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Research Article

A degree sum condition for Hamiltonian graphs

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Abstract

A graph is called Hamiltonian if it has a Hamiltonian cycle, where a Hamiltonian cycle is a cycle containing all vertices of the graph. It is shown in this note that if G is a 2-connected graph of order $n \geq 13$ such that $d(u) + d(v) \geq 2n - \delta - \kappa - 4$ for any pair of nonadjacent vertices u and v in G then either G is Hamiltonian or G belongs to some special families of graphs, where δ and κ are the minimum degree and connectivity of G, respectively.

Keywords: degree sum; minimum degree; connectivity; Hamiltonian graph.

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1. Introduction

In this note, only finite undirected graphs without loops or multiple edges are considered. The notation and terminology not defined here follow those in [2]. For a graph $G = (V, E)$, its order $|V|$ is denoted by n. The complement of a graph G is denoted by G^c. Denote by $\delta(G)$, $\alpha(G)$ and $\kappa(G)$ the minimum degree, independence number and connectivity of a graph G, respectively. For a vertex x in G, $N(x)$ denotes the set of those vertices which are adjacent to x in G. For a nonempty subset S of the vertex set V of G, denote by $G[S]$ the subgraph of G induced by S. Denote by G_r a graph of order r. For two disjoint graphs H and K, denote by $H \vee K$ the join of H and K.

A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A cycle C in a graph G is called a dominating cycle if the order of each component in the graph $G[V(G) - V(C)]$ is less than 2. If C is a cycle of G with a given orientation, we use x^+ to denote the successor of a vertex x on C along the orientation of C. We also use x^{++} to denote the successor of a vertex x^+ on C along the orientation of C. If $A \subseteq V(C)$, A^+ is defined as $\{v^+ : v \in A\}$. For a graph G and an integer s, if $\alpha \geq s$, $\sigma_s(G)$ is defined as

$$\min\{d(u_1) + d(u_2) + \ldots + d(u_s) : \ \{u_1, u_2, \ldots, u_s\} \text{ is an independent set in } G\};$$

and if $\alpha < s$, $\sigma_s(G)$ is defined as $+\infty$. Also, we define

$$\mathcal{A}_\alpha(n) := \left\{G : G \text{ is } G_{\frac{n-2}{2}} \vee \left(K^c_{\frac{n-2}{2}} \cup K_2\right)\right\},$$

$$\mathcal{A}_\beta(n) := \{G : V(G) = A_\beta,\, E(G) = B_\beta\}$$

where

$$A_\beta = V\left(G_{\frac{n-2}{2}} \vee K^c_{\frac{n-2}{2}}\right) \cup \{\, x, y\,\}$$

and

$$B_\beta = E\left(G_{\frac{n-2}{2}} \vee K^c_{\frac{n-2}{2}}\right) \cup \{\, xy\,\} \cup \left\{\, xu : u \in V\left(G_{\frac{n-2}{2}}\right) - \{\, a\,\}, a \in V\left(G_{\frac{n-2}{2}}\right)\right\} \cup \left\{\, yv : v \in V\left(G_{\frac{n-2}{2}}\right)\right\},$$

$$\mathcal{A}_\gamma(n) := \{G : V(G) = A_\gamma,\, E(G) = B_\gamma\,\},$$

where

$$A_\gamma = V\left(G_{\frac{n-2}{2}} \vee K^c_{\frac{n-2}{2}}\right) \cup \{\, x, y\,\}$$

and

$$B_\gamma = E\left(G_{\frac{n-2}{2}} \vee K^c_{\frac{n-2}{2}}\right) \cup \{\, xy\,\} \cup \left\{\, xu : u \in V\left(G_{\frac{n-2}{2}}\right)\right\} \cup \left\{\, yv : v \in V\left(G_{\frac{n-2}{2}}\right) - \{\, b\}, b \in V\left(G_{\frac{n-2}{2}}\right)\right\},$$

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$$\mathcal{A}_\epsilon(n) := \{G : V(G) = A_\epsilon,\, E(G) = B_\epsilon\},$$

where

$$A_\epsilon = V\left(G_{\frac{n-2}{2}} \vee K^c_{\frac{n-2}{2}}\right) \cup \{\, x, y \,\}$$

and

$$B_\epsilon = E\left(G_{\frac{n-2}{2}} \vee (K^c_{\frac{n-2}{2}}\right) \cup \{\, xy \,\} \cup \left\{ xu : u \in V\left(G_{\frac{n-2}{2}}\right) - \{\, a \,\}, a \in V\left(G_{\frac{n-2}{2}}\right) \right\} \cup \left\{ yv : v \in V\left(G_{\frac{n-2}{2}}\right) - \{\, b \,\}, b \in V\left(G_{\frac{n-2}{2}}\right) \right\},$$

provided that $a \neq b$,

$$\mathcal{A}(n) := \mathcal{A}_\alpha(n) \cup \mathcal{A}_\beta(n) \cup \mathcal{A}_\gamma(n) \cup \mathcal{A}_\epsilon(n),$$
$$\mathcal{B}(n) := \left\{ G : G \text{ is } G_{\frac{n-2}{2}} \vee K^c_{\frac{n+2}{2}} \right\},$$
$$\mathcal{C}(n) := \left\{ G : G \text{ is } G_{\frac{n-1}{2}} \vee K^c_{\frac{n+1}{2}} \right\}.$$

In this note, we present the following sufficient condition involving σ_2, δ, and κ for Hamiltonian graphs.

Theorem 1.1. *Let G be a 2-connected graph of order $n \geq 13$. If $\sigma_2 \geq 2n - \delta - \kappa - 4$, then either G is Hamiltonian or G is in $\mathcal{A}(n) \cup \mathcal{B}(n) \cup \mathcal{C}(n)$.*

2. Lemmas

In order to prove Theorem 1.1, we need the following known results. The first one follows from the proof of Theorem 1 in [3].

Lemma 2.1. *Let G be a graph of order $n \geq 3$. If $\alpha \leq \kappa$, then G is Hamiltonian.*

The next lemma can be found in [4]. It was also used in [5].

Lemma 2.2. *Let G be a 2-connected graph. If $d(u) + d(v) \geq n - 1$ for each pair of nonadjacent vertices u, v then either G is Hamiltonian or G is in $\mathcal{C}(n)$.*

The next result follows from Theorem 7 and the proof of Theorem 10 in [1].

Lemma 2.3. *Let G be a 2-connected graph of order n such that $\sigma_3 \geq n + 2$. Then every longest cycle C in G is a dominating cycle and*

$$\max\{\, d(v) : v \in V(G) - V(C) \,\} \geq \frac{\sigma_3}{3}.$$

The next result is Lemma 8 in [1].

Lemma 2.4. *Let G be a graph of order n such that $\delta \geq 2$ and $\sigma_3 \geq n$. Let G contain a longest cycle C which is a dominating cycle. If $v_0 \in V(G) - V(C)$ and $A = N(v_0)$, then $(V(G) - V(C)) \cup A^+$ is an independent set of vertices in G.*

3. Proof of Theorem 1.1

Let G be a graph satisfying the conditions of Theorem 1.1. Suppose that G is not Hamiltonian. From Lemma 2.1, it follows that $\alpha \geq \kappa + 1 \geq 3$. If $\sigma_2 \geq n - 1$ then Lemma 2.2 implies that G is in $\mathcal{C}(n)$. From now on, we assume that $\sigma_2 \leq n - 2$.

Suppose u is a vertex in an independent set I in G with $|I| = \alpha$. Then $N(u)$ is a subset of of $V - I$. Thus,

$$\delta \leq d(u) = |N(u)| \leq |V - I|$$

and hence $\delta \leq n - \alpha$, or $\alpha \leq n - \delta$. Now,

$$\begin{aligned} n - 2 &\geq \sigma_2 \\ &\geq 2n - \delta - \kappa - 4 \\ &\geq n + n - \delta - \kappa - 4 \\ &\geq n + \alpha - \kappa - 4 \\ &\geq n + \kappa + 1 - \kappa - 4 \\ &= n - 3. \end{aligned}$$

Thus, we have three possible cases of $n - \delta = \alpha$ and $\alpha = \kappa + 1$, $n - \delta = \alpha + 1$ and $\alpha = \kappa + 1$, or $n - \delta = \alpha$ and $\alpha = \kappa + 2$.

Let $\{x, y, z\}$ be an independent set in G such that $\sigma_3 = d(x) + d(y) + d(z)$. Then

$$\begin{aligned}\sigma_3 &= d(x) + d(y) + d(z)\\ &= \frac{(d(x) + d(y)) + (d(y) + d(z)) + (d(z) + d(x))}{2}\\ &\geq \frac{\sigma_2 + \sigma_2 + \sigma_2}{2}\\ &= \frac{3\sigma_2}{2}\\ &\geq \frac{3(n-3)}{2}\\ &\geq n + 2.\end{aligned}$$

Let C be a longest cycle in G with a given orientation. From Lemma 2.3, it follows that C is also a dominating cycle. Suppose that $V(G) - V(C) = \{v_0, v_1, \ldots, v_r\}$. Without loss of generality, we assume that $d(v_0) \geq d(v_1) \geq \cdots \geq d(v_r)$. Set $A = N(v_0) = \{z_1, \ldots, z_s\}$. Then, Lemma 2.4 implies that $(V(G) - V(C)) \cup A^+$ is an independent set of vertices in G. Hence,

$$\alpha \geq |(V(G) - V(C)) \cup A^+| = |(V(G) - V(C))| + |A^+| = n - |V(C)| + d(v_0).$$

Next, we divide the remaining proof into the following cases.

Case 1. $n - \delta = \alpha$ and $\alpha = \kappa + 1$.

In this case, we have $n - \delta = \kappa + 1 = \alpha \geq n - |V(C)| + d(v_0) \geq 1 + \delta \geq 1 + \kappa$. Thus, $\kappa = \delta = d(v_0)$ and $n - |V(C)| = 1$. Notice that $n - \delta = \kappa + 1$. We have $\kappa = \delta = d(v_0) = (n-1)/2$. This leads to $n - 1 = 2\delta \leq \sigma_2 \leq n - 2$, a contradiction.

Case 2. $n - \delta = \alpha + 1$ and $\alpha = \kappa + 1$.

In this case, we have $n - \delta - 1 = \kappa + 1 = \alpha \geq n - |V(C)| + d(v_0) \geq 1 + \delta \geq 1 + \kappa$. Thus, $\kappa = \delta = d(v_0)$ and $n - |V(C)| = 1$. Notice that $n - \delta - 1 = \kappa + 1$. We have $\kappa = \delta = d(v_0) = s = (n-2)/2$. Since C is longest cycle in G, C must be in the following form

$$C = z_1 z_1^+ z_2 z_2^+ \cdots z_i z_i^+ z_i^{++} z_{i+1} z_{i+1}^+ \cdots z_s z_s^+ z_1.$$

Set

$$V\left(K^c_{\frac{n-2}{2}}\right) := \{v_0, z_1^+, \ldots, z_{i-1}^+, z_{i+1}^+, \ldots, z_s^+\},$$
$$V\left(G_{\frac{n-2}{2}}\right) := \{z_1, z_2, \ldots, z_s\},$$

$x = z_i^+$, and $y = z_i^{++}$ in $\mathcal{A}(n)$. It can be verified that in this case G belongs to $\mathcal{A}(n)$.

Case 3. $n - \delta = \alpha$ and $\alpha = \kappa + 2$.

In this case, we first note that $\sigma_2 = n - 2$. We further have $n - \delta = \kappa + 2 = \alpha \geq n - |V(C)| + d(v_0) \geq 1 + \delta \geq 1 + \kappa$. We therefore have the following subcases.

Case 3.1. $n - |V(C)| = 2$ and $\kappa = \delta = d(v_0)$.

In this subcase, we have $n - \delta = \kappa + 2$. We further have $\kappa = \delta = d(v_0) = s = (n-2)/2$. Since $d(v_0) + d(v_1) \geq \sigma_2 = n - 2$, $d(v_0) = s = (n-2)/2$, and $d(v_0) \geq d(v_1)$, $d(v_0) = d(v_1) = (n-2)/2$. Since C is longest cycle in G, C must be in the following form

$$C = z_1 z_1^+ z_2 z_2^+ \cdots z_i z_i^+ z_{i+1} z_{i+1}^+ \cdots z_s z_s^+ z_1.$$

Set

$$V\left(K^c_{\frac{n+2}{2}}\right) := \{v_0, v_1, z_1^+, \ldots, z_i^+, \ldots, z_s^+\}$$

and

$$V\left(G_{\frac{n-2}{2}}\right) := \{z_1, z_2, \ldots, z_s\}$$

in $\mathcal{B}(n)$. It is easy to verify that in this subcase G belongs to $\mathcal{B}(n)$.

Case 3.2. $n - |V(C)| = 1$ and $\kappa = \delta$, $d(v_0) = \delta + 1$.

In this subcase, we have $n - \delta = \kappa + 2$. We further have $\kappa = \delta = (n-2)/2$. Thus, $d(v_0) = \delta + 1 = n/2$. This leads to $|V(C)| \geq 2d(v_0) = n$, a contradiction.

Case 3.3. $n - |V(C)| = 1$ and $\kappa + 1 = \delta$, $d(v_0) = \delta$.

In this subcase, we have $n - \delta = \kappa + 2$. We further have $\kappa = (n-3)/2$. Thus, $d(v_0) = \delta = \kappa + 1 = (n-1)/2$. This leads to $n - 1 = 2\delta \leq \sigma_2 = n - 2$, a contradiction.

Hence, the proof of Theorem 1.1 is complete.

Remark. Let k be an integer such that $k \geq 6$. Construct a graph $G = (V, E)$ where

$$V = \{\, x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k, y_{k+1}, y_{k+2} \,\}$$

and

$$E = \{\, x_i y_j : 1 \leq i \leq k, 1 \leq j \leq k+2 \,\} \cup \{\, y_1 y_2, y_{k+1} y_{k+2} \,\}.$$

Then $n = 2k + 2$, $\delta = k$, $\kappa = k$, $\sigma_2(G) = 2k = n - 2 \geq 2n - \delta - \kappa - 4$. So, one can use Theorem 1.1 to conclude that G is Hamiltonian. However, one cannot use the Ore's condition and Lemma 2.2 to decide that G is Hamiltonian.

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