C, D\}$. □

Proposition 3.15. *The group $G = G_2(4)$ is $12Y$-complementary generated for $Y \in \{A, B, C\}$.*

Proof. By Proposition 3.8, the group G is $6A$-complementary generated. Since $(12A)^2 = (12B)^2 = (12C)^2 = 6A$, it follows from Lemma 2.2 that G is $12Y$-complementary generated for $Y \in \{A, B, C\}$. □

Proposition 3.16. *The group $G = G_2(4)$ is $13Z$-complementary generated for $Z \in \{A, B\}$.*

Proof. For $X, Z \in \{A, B\}$ and $Y \in \{A, B, C, D\}$, the group G is $(2X, 13Z, 13Z)$-, $(3X, 13Z, 13Z)$-, $(5Y, 13Z, 13Z)$-, $(7A, 13Z, 13Z)$- and $(13X, 13Y, 13Y)$-generated by Propositions 16, 31, 42, 44 and 45 of [8], respectively. Therefore, the group G is $(pX, 13Z, 13Z)$-generated for every $pX \in T$, and hence, it is $13Z$-complementary generated for $Z \in \{A, B\}$. □

Proposition 3.17. *The group $G = G_2(4)$ is $15Y$-complementary generated for $Y \in \{A, B\}$.*

Proof. We observe from Table 3.2 that every maximal subgroup has an empty intersection with the triple $(pX, 15Y, 13A)$ for all $pX \in T$ and $Y \in \{A, B\}$. Hence, we have $\Delta^*_G(pX, 15Y, 13A) = \Delta_G(pX, 15Y, 13A)$. Now, computations with the GAP system yield

- $\Delta_G(2A, 15Y, 13A) = 273$,
- $\Delta_G(2B, 15Y, 13A) = 4368$,
- $\Delta_G(3A, 15Y, 13A) = 273$,
- $\Delta_G(3B, 15Y, 13A) = 93093$,
- $\Delta_G(5X, 15Y, 13A) = 55965$, $X \in \{A, B, C, D\}$,
- $\Delta_G(7A, 15Y, 13A) = 798525$,
- $\Delta_G(13X, 15Y, 13A) = 1289925$, $X \in \{A, B\}$.

Hence, $\Delta^*_G(pX, 15Y, 13A) = \Delta_G(pX, 15Y, 13A) > 0$. This shows that G is $(pX, 15Y, 13A)$-generated for all $pX \in T$ and $Y \in \{A, B\}$. Therefore, by Proposition 1.1, the group G is $15Y$-complementary generated for $Y \in \{A, B\}$. □

Proposition 3.18. *The group $G = G_2(4)$ is $15X$-complementary generated for $X \in \{C, D\}$.*

Proof. Since, by Proposition 3.7, the group G is $5X$-complementary generated for $X \in \{C, D\}$ and since $(15C)^3 = 5D$ and $(15D)^3 = 5C$, it follows from Lemma 2.2 that G is $15X$-complementary generated for $X \in \{C, D\}$. □

Proposition 3.19. *The group $G = G_2(4)$ is $21X$-complementary generated for $X \in \{A, B\}$.*

Proof. Since, by Proposition 3.10, the group G is $7A$-complementary generated and since $(21A)^3 = (21B)^3 = 7A$, it follows from Lemma 2.2 that G is $21X$-complementary generated for $X \in \{A, B\}$. □

Now, Theorem 1.1 follows from Propositions 3.1–3.19.

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