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Research Article

Irregular domination in trees

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Abstract

For a nonnegative integer r, the r-orbit $O_r(v)$ of a vertex v in a connected graph G of order n and diameter d is the set of vertices at distance r from v. Let $S = \{v_1, v_2, \ldots, v_k\}$ be a set of vertices of G where $3 \le k \le n$ and let $f : S \to \{1, 2, \ldots, d\}$ be a labeling of the vertices of S defined by $f(v_i) = r_i$ for $1 \le i \le k$. If $r_i \ne r_j$ for every pair i, j of integers with $1 \le i, j \le k$ and $\bigcup_{i=1}^{k} O_{r_i}(v_i) = V(G)$, then S is an irregular dominating set for G and f is an irregular dominating labeling for G. The minimum cardinality of an irregular dominating set of G is the irregular domination number $\tilde{\gamma}(G)$ of G. It is known that a nontrivial tree T has an irregular dominating labeling if and only if T is neither a star nor a path of order 2 or 6. In this work, we establish bounds for the irregular domination numbers of trees and present structural characterizations of those trees having a small irregular domination number. Irregular dominating labelings of such trees are also determined.

Keywords: distance; domination; vertex orbits; irregular orbital labeling; trees.

2020 Mathematics Subject Classification: 05C05, 05C12, 05C15, 05C69.

1. Introduction

Domination in graphs has become a popular area of study in recent decades. The book by Haynes, Hedetniemi, and Slater [7] is entirely devoted to this area. While the basic concept of domination is quite simple, many variations and generalizations of domination have been studied over the years. A vertex u in a graph G is said to *dominate* a vertex v if either $u = v$ or $uv \in E(G)$, that is, a vertex u dominates the vertices in its *closed neighborhood* $N[u] = N(u) \cup \{u\}$. A set S of vertices in G is a *dominating set* of G if every vertex of G is dominated by at least one vertex in S. The minimum number of vertices in a dominating set of G is the *domination number* $\gamma(G)$ of G. There are domination parameters defined in terms of distance and vertex orbits in graphs which provides a more general setting for domination in graphs. We refer to the books [4, 7] for graph theory notation and terminology not described here.

Of the many variations of domination that have been introduced, probably the most common and most studied is total domination introduced by Cockayne, Dawes and Hedetniemi [5]. In total domination, a vertex u dominates a vertex v in a graph G if uv is an edge of G. A set S of vertices in a graph G is a *total dominating set* of G if for every vertex v of G, there is a vertex $u \in S$ such that u dominates v. The minimum cardinality of a total dominating set for G is the *total domination number* $\gamma_t(G)$ of G. A graph G has a total domination number if and only if G has no isolated vertices. Here we only consider nontrivial connected graphs.

Total domination and other types of domination can be described in terms of distance in graphs. Let G be a nontrivial connected graph. The *distance* $d(u, v)$ between vertices u and v in G is the minimum number of edges in a $u - v$ path in G. The *eccentricity* $e(v)$ of a vertex v of G is the distance between v and a vertex farthest from v in G. The *radius* $\text{rad}(G)$ of G is the minimum eccentricity among the vertices of G and the *diameter* $\text{diam}(G)$ of G is the maximum eccentricity. Equivalently, the diameter of G is the greatest distance between any two vertices of G. In total domination, a vertex u dominates a vertex v if $d(u, v) = 1$. For a total dominating set S in a nontrivial connected graph G, one can think of assigning each vertex of S the label 1 and assigning no label to the vertices of G not in S. Thus, if $u \in S$, then u is labeled 1, indicating that u dominates all vertices of G whose distance from u is 1.

In [6], a generalization of (total) domination was introduced called *orbital domination*. For a nonnegative integer r, the *r-orbit* $O_r(v)$ of a vertex v in G is the set of vertices at distance r from v. This is sometimes referred to as the *r-step neighborhood* of v. Thus, $O_0(v) = \{v\}$, $O_1(v) = N(v)$, and $\bigcup_{r=0}^{e(v)} O_r(v) = V(G)$. For orbital domination of a nontrivial connected graph G, a set S of vertices of G is sought where there is defined a labeling f of the vertices of S such that for $u \in S$, the label $f(u)$ of u is a positive integer with $f(u) \le e(u)$. The vertex u then dominates a vertex v if $d(u, v) = f(u)$,

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that is, u dominates all vertices of G whose distance from u is $f(u)$. Equivalently, u dominates all vertices in the $f(u)$-orbit of u. The minimum cardinality of such a dominating set of G is the *orbital domination number* $\gamma_o(G)$ of G. Since 1 is a possible label for any vertex of G, the orbital domination number exists for every nontrivial connected graph G; indeed, $\gamma_o(G) \le \gamma_t(G)$.

In [2], a more restricted version of orbital domination was introduced called *irregular domination*, which deals with the area of irregularity in graphs discussed in [1]. For irregular domination of a nontrivial connected graph G, a set S of vertices of G is sought where there is defined a labeling f of the vertices of S with *distinct* positive integers in such a way that for every v of G, there is $u \in S$ such that $d(u, v) = f(u)$. More formally, for a connected graph G of order $n \ge 3$, we seek a set $S = \{v_1, v_2, \ldots, v_k\}$ of vertices of G where $3 \le k \le n$ on which can be defined a labeling (a one-to-one function) f on S by $f(v_i) = r_i$ such that $r_i \le e(v_i)$ for $1 \le i \le k$ such that $\bigcup_{i=1}^{k} O_{r_i}(v_i) = V(G)$. Thus, if $d = \text{diam}(G)$, then f assigns distinct labels from the set $[d] = \{1, 2, \ldots, d\}$ to the vertices of S where the remaining vertices of G are unlabeled. Such a set S is called an *irregular orbital dominating set* or, more simply, an *irregular dominating set* for G and the labeling f is an *irregular* (*orbital*) *dominating labeling* of G. Unlike total domination and the more general orbital domination, however, irregular dominating labelings are not defined for all nontrivial connected graphs. The primary goal is investigating graphs that possess irregular dominating labelings. While there are well-known classes of graphs that do not possess irregular dominating labelings, such as vertex-transitive graphs, such labelings exist for nearly all trees. We write P_n for the path of order n.

Theorem 1.1. [2] *A nontrivial tree T has an irregular dominating labeling if and only if T is none of P_2, P_6 or a star.*

For a nontrivial connected graph G possessing irregular dominating sets, the minimum cardinality of an irregular dominating set of G is referred to as the *irregular domination number* of G, denoted by $\tilde{\gamma}(G)$. A irregular dominating set of cardinality $\tilde{\gamma}(G)$ is a *minimum irregular dominating set* and its corresponding irregular dominating labeling is a *minimum irregular dominating labeling* of G. In this work, we establish bounds for irregular dominating numbers of trees and present structural characterizations of trees having small irregular domination numbers. Minimum irregular dominating labelings of these trees are also determined.

2. Irregular domination numbers of trees

The proof of Theorem 1.1 in [2] gives rise to the following result.

Proposition 2.1. *If T is a tree of diameter $d \ge 3$ and $d \ne 5$, then $\tilde{\gamma}(T) \le \tilde{\gamma}(P_{d+1})$.*

Consequently, it would be useful to know the value of irregular domination numbers of paths. In order to obtain information on irregular domination numbers of paths, we first state an immediate observation.

Observation 2.1. *If a connected graph G possessing an irregular dominating set, then*

$$3 \le \tilde{\gamma}(G) \le \text{diam}(G).$$

Furthermore, if $\text{diam}(G) = 3$, then every irregular dominating labeling uses the labels $1, 2, 3$.

It can be shown that $\tilde{\gamma}(P_4) = 3$, $\tilde{\gamma}(P_5) = 4$, $\tilde{\gamma}(P_7) = 5$ and $\tilde{\gamma}(P_n) = 6$ for $n = 8, 9, 10$. A minimum irregular dominating labeling is shown in Figure 1 for each of these paths.

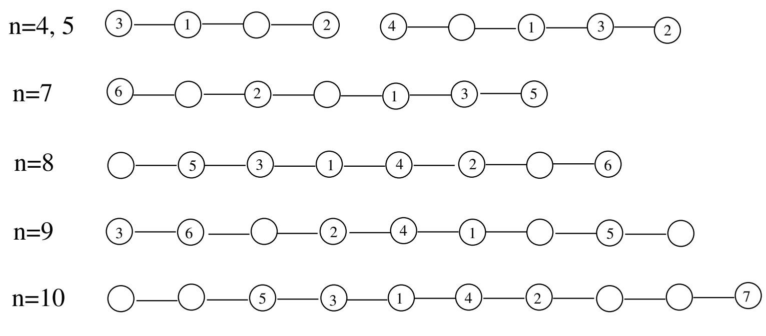

Figure 1: Minimum irregular dominating labelings of P_n for $n = 4, 5, 7, 8, 9, 10$.

For the paths P_n of order $n \ge 11$, we present a lower bound for $\tilde{\gamma}(P_n)$ in terms of n. To establish this bound, we first present two useful observations on irregular dominating labelings of trees.

Observation 2.2. *Let U and W be the partite sets of a nontrivial tree T and let f be an irregular dominating labeling of T. If x is a labeled vertex with $f(x) = r$, then (i) $O_r(x) \subseteq U$ or $O_r(x) \subseteq W$ and (ii) x and $O_r(x)$ belong to the same partite set if and only if r is even. Consequently, a labeled vertex cannot dominate two vertices u and w if $d(u, w)$ is odd.*

Observation 2.3. *Let P be a path of length 2 or more in a tree T. For every irregular dominating labeling of T, a labeled vertex of T dominates at most two vertices of P.*

Theorem 2.1. *For each integer $n \ge 11$, $\tilde{\gamma}(P_n) \ge \lceil (n+3)/2 \rceil$.*

Proof. Let $P_n = (u_0, u_1, \dots, u_{n-1})$ be a path of order $n \ge 11$. Assume, to the contrary, there is either an odd integer $n \ge 11$ such that $\tilde{\gamma}(P_n) < \frac{n+3}{2}$ or an even integer $n \ge 12$ such that $\tilde{\gamma}(P_n) < \frac{n+4}{2}$. We consider these two cases.

Case 1. *n is odd*. Then $n = 2k+1$ for some integer $k \ge 5$ and $\lceil \frac{n+3}{2} \rceil = \frac{n+3}{2} = k+2$. Thus, there is a minimum irregular dominating labeling f of P_{2k+1} using at most $\frac{n+1}{2} = k+1$ labels from the set $[2k]$. If $f(v) \in [k]$, then v dominates one or two vertices of P_{2k+1} (according to the location of v). If $f(v) \in \{k+1, k+2, \dots, 2k\} = [k+1, 2k]$, then v dominates only one vertex of P_{2k+1}. This implies that f uses at least $k+1$ labels and so f uses exactly $k+1$ labels. In order for $k+1$ labeled vertices to dominate $2k+1$ vertices, there must be k vertices labeled $1, 2, \dots, k$ that dominate $2k$ vertices of P_{2k+1} and one vertex whose label belongs to $[k+1, 2k]$ that dominates exactly one vertex of of P_{2k+1}. Furthermore, any subset of ℓ vertices labeled from elements of the set $[k]$ must dominate 2ℓ vertices of P_{2k+1}. If $f(v) = k$ and v dominates two vertices of P_{2k+1}, then $v = u_k$ and v dominates u_0 and u_{2k}. If $f(w) = k-1$ and w dominates two vertices of P_{2k+1}, then $w \ne u_k$ and so $w \in \{u_{k-1}, u_{k+1}\}$. However, if $f(w) = k-1$ and $w \in \{u_{k-1}, u_{k+1}\}$, then the two vertices labeled k and $k-1$ dominate exactly three vertices of P_{2k+1}. This is a contradiction.

Case 2. *n is even*. Let $n = 2k+2$ for some integer $k \ge 5$. Then $\lceil \frac{n+3}{2} \rceil = \frac{n+4}{2} = k+3$. Thus, there is a minimum irregular dominating labeling f of P_{2k+2} using at most $k+2$ labels from the set $[2k+1]$. If $f(v) \in [k]$, then v dominates one or two vertices of P_{2k+2}. If $f(v) \in [k+1, 2k+1]$, then v dominates only one vertex of P_{2k+2}. This implies that f uses at least $k+2$ labels and so f uses exactly $k+2$ labels. In order for $k+2$ labeled vertices to dominate $2k+2$ vertices of P_{2k+2}, there must be k vertices labeled $1, 2, \dots, k$ that dominate $2k$ vertices of P_{2k+2} and two vertices whose label belong to $[k+1, 2k+1]$ that dominate two vertices of P_{2k+2}. Furthermore, any subset of ℓ vertices labeled from elements of the set $[k]$ must dominate 2ℓ vertices of P_{2k+2}. If $f(v) = k$ and v dominates two vertices of P_{2k+2}, then $v \in \{u_k, u_{k+1}\}$. By symmetry, we may assume that $f(u_k) = k$ and u_k dominates u_0 and u_{2k}. There are three situations, according to $k = 5$, $k = 6$ or $k \ge 7$.

First, suppose that $k = 5$ and $n = 12$. Since $f(u_5) = 5$, this forces $f(u_7) = 4$, $f(u_4) = 3$, and $f(u_6) = 2$. However, no unlabeled vertex can be labeled 1 to dominate two vertices not already dominated, which is a contradiction. Next, suppose that $k = 6$ and $n = 14$. Since $f(u_6) = 6$, this forces $f(u_8) = 5$, $f(u_5) = 4$, $f(u_7) = 3$, and $f(u_9) = 2$. However, no unlabeled vertex can be labeled 1 and dominates two vertices not already dominated, which is a contradiction. Finally, suppose that $k \ge 7$ and $n = 2k+2 \ge 16$. Since $f(u_k) = k$, it follows that u_k dominates u_0 and u_{2k}, which forces $f(u_{k+2}) = k-1$ and u_{k+2} dominates u_3 and u_{2k+1}, $f(u_{k-1}) = k-2$ and u_{k-1} dominates u_1 and u_{2k-3}, and $f(u_{k+1}) = k-3$ and u_{k+1} dominates u_4 and u_{2k-2}. Then either $f(u_{k+3}) = k-4$ or $f(u_{k-2}) = k-4$. If $f(u_{k+3}) = k-4$, then u_{k+3} dominates u_7 and u_{2k-1}. This forces $f(u_{k-3}) = k-5$ and u_{k-3} dominates u_2 and u_{2k-8}. If $f(u_{k-2}) = k-4$, then u_{k-2} dominates u_2 and u_{2k-6}. This forces $f(u_{k+4}) = k-5$ and u_{k+4} dominates u_9 and u_{2k-1}. In either case, if $i \in \{0, 1, 2, 3, 4\} \cup \{2k-3, 2k-2, 2k-1, 2k, 2k+1\}$, then u_i is already dominated. Hence, there is no unlabeled vertex that can be labeled $k-6$ and dominates two vertices not already dominated, which is impossible. □

It can be shown that equality holds in Theorem 2.1 for $11 \le n \le 26$. In fact, we conjecture that equality holds in Theorem 2.1 for all $n \ge 11$.

By Proposition 2.1, if T is a tree of diameter $d \ge 3$ and $d \ne 5$, then $\tilde{\gamma}(T) \le \tilde{\gamma}(P_{d+1})$. Hence, as expected, the diameter of a tree T plays an important role in determining the irregular domination number $\tilde{\gamma}(T)$ of T. A path P in a tree T is *diametrical* if the length of P is the diameter of T. A tree T is an *irregular minimal tree with respect to* (1) *its diameter and* (2) *its irregular domination number*, or more simply a *minimal tree*, if there is a *minimum* irregular dominating labeling of T such that for every unlabeled end-vertex v of T we have $\text{diam}(T - v) < \text{diam}(T)$. Consequently, this *minimum* irregular dominating labeling of T must assign a label to each end-vertex of T not belonging to some diametrical path of T. For example, Figure 2 shows two minimal trees of diameter 4 and one minimal tree of diameter d for each $d \in \{5, 6, 7\}$, together with a minimum irregular dominating labeling for each of these five trees.

The next two results provide a sharp upper bound for the diameter of a tree T with a given irregular domination number $\tilde{\gamma}(T)$, according to the parity of $\tilde{\gamma}(T)$.

Theorem 2.2. *If T is a tree with $\tilde{\gamma}(T) = k$ for some odd integer $k \ge 3$, then $\text{diam}(T) \le 2k-2$. Furthermore, for each odd integer $k \ge 3$, there is a minimal tree T_k of diameter $2k-2$ such that $\tilde{\gamma}(T_k) = k$.*

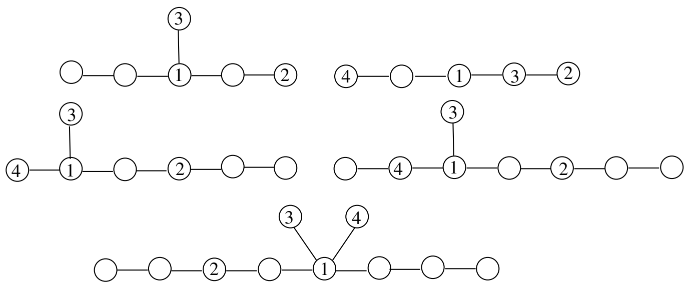

Figure 2: Examples of minimal trees of diameter d with $4 \le d \le 7$.

Proof. Assume, to the contrary, that there is a tree with $\tilde{\gamma}(T) = k$ for some odd integer $k \ge 3$ such that $\mathrm{diam}(T) \ge 2k-1$. Then T contains a path $P = (u_1, u_2, \dots, u_{2k})$ of order $2k$. Let $U_1 = \{u_1, u_3, \dots, u_{2k-1}\}$ and $U_2 = \{u_2, u_4, \dots, u_{2k}\}$. Thus, $|U_1| = |U_2| = k$. Since $\tilde{\gamma}(T) = k$, there is an irregular dominating labeling of T using exactly k distinct labels from the set $[2k-1]$. Necessarily, each labeled vertex of T must dominate two vertices of P and the set of pairs of vertices of P dominated by a labeled vertex of T must result in a partition of $V(P)$. If $\{u_i, u_j\}$ is a pair of vertices dominated a labeled vertex of T, then $d(u_i, u_j) = |i-j|$ must be even. Therefore, either $\{u_i, u_j\} \subseteq U_1$ or $\{u_i, u_j\} \subseteq U_2$. However, since both U_1 and U_2 consist of an odd number of vertices, such a labeling is impossible.

Next, let $k \ge 3$ be an odd integer. We saw in Figure 2 that there is a minimal tree T_3 of diameter 4 with $\tilde{\gamma}(T_3) = 3$. For an odd integer $k \ge 5$, let T_k be the tree obtained from the path $P_{2k-1} = (u_1, u_2, \dots, u_{2k-1})$ of order $2k-1$ by adding $k-2$ pendant edges $u_k v_i$ for $3 \le i \le k$ at u_k. Then $\mathrm{diam}(T_k) = 2k-2$ (and P_{2k-1} is the only path of length $\mathrm{diam}(T_k) = 2k-2$ in T_k). Hence, $\tilde{\gamma}(T_k) \ge k$. To show that $\tilde{\gamma}(T_k) \le k$, we define a labeling f of T_k by $f(u_k) = 1$, $f(u_{k+2}) = 2$, and $f(v_i) = i$ for $3 \le i \le k$ with all remaining vertices of T_k unlabeled. In particular, u_1 and u_{2k-1} are the only unlabeled end-vertices of T. Observe that

$$\begin{aligned} O_1(u_k) &= \{u_{k-1}, u_{k+1}\} \cup \{v_3, v_4, \dots, v_k\}, \\ O_2(u_{k+2}) &= \{u_k, u_{k+4}\} \text{ (and so } u_k \in O_2(u_{k+2})), \\ O_i(v_i) &= \{u_{k-i+1}, u_{k+i-1}\} \text{ for } 3 \le i \le k. \end{aligned}$$

Hence,

$$O_1(u_k) \bigcup O_2(u_{k+2}) \bigcup \left(\bigcup_{i=3}^{k} O_i(v_i) \right) = V(T)$$

and so f is an irregular dominating labeling of T_k using exactly k labels from the set $[2k-2]$. Therefore, $\tilde{\gamma}(T_k) = k$ and f is a minimum irregular dominating labeling of T_k. Since P_{2k-1} is the only diametrical path in T_k and $\mathrm{diam}(T_k - v) < \mathrm{diam}(T_k)$ if $v \in \{u_i, u_{2k-1}\}$, it follows that T_k is a minimal tree of diameter $2k-2$ such that $\tilde{\gamma}(T_k) = k$. □

Theorem 2.3. *If T is a tree with $\tilde{\gamma}(T) = k$ for some even integer $k \ge 4$, then $\mathrm{diam}(T) \le 2k-1$. Furthermore, for each even integer $k \ge 4$, there is a minimal tree T_k of diameter $2k-1$ such that $\tilde{\gamma}(T_k) = k$.*

Proof. Let T be a tree with $\tilde{\gamma}(T) = k$ for some even integer $k \ge 4$ and let f be an irregular dominating labeling of T. Since each labeled vertex dominates at most two vertices on a path of order $\mathrm{diam}(T)+1$, it follows that $\mathrm{diam}(T)+1 \le 2k$ and so $\mathrm{diam}(T) \le 2k-1$.

Next, let $k \ge 4$ be an even integer. We saw in Figure 2 that there is a tree T_4 of diameter 7 with $\tilde{\gamma}(T_4) = 4$. A tree T_6 of diameter 11 with $\tilde{\gamma}(T_6) = 6$ is shown in Figure 3 with a minimum irregular dominating labeling using exactly six labels from the set $[11]$.

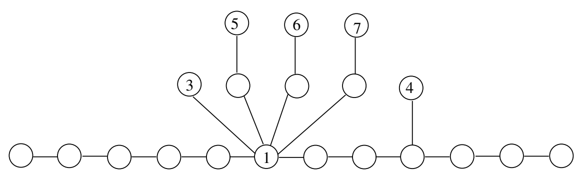

Figure 3: A tree T_6 with $\mathrm{diam}(T) = 11$ and $\tilde{\gamma}(T) = 6$.

Thus, we may assume that $k \ge 8$ and we can write $k = 2\ell$ where $\ell \ge 4$. Let T_k be the tree obtained from the path $P_{4\ell} = (u_1, u_2, \dots, u_{4\ell})$ of order 4ℓ by adding (1) $\ell - 3$ pendant edges $u_{2\ell} v_i$ for $3 \le i \le \ell - 1$ at $u_{2\ell}$ and (2) $\ell + 1 \ge 5$ pendant paths $(u_{2\ell}, v_j, w_j)$ of order 3 for $\ell + 1 \le j \le 2\ell + 1$ at $u_{2\ell}$. Then $\mathrm{diam}(T_k) = 2k-1 = 4\ell - 1$ and $P_{4\ell}$ is the only path of length $\text{diam}(T_k) = 4\ell - 1$ in T_k. We show that $\tilde{\gamma}(T_k) = k = 2\ell$. Since $\tilde{\gamma}(T_k) \geq 2\ell$, it suffices to show that $\tilde{\gamma}(T_k) \leq 2\ell$. We define a labeling f of T_k by $f(u_{2\ell}) = 1$, $f(v_i) = i$ for $3 \leq i \leq \ell - 1$, $f(u_{3\ell}) = \ell$, and $f(w_j) = j$ for $\ell + 1 \leq j \leq 2\ell + 1$ with all remaining vertices of T_k unlabeled. In particular, the two end-vertices u_1 and $u_{4\ell}$ of $P_{4\ell}$ are not labeled. (In fact, u_1 and $u_{4\ell}$ are the only unlabeled end-vertices of T.) The set of labels used by f is $L_f = [2\ell + 1] - \{2\}$ and so $|L_f| = 2\ell$. Observe that

$$\begin{aligned}
O_1(u_{2\ell}) &= \{u_{2\ell-1}, u_{2\ell+1}\} \cup \{v_i : 3 \leq i \leq \ell - 1\} \cup \{v_j : \ell + 1 \leq j \leq 2\ell + 1\} \\
O_\ell(u_{3\ell}) &= \{u_{2\ell}, u_{4\ell}\} \\
O_3(v_3) &= \{u_{2\ell-2}, u_{2\ell+2}\} \cup \{w_j : \ell + 1 \leq j \leq 2\ell + 1\} \\
O_i(v_i) &= \{u_{2\ell-i+1}, u_{2\ell+i-1}\} \text{ for } 4 \leq i \leq \ell - 1 \\
O_j(w_j) &= \{u_{2\ell-j+2}, u_{2\ell+j-2}\} \text{ for } \ell + 1 \leq j \leq 2\ell + 1.
\end{aligned}$$

Hence, f is an irregular dominating labeling of T_k using exactly $k = 2\ell$ labels from the set $[4\ell-1]$. Therefore, $\tilde{\gamma}(T_k) = k$ and f is a minimum irregular dominating labeling of T_k. Since $P_{4\ell}$ is the only diametrical path in T_k and $\text{diam}(T_k - v) < \text{diam}(T_k)$ for each unlabeled end-vertex of $P_{4\ell}$, it follows that the tree T_k is a minimal tree of diameter $2k - 1$ such that $\tilde{\gamma}(T_k) = k$. □

By Theorem 2.3, there exists a minimal tree T of diameter $2k - 1$ having $\tilde{\gamma}(T) = k$ for each even integer $k \geq 4$. However, more can be said.

Theorem 2.4. *For an even integer $2k \geq 4$, let $\mathcal{T}_{2k}$ denote the set of all non-isomorphic minimal trees T of diameter $4k - 1$ and $\tilde{\gamma}(T) = 2k$. Then $\lim_{k\to\infty} |\mathcal{T}_{2k}| = \infty$.*

Proof. Let p be a positive integer. We show that there exists a positive integer j such that for every even integer $2k \geq j$, we have $|\mathcal{T}_{2k}| \geq p$. Let $j = 4p$ and let $2k$ be any even integer such that $2k \geq j = 4p$. Thus, either $2k = 4\ell$ or $2k = 4\ell + 2$ for some positive integer ℓ. In either case, $\ell \geq p$. We show that $|\mathcal{T}_{2k}| \geq \ell$. We consider two cases, according to whether $2k = 4\ell$ or $2k = 4\ell + 2$.

Case 1. $2k = 4\ell$. Let T_1 be the tree consisting of the path $P = (u_1, u_2, \ldots, u_{8\ell})$ of length $8\ell - 1 = 4k - 1$, where at the vertex $u_{4\ell}$ are placed the $4\ell - 2$ pendant paths $Q_1, Q_2, \ldots, Q_{4\ell-2}$ of lengths $1, 2, \ldots, 4\ell - 2$, respectively. Let Q_j be a $u_{4\ell} - v_j$ path of length j for $1 \leq j \leq 4\ell - 2$. For $2 \leq i \leq \ell$, the tree T_i is obtained from T_1 by placing a pendant $u_{6\ell} - w_i$ path Q of length $2i - 2$ at the vertex $u_{6\ell}$. Observe that P is the only diametrical path in T_i for $1 \leq i \leq \ell$. Next, we define a labeling f_i of T_i for $1 \leq i \leq \ell$ as follows. For $i = 1$, let $f_1(u_{4\ell}) = 1$, $f_i(v_j) = 2j + 1$ for $1 \leq j \leq 4\ell - 2$, and $f(u_{6\ell}) = 2\ell$. For $2 \leq i \leq \ell$, let $f_i(u_{4\ell}) = 1$, $f_i(v_j) = 2j + 1$ for $1 \leq j \leq 4\ell - 2$, and $f_i(w_i) = 2\ell + 2i - 2$. In particular, the end-vertices u_1 and $u_{8\ell}$ of P are not labeled by f_i for $1 \leq i \leq \ell$. In fact, u_1 and $u_{8\ell}$ are the only unlabeled end-vertices in T_i for $1 \leq i \leq \ell$. Observe that

$$\begin{aligned}
O_1(u_{4\ell}) &= \{u_{4\ell-1}, u_{4\ell+1}\} \cup \left(\bigcup_{i=1}^{4\ell-2} \{x \in V(Q_i) : d(x, u_{4\ell}) = 1\} \right) \\
O_{2j+1}(v_j) &= \{u_{4\ell-j-1}, u_{4\ell+j+1}\} \cup \left(\bigcup_{i=1}^{4\ell-2} \{x \in V(Q_i) : d(x, u_{4\ell}) = i + 1\} \right), 1 \leq j \leq 4\ell - 2, \\
O_{2\ell}(u_{6\ell}) &= \{u_{4\ell}, u_{8\ell}\} \text{ if } i = 1 \\
O_{2\ell+2i-2}(w_i) &= \{u_{4\ell}, u_{8\ell}\}, 2 \leq i \leq \ell.
\end{aligned}$$

For $1 \leq i \leq \ell$, this labeling f_i is an irregular dominating labeling of T_i consisting of $4\ell = 2k$ labels from the set $[8\ell - 1]$ and so $\tilde{\gamma}(T_i) = 2k$. Hence, f_i is a minimum irregular dominating labeling of T_i for $1 \leq i \leq \ell$. Since P is the only diametrical path in T_i and $\text{diam}(T_i - v) < \text{diam}(T_i)$ for each unlabeled end-vertex v of P, it follows that T_i is a minimal tree of diameter $4k - 1$ and $\tilde{\gamma}(T_i) = 2k$ for $1 \leq i \leq \ell$. Consequently, $|\mathcal{T}_{2k}| \geq \ell \geq p$.

Case 2. $2k = 4\ell + 2$. For $1 \leq i \leq \ell$, let T_i be the tree consisting of the path $P = (u_1, u_2, \ldots, u_{8\ell+4})$ of length $8\ell + 3 = 4k - 1$, where at the vertex $u_{4\ell+2}$ are placed 4ℓ pendant paths Q_1, Q_2, …, $Q_{4\ell}$ of lengths $1, 2, \ldots, 4\ell$, respectively and a pendant $u_{6\ell+3} - w_i$ path Q of length $2i - 1$ at the vertex $u_{6\ell+3}$. For $1 \leq j \leq 4\ell$, let Q_j be a $u_{4\ell} - v_j$ path of length j. Thus, P is the only diametrical path in T_i for $1 \leq i \leq \ell$. Next, we define a labeling f_i of T_i for $1 \leq i \leq \ell$ by $f_i(u_{4\ell+2}) = 1$, $f_i(v_j) = 2j + 1$ for $1 \leq j \leq 4\ell$, and $f_i(w_i) = 2\ell + 2i$. In particular, u_1 and $u_{8\ell+4}$ are not labeled by f_i for $1 \leq i \leq \ell$. In fact, u_1 and $u_{8\ell+4}$ are the only unlabeled end-vertices in T_i for $1 \leq i \leq \ell$. Observe that

$$\begin{aligned}
O_1(u_{4\ell+2}) &= \{u_{4\ell+1}, u_{4\ell+3}\} \cup \left(\bigcup_{i=1}^{4\ell} \{x \in V(Q_i) : d(x, u_{4\ell+2}) = 1\} \right) \\
O_{2j+1}(v_j) &= \{u_{4\ell-j+1}, u_{4\ell+j+3}\} \cup \left(\bigcup_{i=1}^{4\ell} \{x \in V(Q_i) : d(x, u_{4\ell+2}) = i + 1\} \right), 1 \leq j \leq 4\ell,
\end{aligned}$$

$$O_{2\ell+2i}(w_i) = \{u_{4\ell+2}, u_{8\ell+4}\}, 1 \le i \le \ell.$$

For $1 \le i \le \ell$, this labeling f_i is an irregular dominating labeling of T_i consisting of $4\ell+2 = 2k$ labels from the set $[8\ell+3]$ and so $\tilde{\gamma}(T_i) = 2k$. Hence, f_i is a minimum irregular dominating labeling of T_i for $1 \le i \le \ell$. Since P is the only diametrical path in T_i and $\operatorname{diam}(T_i - v) < \operatorname{diam}(T_i)$ for each unlabeled end-vertex v of P, it follows that T_i is a minimal tree of diameter $4k-1$ and $\tilde{\gamma}(T_i) = 2k$ for $1 \le i \le \ell$. Consequently, $|\mathcal{T}_{2k}| \ge \ell \ge p$. □

We saw in the proof of Theorem 2.4 that for each positive integer ℓ, (1) there are ℓ non-isomorphic minimal trees T with $\operatorname{diam}(T) = 8\ell-1$ and $\tilde{\gamma}(T) = 4\ell$ and (2) there are ℓ non-isomorphic minimal trees T with $\operatorname{diam}(T) = 8\ell+3$ and $\tilde{\gamma}(T) = 4\ell+2$. In general, there are typically more than ℓ such non-isomorphic trees. For example, while the proof of Theorem 2.4 shows that there are three non-isomorphic minimal trees T with $\operatorname{diam}(T) = 23$ and $\tilde{\gamma}(T) = 12$, all four trees shown in Figure 4 also have diameter 23, irregular domination number 12 and are minimal, where $P = (u_1, u_2, \ldots, u_{24})$ is the longest path in each of these trees. In fact, using this procedure, we can construct four additional minimal trees T with $\operatorname{diam}(T) = 23$ and $\tilde{\gamma}(T) = 12$. One such tree consists of P where at the vertex u_{12} are placed ten pendant paths $Q_1, Q_2, \ldots, Q_{10}$ such that the length of Q_i is i for $1 \le i \le 10$. An irregular dominating labeling of this tree assigns the label 1 to u_{12}, the label 6 to u_8, and the label $2i+1$ to the end-vertex of Q_i for $1 \le i \le 10$.

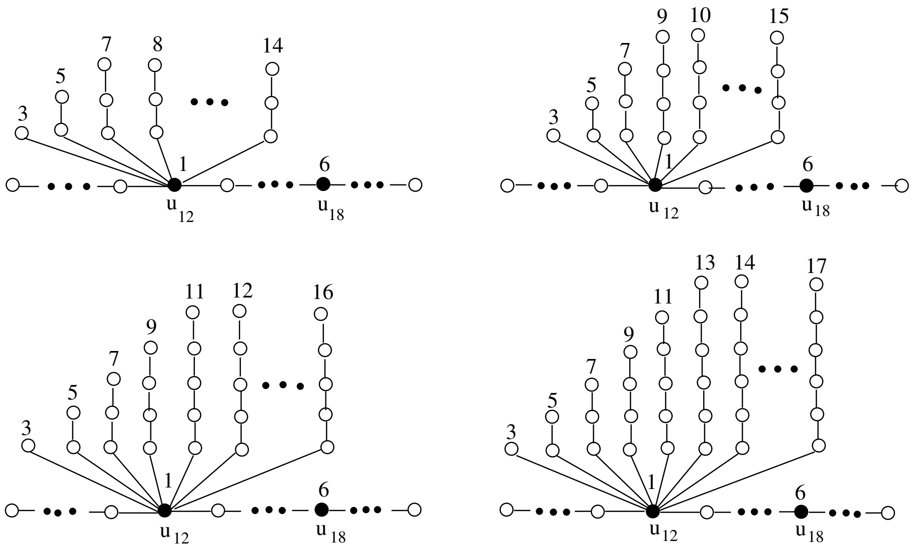

Figure 4: Four non-isomorphic minimal trees T with $\operatorname{diam}(T) = 23$ and $\tilde{\gamma}(T) = 12$.

3. Trees of small irregular domination number

We have seen in Observation 2.1 that if G is a connected graph of diameter 3 or more having an irregular dominating labeling, then $\tilde{\gamma}(G) \ge 3$. We now characterize those trees T for which $\tilde{\gamma}(T) = 3$ or $\tilde{\gamma}(T) = 4$. It is convenient to introduce some terminology. Here, we consider a star to be a graph of type $K_{1,t}$ where $t \ge 1$. A vertex of degree t in a star $K_{1,t}$ is referred to as its *center*, where either vertex of the star $K_{1,1} = K_2$ is a center. By *attaching a pendant star* $K_{1,t}$ *to a vertex* v of a graph, we mean identifying v with the center of the star. If T is a tree obtained by attaching a pendant star to at least one end-vertex of a star S, then S is referred to as the *defining star* of T. Let $\mathcal{T}$ be the set of all trees obtained by attaching a pendant star to at least two but not all end-vertices of a defining star of size 3 or more. Thus, if $T \in \mathcal{T}$ with partite sets U and W such that U contains the center v of the defining star of T, then v is adjacent to every vertex of W and v is the only vertex in U that is not an end-vertex of T. Thus, the diameter of T is 4 and v is the central vertex of T.

Theorem 3.1. *A tree T has $\tilde{\gamma}(T) = 3$ if and only if T is a double star or $T \in \mathcal{T}$.*

Proof. We have seen that every double star has irregular domination number 3. Let $T \in \mathcal{T}$, where S is the defining star of T and let v be the central vertex of S. Assigning the label 1 to v, the label 2 to any vertex at distance 2 from v, and the label 3 to any neighbor of v that is an end-vertex of T results in an irregular dominating labeling of T. Thus, $\tilde{\gamma}(T) = 3$.

It remains to show that if T is a tree with $\tilde{\gamma}(T) = 3$, then T is a double star or $T \in \mathcal{T}$. Suppose that T is not a double star. Since $\tilde{\gamma}(T) = 3$, it follows that (1) $\operatorname{diam}(T) = d \ge 4$ and (2) there is an irregular dominating labeling g of T that assigns three distinct elements of $[d]$ of T to three vertices of T. Let U and W be the partite sets of T. By Observation 2.2, if x is a labeled vertex and $g(x) = r$, then (i) $O_r(x) \subseteq U$ or $O_r(x) \subseteq W$ and (ii) x and $O_r(x)$ belong to the same partite set if and only if r is even. Since there are exactly three labeled vertices of T, some labeled vertex must dominate all vertices of U or all vertices of W.

We may assume that v is a labeled vertex that dominates all vertices of W. Since v does not dominate itself, $v \in U$. Because every neighbor of v belongs to W and the distance from v to these vertices is 1, the vertex v must be labeled 1, implying that v is adjacent to every vertex of W. This in turn implies that the distance between every two vertices of T is at most 4 and so $d \leq 4$. Therefore, $d = 4$. Thus, all three labels of g come from the set $\{1, 2, 3, 4\}$. At this stage, every vertex of W is dominated, in fact by v, and no vertex of U is dominated. In particular, v is not dominated.

Because every vertex of W is adjacent to v and the label 1 has already been assigned, no vertex of W can dominate v. Therefore, v can only be dominated by a vertex of U. Since the distance from a vertex of $U - \{v\}$ to v is 2, some vertex of $U - \{v\}$ must be labeled 2. Let $u \in U - \{v\}$ such that $g(u) = 2$. Thus, u dominates all vertices of U at distance 2 from u. In particular, u dominates v. Necessarily, u is adjacent to only one vertex x of W, for if u is also adjacent to a vertex $y \in W - \{x\}$, then (u, x, v, y, u) is a cycle of T, which is impossible. Therefore, u and, in fact, every vertex of $U - \{v\}$ is an end-vertex of T. Let (v, x, u) be the $v - u$ path in T where $x \in W$. At this stage, each vertex in the set $U - O_2(u)$ is not dominated. In particular, u is not dominated.

Since $d = 4$, it follows that W must contain at least two vertices adjacent to vertices of $U - \{v\}$. Consequently, for every vertex $s \in U - \{v\}$, there is $t \in U - \{v\}$ such that $d(s, t) = 4$. Therefore, there is a vertex $z \in U$ such that $d(u, z) = 4$. Thus, labeling a vertex $z' \in U - \{v\}$ with the label 4 in order to dominate both u and z would mean that $d(u, z') = 4$ and so z' would be undominated by the two vertices labeled 1 and 2, namely v and u. Thus, no vertex can be labeled 4.

Since $\tilde{\gamma}(T) = 3$, the only conclusion is that there must be a vertex $w \in W - \{x\}$ that is labeled 3 and dominates the remaining vertices of U which includes u and z. Thus, w is adjacent to neither u nor z. Moreover, w cannot be adjacent any vertex $y \in U - \{v\}$; for if this were the case, then since $d(y, v) = 2$ and $d(y, z) = d(y, u) = 4$, it follows that y is not dominated by any of v, u, and w. Therefore, w must be an end-vertex of T. Assigning the label 3 to w then has the effect of having w dominate all vertices of $U - \{v\}$. Consequently, every vertex of T is dominated by at least one of u, v, and w. This says that T is a tree obtained by the defining star $K_{1,|W|}$, where $|W| \geq 3$, with the center v and so $T \in \mathcal{T}$. □

The proof of Theorem 3.1 also provides the following result.

Theorem 3.2. *A tree T has $\tilde{\gamma}(T) = 3$ if and only if the following labeling is an irregular dominating labeling of T: (1) a central vertex u of T is labeled 1, (2) an end-vertex v at distance 2 from u is labeled 2, and (3) an end-vertex w at distance 3 from v is labeled 3.*

The following is a consequence of Theorem 3.2, which characterizes the structure of a minimum irregular dominating labeling of a tree T with $\tilde{\gamma}(T) = 3$.

Corollary 3.1. *If T is a tree with $\tilde{\gamma}(T) = 3$, then every minimum irregular dominating labeling of T produces one of the labeled minimal subtrees in Figure 5 depending on the diameter of T.*

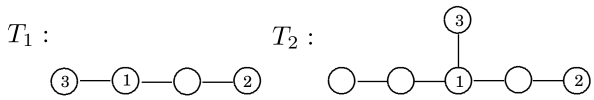

Figure 5: Minimal trees in a tree T with $\tilde{\gamma}(T) = 3$.

A tree $T \in \mathcal{T}$ can also be described as a tree of diameter 4 such that its central vertex is adjacent to at least one end-vertex of T. Hence, Theorem 3.1 can be restated as follows.

Theorem 3.3. *A tree T has $\tilde{\gamma}(T) = 3$ if and only if (i) $\operatorname{diam}(T) = 3$ or (ii) $\operatorname{diam}(T) = 4$ and the central vertex of T is adjacent to at least one end-vertex of T.*

Since $\tilde{\gamma}(P_5) = 4$, the following is a consequence Theorem 3.3 and Proposition 2.1.

Corollary 3.2. *If T is a tree of diameter 4, then $\tilde{\gamma}(T) = 3$ or $\tilde{\gamma}(T) = 4$. Furthermore, $\tilde{\gamma}(T) = 3$ if and only if the central vertex of T is adjacent to at least one end-vertex of T. Consequently, $\tilde{\gamma}(T) = 4$ if and only if the central vertex of T is not adjacent to any end-vertex of T.*

Next, we present a structural characterization of a minimum irregular dominating labeling of a tree T with $\operatorname{diam}(T) = \tilde{\gamma}(T) = 4$.

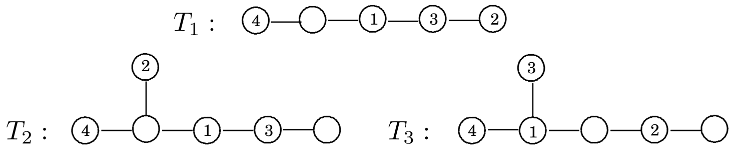

Figure 6: Three trees of diameter 4.

Theorem 3.4. *If T is a tree with $\mathrm{diam}(T) = \tilde{\gamma}(T) = 4$, then every minimum irregular dominating labeling of T produces one of the three labeled minimal subtrees in Figure 6.*

Proof. Since $\mathrm{diam}(T) = \tilde{\gamma}(T) = 4$, the four labels used by a minimum irregular dominating labeling f of G are $1, 2, 3, 4$. Let $u \in V(G)$ such that $f(u) = 4$. Since u must dominate at least one vertex u' of T, it follows that $d(u, u') = 4$ and so u and u' are peripheral vertices of T. Thus, u and u' are end-vertices of T. Let $P = (u = u_0, u_1, u_2, u_3, u_4 = u')$ be the unique $u - u'$ diametrical path in T. Since u_0 cannot dominate itself, u_0 is dominated by a vertex v labeled 1, 2 or 3. We consider these three cases.

Case 1. *The vertex u_0 is dominated by a vertex v labeled* 1. Then $d(u_0, v) = 1$ and so $u_0v \in E(G)$. Since u_0 is an end-vertex of T, it follows that v is the only neighbor of u_0. Thus, $v = u_1$ and $f(u_1) = 1$. Since u_1 is not dominated by u_0 or u_1, it follows that u_1 must be dominated by a vertex labeled 2 or 3.

- ⋆ First, suppose that u_1 is dominated by a vertex w labeled 2. Thus, $d(u_1, w) = 2$ and u_1 and w belong to the same partite set of T. This implies that $d(w, u_0) = 1$ or $d(w, u_0) = 3$. Since $w \neq u_1$, it follows that, $d(w, u_0) = 3$. We may therefore assume that $w = u_3$, where the vertices of T can be relabeled if necessary. Since u_3 is not dominated by u_0, u_1, or u_3, the vertex u_3 must be dominated by a vertex x labeled 3 and $d(u_3, x) = 3$. Thus, x belongs to the partite set of T not containing u_3. Hence, $d(x, u_0) = 4$ or $d(x, u_0) = 2$. If $d(x, u_0) = 4$, then T must contain a path (x, y, u_2, u_3), where $x \neq u_4$ and $y \neq u_3$. Since u_4 is not labeled, $\mathrm{diam}(T - u_4) = 3$ and so the resulting subtree $T - u_4$ is not minimal, a contradiction. Thus, $d(x, u_0) = 2$ and T contains the path (x, u_1, u_2, u_3). This is the subtree T_3 in Figure 6.
- ⋆ Next, suppose that u_1 is dominated by a vertex w labeled 3. Thus, $d(u_1, w) = 3$ and we may assume that $w = u_4$ and $f(u_4) = 3$. Since u_3 is not dominated by u_0, u_1, or $w = u_4$, the vertex u_3 must be dominated by a vertex x labeled 2. Thus, $d(x, u_3) = 2$ and x and u_3 belong to the same partite set of T. Hence, $d(x, u_0) = 1$ or $d(x, u_0) = 3$. Since $x \neq u_1$ (where $f(u_1) = 1$), it follows that $d(x, u_0) = 3$. Because $d(x, u_3) = 2$ and $\mathrm{diam}(T) = 4$, it follows that (x, u_2, u_3) must be a path in T and so T must contain the minimal labeled tree T' obtained by adding a pendant edge to P at u_2. However then, $\tilde{\gamma}(T') = 3$, which is a contradiction, and so u_1 cannot be dominated a vertex labeled 3,.

Case 2. *The vertex u_0 is dominated by a vertex v labeled* 2 *and by no vertex labeled* 1. Then $f(v) = 2$ and $d(u_0, v) = 2$. Therefore, either (1) $v = u_2$ or (2) v is an end-vertex of T adjacent to u_1 since $\mathrm{diam}(T) = 4$.

- ⋆ Suppose first that $v = u_2$ and so $f(u_2) = 2$. Since u_2 is not dominated by u_0 or u_2, it follows that u_2 is dominated by a vertex x labeled 1 or 3. If u_2 is dominated by a vertex x labeled 3, then T must contain a path (x, y, u_1, u_2), where $x, y \notin V(P)$. However, this is impossible since $\mathrm{diam}(T) = 4$. Thus, u_2 is dominated by a vertex x labeled 1. However, $x \neq u_1$ because u_0 is not dominated by a vertex z labeled 1. If z is an end-vertex adjacent to u_2, then T must contain the minimal labeled tree T' obtained by adding a pendant edge to P at u_2. However then, $\tilde{\gamma}(T') = 3$, which is a contradiction, Thus, $z = u_3$ and $f(u_3) = 1$. Since u_3 is not dominated by u_0, u_2, or u_3, it follows that u_3 must be dominated by a vertex w labeled 3. Thus, (w, u_1, u_2, u_3) must be path in T. However then, u_1 is not dominated by any of the four labeled vertices, which is impossible.
- ⋆ Next, suppose that $v \neq u_2$ and so v is an end-vertex adjacent to u_1. Since v is not dominated by u_0 or v, it follows that v must be dominated by a vertex labeled 1 or 3. Since u_0 is not dominated by a vertex labeled 1, we have $f(u_1) \neq 1$ and v cannot be dominated by a vertex labeled 1. Thus, v must be dominated by a vertex w labeled 3. Either $w = u_3$ or $w \neq u_3$ and (w, u_2, u_1, v) is a path in T. If $w \neq u_3$ and $f(w) = 3$, then since neither w nor u_1 is dominated by u_0, v, or w, it follows that w and u_1 must be dominated by a vertex labeled 1. Thus, $f(u_2) = 1$. This implies that w is an end-vertex of T and so T contains the minimal labeled tree T' obtained by adding a pendant edge wu_2 to P at u_2. Since $\tilde{\gamma}(T') = 3$, however, this is impossible. Therefore, $w = u_3$ and $f(u_3) = 3$. At this stage, u_1 and u_3 are not dominated, but this can be accomplished by labeling u_2 with 1. This is the tree T_2 in Figure 6.

Case 3. *The vertex u_0 is dominated by a vertex v labeled* 3 *and by no vertex labeled* 1 *or* 2. Then $f(v) = 3$ and $d(u_0, v) = 3$. Therefore, either (1) $v \neq u_3$ and T contains the path (v, u_2, u_1, u_0) or (2) $v = u_3$.

- ⋆ First, suppose that $v \neq u_3$ and T contains the path (v, u_2, u_1, u_0) where $f(v) = 3$. Since v is not dominated by u_0 or v, it follows that v must be dominated by a vertex x labeled 1 or 2. If v is dominated by a vertex x labeled 2, then $x = u_1$ or $x = u_3$. In either case, there are two adjacent undominated vertices on P, not both of which can be dominated by the remaining labeled vertex, a contradiction. So, v must be dominated by a vertex y labeled 1. Therefore, either $y = u_2$ or y is an end-vertex of T adjacent to v. If y is an end-vertex of T adjacent to v and $f(y) = 1$, then u_1, u_2 and u_3 are undominated vertices on P and cannot be dominated by any remaining labeled vertex. So, $y = u_2$ and $f(u_2) = 1$. In this case, v must be an end-vertex not on P adjacent at u_2 and so T contains a subtree T' consisting of P and a pendant edge vu_2. Since $\tilde{\gamma}(T') = 3$, this is impossible.

- ⋆ Next, suppose that $v = u_3$. and $f(u_3) = 3$. Since neither u_0 nor u_3 dominates u_3, it follows that u_3 must be dominated by a vertex w labeled 1 or 2. Suppose that $f(w) = 2$. Thus, either $w = u_1$ or $w \neq u_1$ and $wu_2 \in E(T)$. Regardless of which of these occurs, there are two adjacent undominated vertices on T, not both of which can be dominated by the remaining labeled vertex, a contradiction. Therefore, $f(w) = 1$ and $w = u_2$, $w = u_4$, or $w \notin V(P)$ and $wu_3 \in E(T)$. If $w \neq u_2$ and $f(w) = 1$, then u_1 and u_2 are adjacent undominated vertices on T, not both of which can be dominated by the remaining labeled vertex. So, $w = u_2$. By defining $f(u_4) = 2$, we have an irregular dominating labeling of T containing T_1 in Figure 6.

□

The following is an immediate consequence of Theorem 3.1 and Corollary 3.2.

Corollary 3.3. *If T is a tree of diameter 5 or more, then $\tilde{\gamma}(T) \geq 4$.*

We now show that every tree T of diameter 5 that is not P_6 has irregular domination number 4.

Theorem 3.5. *If T is a tree with $\text{diam}(T) = 5$ and $T \neq P_6$, then $\tilde{\gamma}(T) = 4$.*

Proof. Let T be tree with $\text{diam}(T) = 5$ and $T \neq P_6$. Since $\tilde{\gamma}(T) \geq 4$ by Corollary 3.3, it remains to show that $\tilde{\gamma}(T) \leq 4$. Let $P = (u_0, u_1, u_2, \ldots, u_5)$ be a longest path in T. Since $T \neq P_6$, some interior vertex of P has degree 3 or more. We may assume that $\deg u_1 \geq 3$ or $\deg u_2 \geq 3$. If $\deg u_1 \geq 3$, then T contains the tree T_1 of Figure 7 as a subtree. If $\deg u_2 \geq 3$ and a neighbor w_2 of u_2 not on P is an end-vertex of T, then T contains the tree T_2 of Figure 7 as a subtree; while if $\deg u_2 \geq 3$ and a neighbor w_2 of u_2 not on P is not an end-vertex of T, then T contains the tree T_3 of Figure 7 as a subtree.

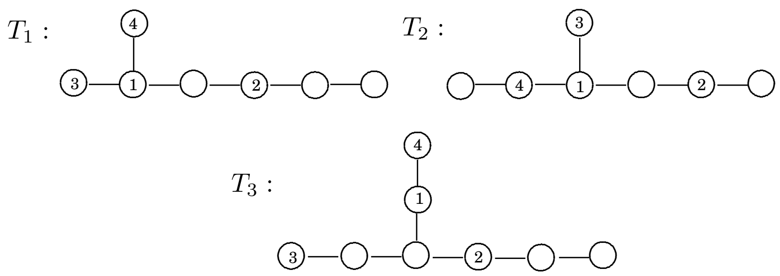

Figure 7: Three subtrees T_1, T_2, and T_3 in a tree of diameter 5 and order 7 or more.

For $i = 1, 2, 3$, Figure 7 also shows an irregular dominating labeling of T_i with exactly four labels $1, 2, 3, 4$. Consequently, if $T_i \subseteq T$ where $i = 1, 2, 3$, then an irregular dominating labeling of T can be defined by assigning labels only to the vertices of the subtree T_i as indicated in Figure 7 (and so all other vertices of T are not labeled). Thus, $\tilde{\gamma}(T) \leq 4$ and so $\tilde{\gamma}(T) = 4$. □

By Theorem 3.5, if T is a tree with $\text{diam}(T) = 5$ and $T \neq P_6$, then $\tilde{\gamma}(T) = 4$. Thus, every minimum irregular dominating labeling of a tree T of diameter 5 (that is not a path) uses exactly four labels from the set $[5]$. Next, we present a structural characterization of minimum irregular dominating labelings of non-path trees of diameter 5.

Theorem 3.6. *If T is a tree of diameter 5 and $T \neq P_6$, then every minimum irregular dominating labeling of T produces one of the labeled trees in Figure 8.*

Proof. Let T is a tree of order n, $\text{diam}(T) = 4$ and $\tilde{\gamma}(T) = 4$ and let $P = (u_1, u_2, \ldots, u_6)$ be a path of length 6 in T. Since $T \neq P_6$, it follows that $V(T) - V(P) \neq \emptyset$. Define

$$W_i = \{w \in V(T) - V(P) : d(u_i, w) = 1\} \quad \text{if } 2 \leq i \leq 5$$
$$X_i = \{x \in V(T) - V(P) : d(u_i, x) = 2\} \quad \text{if } 3 \leq i \leq 4$$

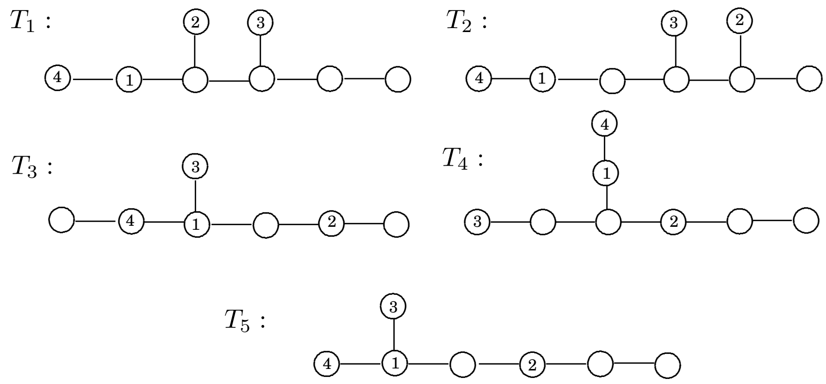

Figure 8: Minimal trees in a tree T with $\operatorname{diam}(T) = 5$ and $\tilde{\gamma}(T) = 4$.

where some of W_i and X_i may be empty. Suppose that f is a minimum irregular dominating labeling of T where L_f is the set of labeled vertices of T. Thus, $|L_f| = 4$ and $\{i : u_i \in L_f\} \subseteq [5]$. Hence, there are at least two vertices in L_f, each of which dominates exactly two vertices of P such that exactly four vertices of P are dominated by these two vertices. If a labeled vertex dominates two vertices u_i and u_j on P, then $|i - j|$ is even. So, there is $x \in L_f$ that dominates two vertices of $\{u_1, u_3, u_5\}$ and there is $y \in L_f$ that dominates two vertices of $\{u_2, u_4, u_6\}$. Consequently, each of the sets $\{u_1, u_3, u_5\}$ and $\{u_2, u_4, u_6\}$ is partitioned into two sets, one of which is a 2-element set and the other is a singleton. The following are the six distinct ways to do this and so we consider these six cases. (1) $\{u_1, u_3\}$, $\{u_5\}$, $\{u_2, u_4\}$, $\{u_6\}$, (2) $\{u_1, u_3\}$, $\{u_5\}$, $\{u_4, u_6\}$, $\{u_2\}$, (3) $\{u_1, u_3\}$, $\{u_5\}$, $\{u_2, u_6\}$, $\{u_4\}$, (4) $\{u_1, u_5\}$, $\{u_3\}$, $\{u_2, u_4\}$, $\{u_6\}$, (5) $\{u_1, u_5\}$, $\{u_3\}$, $\{u_4, u_6\}$, $\{u_2\}$, and (6) $\{u_1, u_5\}$, $\{u_3\}$, $\{u_2, u_6\}$, $\{u_4\}$.

Case 1. *The set* $\{u_1, u_3, u_5\} \cup \{u_2, u_4, u_6\}$ *is partitioned into the four subsets* $\{u_1, u_3\}$, $\{u_5\}$, $\{u_2, u_4\}$, $\{u_6\}$. In order for a labeled vertex to dominate u_1 and u_3, either u_2 is labeled 1 or an end-vertex neighbor w_2 of u_2 is labeled 2. Similarly, in order for a labeled vertex to dominate u_2 and u_4, either u_3 is labeled 1, an end-vertex neighbor w_3 of u_3 is labeled 2, or an end-vertex x_3 on a pendant path (x_3, w_3, u_3) of length 2 at u_3 is labeled 3.

- ⋆ First, assume that $f(u_2) = 1$ and u_1 and u_3 are dominated. In order to dominate u_2 and u_4, either an end-vertex neighbor w_3 of u_3 is labeled 2 or an end-vertex x_3 on a pendant path (x_3, w_3, u_3) of length 2 at u_3 is labeled 3. If $f(w_3) = 2$, then assigning the label 3 to w_4 (to dominate w_2 and u_6) and the label 4 to u_1 produces a minimal subtree T_1 of T shown in Figure 8. (This is the only possible irregular dominating labeling when $f(u_2) = 1$ and $f(w_3) = 2$.) If $f(x_3) = 3$, then this forces $f(x_4) = 4$ (to dominate w_3 and u_6) and so x_4 and w_4 cannot be dominated by a 4th labeled vertex, a contradiction.
- ⋆ Next, assume that $f(w_2) = 2$ and u_1 and u_3 are dominated. In order to dominate u_2 and u_4, either $f(u_3) = 1$ or an end-vertex x_3 on a pendant path (x_3, w_3, u_3) of length 2 at u_3 is labeled 3. If $f(u_3) = 1$, then this forces $f(w_3) = 3$ and $f(x_4) = 4$ and so x_4 is not dominated, a contradiction. If $f(x_3) = 3$, then the additional two labeled vertices cannot dominate the five undominated vertices u_6, u_5, w_2, w_3 and x_3, a contradiction.

Case 2. *The set* $\{u_1, u_3, u_5\} \cup \{u_2, u_4, u_6\}$ *is partitioned into the four subsets* $\{u_1, u_3\}$, $\{u_5\}$, $\{u_4, u_6\}$, $\{u_2\}$. In order for a labeled vertex to dominate u_1 and u_3, either u_2 is labeled 1 or an end-vertex neighbor w_2 of u_2 is labeled 2. Similarly, in order for a labeled vertex to dominate u_4 and u_6, either u_5 is labeled 1 or an end-vertex neighbor w_5 of u_5 is labeled 2. By symmetry, we may assume that $f(u_2) = 1$ and $f(w_5) = 2$. Assigning the label 3 to w_4 and the label 4 to u_1 produces a minimal subtree T_2 of T shown in Figure 8. This is the only possible minimal subtree of T in Case 2 (up to isomorphism).

Case 3. *The set* $\{u_1, u_3, u_5\} \cup \{u_2, u_4, u_6\}$ *is partitioned into the four subsets* $\{u_1, u_3\}$, $\{u_5\}$, $\{u_2, u_6\}$, $\{u_4\}$. In order for a labeled vertex to dominate u_1 and u_3, either u_2 is labeled 1 or an end-vertex neighbor w_2 of u_2 is labeled 2. Similarly, in order for a labeled vertex to dominate u_2 and u_6, either u_4 is labeled 2, an end-vertex neighbor w_4 of u_4 is labeled 3, or an end-vertex x_4 on a pendant path (x_4, w_4, u_4) of length 2 at u_4 is labeled 4.

- ⋆ First, assume that $f(u_2) = 1$ and u_1 and u_3 are dominated. If $f(u_4) = 2$, then assigning the label 3 to w_2 and the label 4 to u_1 produces a minimal subtree T_5 of T shown in Figure 8. If $f(w_4) = 3$, then assigning the label 4 to u_1 and the label 2 to u_6 produces a minimal subtree T_3 of T shown in Figure 8. If $f(x_4) = 4$, then this forces $f(u_3) = 2$ and u_3 dominates w_4 and u_5. However then, a 4th labeled vertex (not labeled 1) cannot dominate u_4 and x_4, a contradiction.
- ⋆ Next, assume that $f(w_2) = 2$ and u_1 and u_3 are dominated. If $f(w_4) = 3$, then this forces $f(u_1) = 4$ or $f(x_3) = 4$ and a 4th labeled vertex cannot dominate u_4 and w_2, a contradiction. If $f(x_4) = 4$, then assigning the label 3 to w_3 and the

label 1 to w_4 produce an irregular dominating labeling. Since u_1 is not labeled and u_1 and w_2 are similar, a minimal subtree of T produced here is isomorphic T_2 of Figure 8.

Case 4. *The set* $\{u_1, u_3, u_5\} \cup \{u_2, u_4, u_6\}$ *is partitioned into the four subsets* $\{u_1, u_5\}$, $\{u_3\}$, $\{u_2, u_4\}$, $\{u_6\}$. In order for a labeled vertex to dominate u_1 and u_5, either u_3 is labeled 2, an end-vertex neighbor w_3 of u_3 is labeled 3, or an end-vertex x_3 on a pendant path (x_3, w_3, u_3) of length 2 at u_3 is labeled 4. Similarly, in order for a labeled vertex to dominate u_2 and u_4, either u_3 is labeled 1, an end-vertex neighbor w_3 of u_3 is labeled 2, or an end-vertex x_3 on a pendant path (x_3, w_3, u_3) of length 2 at u_3 is labeled 3,

- ⋆ First, suppose that $f(u_3) = 2$ and u_1 and u_5 are dominated by u_3. This forces $f(x_3) = 3$ (and x_3 dominates u_2 and u_4), which in turn forces $f(w_3) = 1$ (and w_3 dominates u_3 and x_3). This then forces $f(x_4) = 4$ and x_4 dominates u_6 and w_2. However then, x_4 is not dominated, a contradiction.
- ⋆ Next, suppose that $f(w_3) = 3$ and u_1 and u_5 are dominated by w_3. If $f(u_3) = 1$, then assigning the label 2 to u_4 and the label 4 to u_2 produces a minimal subtree T_3 of T shown in Figure 8. This is the only minimal subtree produced when $f(u_3) = 1$ and $f(w_3) = 3$. If $f(w_3') = 2$ where $w_3' \neq w_3$, this forces $f(x_4) = 4$ and x_4 dominates $w_3.w_3'$ and u_6. However then x_4 is not dominated, a contradiction.
- ⋆ Finally, suppose that $f(x_3) = 4$ where (x_3, w_3, u_3) is a pendant path at u_3 and u_1 and u_5 are dominated by x_3. If $f(u_3) = 1$, then x_3, u_3 and u_6 cannot be dominated by the two remaining labeled vertices, a contradiction. If $f(w_3) = 2$, then this forces $f(w_4) = 3$ and w_4 dominates w_3 and u_6. Then a 4th labeled vertex cannot dominate x_3 and u_3. If $f(w_3') = 2$, then this forces $f(w_3) = 1$ and $f(u_1) = 3$. Then u_6 is not dominated, a contradiction. If $f(x_3') = 3$ where (x_3', w_3', u_3) is a pendant path at u_3 and $w_3' \neq w_3$, then assigning the label 1 to w_3 and the label 2 to u_4 produces a minimal subtree T_4 of T shown in Figure 8. If $f(x_3'') = 3$ where (x_3'', w_3, u_3) is a pendant path at u_3, then assigning the label 1 to w_3 and the label 2 to u_4 produces a minimal subtree T_5 of T shown in Figure 8.

Case 5. *The set* $\{u_1, u_3, u_5\} \cup \{u_2, u_4, u_6\}$ *is partitioned into the four subsets* $\{u_1, u_5\}$, $\{u_3\}$, $\{u_4, u_6\}$, $\{u_2\}$. In order for a labeled vertex to dominate u_1 and u_5, either u_3 is labeled 2, an end-vertex neighbor w_3 of u_3 is labeled 3, or an end-vertex x_3 on a pendant path (x_3, w_3, u_3) of length 2 at u_3 is labeled 4. Similarly, in order for a labeled vertex to dominate u_4 and u_6, either u_5 is labeled 1 or an end-vertex neighbor w_5 of u_5 is labeled 2.

- ⋆ First, suppose that $f(u_5) = 1$ and u_4 and u_6 by u_5. If $f(u_3) = 2$, then in order to dominate u_2, either $f(x_3) = 3$, $f(w_4) = 3$ or $f(w_5) = 4$. Then a 4th labeled vertex cannot dominate u_3 and a vertex in $\{w_3, w_4, w_5\}$ in each situation, a contradiction. If $f(w_3) = 3$, then assigning the label 4 to u_6 and the label 2 to any vertex in $\{u_1, w_2, w_4\}$ produces a minimal subtree of T shown in Figure 8. So, there are three distinct minimal subtrees T_1, T_2, and T_3 of T of Figure 8 if $f(u_5) = 1$ and $f(w_3) = 3$. If $f(x_3) = 4$, this forces $f(u_4) = 2$ and u_4 dominates w_3 and u_2. Since the labeled 1 is used, a 4th label vertex cannot dominate x_3 and u_3, a contradiction.
- ⋆ Next, suppose that $f(w_5) = 2$ and u_4 and u_6 by w_5. If $f(w_3) = 3$ (and w_3 dominates u_1 and u_5), then in order to dominate u_2 and w_3, either $f(u_3) = 1$ or $f(u_6) = 4$. In either situation, a 4th labeled vertex cannot dominate u_3 and w_5, a contradiction. If $f(x_3) = 4$ (and x_3 dominates u_1 and u_5), then in order to dominate u_2 and w_3, either $f(u_3) = 1$ or $f(u_5) = 3$. In either situation, a 4th labeled vertex cannot dominate u_3 and w_5, a contradiction.

Case 6. *The set* $\{u_1, u_3, u_5\} \cup \{u_2, u_4, u_6\}$ *is partitioned into the four subsets* $\{u_1, u_5\}$, $\{u_3\}$, $\{u_2, u_6\}$, $\{u_4\}$. In order for a labeled vertex to dominate u_1 and u_5, either u_3 is labeled 2, an end-vertex neighbor w_3 of u_3 is labeled 3, or an end-vertex x_3 on a pendant path (x_3, w_3, u_3) of length 2 at u_3 is labeled 4. Similarly, in order for a labeled vertex to dominate u_2 and u_6, either u_4 is labeled 2, an end-vertex neighbor w_4 of u_4 is labeled 3, or an end-vertex x_4 on a pendant path (x_4, w_4, u_4) of length 2 at u_4 is labeled 4. If u_1, u_5, u_2, u_6 are dominated by the two vertices labeled 2 or 3, say $f(u_3) = 2$ and $f(w_4) = 3$, then this forces $f(u_4) = 1$ (and u_4 dominates u_3 and w_4). However then, u_4 (or u_4 and an additional added vertex) cannot be dominated by a 4th labeled vertex, a contradiction. If u_1, u_5, u_2, u_6 are dominated by the two vertices labeled 2 or 4, say $f(u_3) = 2$ and $f(x_4) = 3$, then assigning the label 1 to w_4 and the label 3 to u_6 produces a minimal subtree T_4 of T shown in Figure 8. If u_1, u_5, u_2, u_6 are dominated by the two vertices labeled 3 or 4, say $f(w_3) = 3$ and $f(x_4) = 4$, then assigning the label 1 to w_4 and the label 2 to u_5 produces a minimal subtree T_2 of T shown in Figure 8. □

An extensive case-by-case analysis gives us the following two results.

Theorem 3.7. *If T is a tree of diameter 6 with $\tilde{\gamma}(T) = 4$, then every minimum irregular dominating labeling of T produces one of the labeled minimal subtrees in Figure 9.*

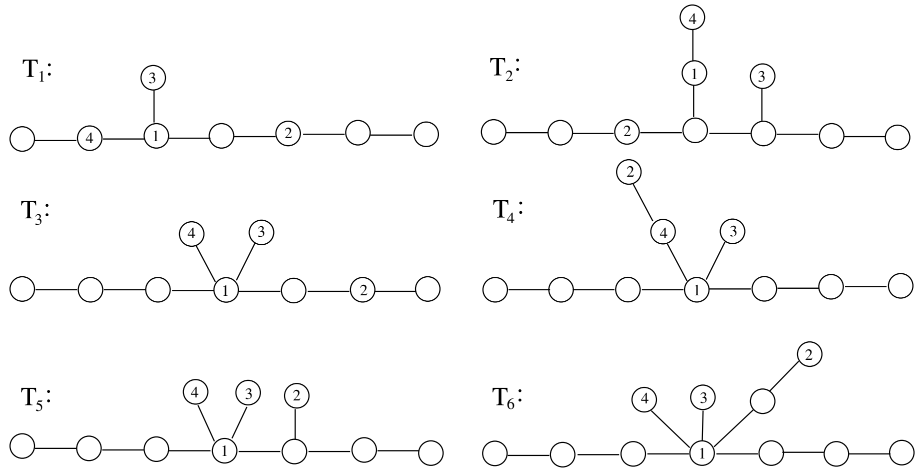

Figure 9: Minimal trees in a tree T with $\mathrm{diam}(T) = 6$ and $\tilde{\gamma}(T) = 4$.

Theorem 3.8. *If T is a tree of diameter 7 with $\tilde{\gamma}(T) = 4$, then every minimum irregular dominating labeling of T produces one of the labeled minimal subtrees in Figure 10.*

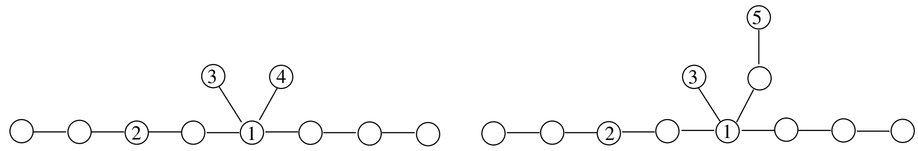

Figure 10: Minimal trees in a tree T with $\mathrm{diam}(T) = 7$ and $\tilde{\gamma}(T) = 4$.

References

[1] A. Ali, G. Chartrand, P. Zhang, *Irregularity in Graphs*, Springer, New York, 2021.
[2] P. Broe, G. Chartrand, P. Zhang, Irregular orbital domination in graphs, Preprint.
[3] G. Chartrand, M. A. Henning, K. Schultz, On orbital domination numbers of graphs, *J. Combin. Math. Combin. Comput.* **37** (2001) 3–26.
[4] G. Chartrand, P. Zhang, *Chromatic Graph Theory*, Second Edition, Chapman & Hall/CRC Press, Boca Raton, 2020.
[5] E. J. Cockayne, R. M. Dawes, S. T. Hedetniemi, Total domination in graphs, *Networks* **10** (1977) 211–219.
[6] L. Hayes, K. Schultz, J. Yates, Universal domination sequences of graphs, *Util. Math.* **54** (1998) 193–209.
[7] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.