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Research Article

Generalized functions and the expansion of moments

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Abstract

In this paper we show how generalized functions can be used to study the asymptotic behavior of moments of integrals and series in the real and complex domain. We explain how an asymptotic expansion can be obtained in the case of distributions, but show that no such results are possible for analytic functionals.

Keywords: moments; distributions; analytic functionals; generalized functions; rearrangements of series.

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1. Introduction

The aim of this article is to show how one can study the asymptotic behavior of moments of the type

$$M_n(f) = \int_X (f(x))^n \, \mathrm{d}x \, , \tag{1}$$

as $n \to \infty$, where f is a bounded measurable function defined in a measurable space (X, μ) with values in $\mathbb{R}$ or $\mathbb{C}$, by assigning to f a generalized function F in such a way that the moments $M_n(f)$ coincide with the moments[‡] of F,

$$\mu_n(F) = \langle F(u), u^n \rangle \, , \tag{2}$$

for n large. In particular, if $X = \mathbb{N}$ with the counting measure, we may consider the behavior of moment series, $M_n(f) = M_p(\{\xi_q\}) = \sum_{q=0}^{\infty} \xi_q^p$, where $\{\xi_q\}_{q=0}^{\infty}$ is a bounded sequence of complex numbers.

Depending on the problem, the generalized function F could be a distribution, an analytic functional, or a hyperfunction. Interestingly, the results are quite different for each type of generalized function.

The study of the behavior of the moments $M_n(f)$ for n large is very important in many areas. We mention the recent work of Schlage-Puchta [30], who motivated by problems in signal analysis [4] obtained the asymptotic expansion

$$\int_0^{\infty} \left(\frac{\sin x}{x} \right)^n \mathrm{d}x \sim \sqrt{\frac{3\pi}{2n}} \left(1 - \frac{3}{20n} - \frac{13}{1120n^2} + \cdots \right) .$$

This formula illustrates a basic but important point about the moments $M_n(f)$. Namely, if a function with real values can be both positive and negative, then there is no simple or useful way to define powers of the type $(f(x))^{\lambda}$ if λ is not an integer, and thus integrals of the type $\int_X (f(x))^{\lambda} \, \mathrm{d}x$ do *not* make sense unless λ is an integer. Actually, when $f(x) > 0$ for all x, one can obtain the asymptotic expansion of the now well defined integral $\int_X (f(x))^{\lambda} \, \mathrm{d}x$ as the *real* number $\lambda \to \infty$ by using a change of variables in the Laplace asymptotic formula [10, (3.133)], so that in particular

$$\int_{-\infty}^{\infty} (f(x))^{\lambda} \phi(x) \, \mathrm{d}x \sim f(x_0)^{\lambda + 1/2} \sqrt{\frac{-2\pi}{\lambda f''(x_0)}} \phi(x_0) \, , \tag{3}$$

as $\lambda \to \infty$ if the *positive* function f has a single maximum at the interior point x_0 where $f''(x_0) < 0$. However, (3) cannot be applied if f changes signs as in the case when $f(x) = \sin x / x$.

The study of the moments for functions with complex values has also attracted a lot of attention. We mention the work of Duistermaat and van der Kallen [6] who considered the moments of rational functions over the unit disc and employing

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‡Unfortunately, the same term "moments" is applied to both (1) and (2) in the literature. We have tried to avoid any confusion caused by this practice.

this were able to solve a conjecture of Mathieu [5, 23]. Recently Müger and Tuset [25] considered the behavior of the moments of a non constant complex polynomial f, showing that

$$\limsup_{n\to\infty} |M_n(f)|^{1/n} > 0\,,$$

or, equivalently, that the power series $\sum_{n=0}^{\infty} M_n(f)\, z^n$ does not define an entire function. Other properties of the moments of complex polynomials were given by Markowsky and Phung [22]. Corresponding results for non periodic real analytic functions were given in [9]. The case of moment series was studied by Boudabra and Markowsky [3]. Notice that while the method of steepest descent [10, Section 3.6] allows one to obtain the asymptotic behavior of several path integrals of the form $\int_\Gamma (f(z))^\lambda \,\mathrm{d}z$ as the *complex* number $\lambda \to \infty$ when f is analytic and zero free, these results do not apply in the mentioned works.

Another place where moments are important is in the study of the Cauchy transform. Indeed, it is interesting to observe that if F is an analytic functional with compact support, that is $T \in \mathfrak{O}'(\mathbb{C})$, where $\mathfrak{O}(\mathbb{C})$ is the space of entire functions, with its canonical Fréchet topology, then its Cauchy representation

$$f(z) = \mathcal{C}\{T(\omega)\,; z\} = \frac{1}{2\pi i}\left\langle T(\omega)\,, \frac{1}{\omega - z}\right\rangle,$$

$z \in \mathbb{C}\setminus K$, where K is a carrier [2, 24] of T, has the Laurent expansion

$$\mathcal{C}\{T(\omega)\,; z\} = -\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{\mu_n(T)}{z^{n+1}}\,, \qquad |z| > \rho\,, \tag{4}$$

in terms of the moments $\mu_n(T) = \langle T(\omega)\,, \omega^n\rangle$. Here $\rho = \max\{|z| : z \in K\}$.[§]

The plan of this article is the following. In Section 2 we give a general method for the construction of distributions associated to probably divergent integrals of the type $\int_X \phi(f(x))\,\mathrm{d}\mu(x)$, where ϕ is a test function in $\mathbb{R}^d$ and where f is a bounded measurable function defined in a measurable space (X, μ) and with values in $\mathbb{R}^d$. We then discuss the Stirling numbers and the asymptotic expansion of quotients of gamma functions in Section 3 in order to give the expansion of the moments of distributions of one variable with compact support in Section 4. We then consider moments of the type (1) of real valued functions in Section 5. Finally we study expansions of complex valued series in Section 6; we concentrate on series since the study of moments of integrals by employing analytic functionals and hyperfunctions has been already presented in [9].

2. Distributions: a general construction

In this article we will need to employ a method of construction of distributions as regularizations of integrals of the type

$$\int_X \phi(f(x))\,\mathrm{d}\mu(x)\,, \tag{5}$$

where ϕ is a test function in $\mathbb{R}^d$ and where f is a bounded measurable function defined in a measurable space (X, μ) with values in $\mathbb{R}^d$. In general the integrals in (5) are divergent; notice, in particular that if $\phi = 1$ and X has infinite measure the integral is divergent.

Similar considerations [9] allow us to construct analytic functionals from complex valued functions f.

2.1. Preliminaries

We refer to the textbooks [1, 17, 31, 34] for basic ideas about distributions. The spaces of test functions $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{S}$ and the corresponding spaces of distributions are well known. In general [35], we call a topological vector space $\mathcal{A}$ a space of test functions if $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{E}$, where the inclusions are continuous, and if $\frac{d}{dx}$ is a continuous operator of $\mathcal{A}$.

The various methods for the regularization of integrals and series can also be found in these textbooks; see also [26] and [10].

Basic ideas about analytic functionals can be found in the textbooks [2, 24]. We will denote by $\mathfrak{O}(\mathbb{C})$ the Fréchet space of entire functions and by $\mathfrak{O}'(\mathbb{C})$ its dual, the space of analytic functionals with compact support.

We shall also employ results from functional analysis freely and refer the reader to the textbooks [15, 32].

[§]Observe that $\mathcal{C}\{T(\omega)\,; z\}$ is defined if $z \in \overline{\mathbb{C}}\setminus K$, but the series in (4) is divergent if $|z| < \rho$. It will be exterior Euler summable [11] if $z \notin K$, where K is the minimal convex carrier of T.

2.2. One variable

We now will consider a general construction of distributions from divergent integrals of the type of (5); it is convenient to start in the case of real valued functions. Let $\mathcal{A}$ be a space of test functions on $\mathbb{R}$ or on $[0,\infty)$, such as $\mathcal{A}=\mathcal{D}$, $\mathcal{S}$, or $\mathcal{E}$. Following Grafakos and Teschl [13] and Estrada [8], we shall denote as $\mathcal{R}_n=\mathcal{R}_n(\mathcal{A})$ the subspace

$$\mathcal{R}_n=\left\{\phi\in\mathcal{A}:\phi^{(j)}(0)=0\,,\;0\le j\le n-1\right\}\,,$$

for $n=1,2,\ldots$, or for $n=\infty$. Since $\mathcal{R}_n$ is a closed subspace of $\mathcal{A}$ for any n, it follows from the Hahn-Banach theorem that any distribution $f_n\in\mathcal{R}_n'$ has extensions $f\in\mathcal{A}'$. If f_* is an extension, then the general form of all extensions is

$$f(x)=f_*(x)+\sum_{j=0}^{n-1}c_j\delta^{(j)}(x)\,, \tag{6}$$

where the c_j are arbitrary constants. For $\mathcal{R}_\infty$ it would be (6) but for an arbitrary n.

Notice, however, that the extension from $\mathcal{R}_\infty$ can be done in two steps. Let us work in $\mathcal{R}_\infty(\mathcal{E})$ to fix the ideas; the analysis in other spaces of test functions being similar. First, because of the way the topology of the space of test functions $\mathcal{E}(\mathbb{R})$ is defined, in terms of the seminorms of the type

$$\|\phi\|_{a,n}=\max_{|x|\le a,\,j\le n}\left|\phi^{(j)}(x)\right|\,,$$

it follows that if $f_\infty\in\mathcal{R}_\infty'$ then there exists $n\in\mathbb{N}$ such that f_∞ admits a continuous extension to $\mathcal{R}_n'$, that is, f_∞ is continuous with respect to this seminorm:

$$|\langle f_\infty(x),\phi(x)\rangle|\le M\|\phi\|_{a,n}\,,$$

whenever $\phi\in\mathcal{R}_\infty$ has support included in $[-a,a]$. Hence f_∞ can be extended (in a unique way) to this $\mathcal{R}_n$. Another extension where one actually uses the Hahn-Banach theorem gives extensions to $\mathcal{A}'$; this second extension contains n arbitrary constants.

We may employ these ideas for distributions defined by integrals. Indeed, we immediately obtain the following result.¶

Lemma 2.1. *Suppose that* f *is a measurable function defined in* $\mathbb{R}$ *such that* $\mathsf{f}\chi_{\mathbb{R}\setminus(-a,a)}$ *is a regular distribution of* $\mathcal{A}'$ *for all* $a>0$*, and such that the integral*

$$\int_{-\infty}^{\infty}\mathsf{f}(x)\,\phi(x)\,\mathrm{d}x\,, \tag{7}$$

exists (as a $(\mathfrak{Z})$ *integral) for each* $\phi\in\mathcal{R}_\infty$*. Then there exists* $n\in\mathbb{N}$ *such that the integral will exist in the same sense for all* $\phi\in\mathcal{R}_n$ *and there exist distributions (not unique, depending on* n *arbitrary constants)* $f\in\mathcal{A}'$ *such that*

$$\langle f(x),\phi(x)\rangle=\int_{-\infty}^{\infty}\mathsf{f}(x)\,\phi(x)\,\mathrm{d}x\,,\quad\phi\in\mathcal{R}_n\,.$$

The $(\mathfrak{Z})$ sense could be absolute convergence at 0, that is Lebesgue integrals, or conditionally convergent at 0, that is, improper Riemann integrals at the origin, or others such as Denjoy integrals, etc. In general one needs to use the *same* integration sense when extending to $\mathcal{R}_n$, as the next example shows.

Example 2.1. *Consider the function* $\mathsf{f}(x)=e^{1/x^2}x^{-2}\cos\left(e^{1/x^2}\right)$*. In this case the integrals (7) exist whenever* $\phi\in\mathcal{R}_1(\mathcal{D})$ *and are conditionally convergent, in general, but they might not be absolutely convergent, even if* $\phi\in\mathcal{R}_\infty$*.*

Our aim is to use these ideas to define distributions not from integrals of the form (7) but from integrals of the type $\int_X\phi(f(x))\,\mathrm{d}\mu(x)$, where (X,μ) is a measure space and $f\in L^\infty(X)$ is real-valued. In the next result we can take $\mathcal{A}=\mathcal{E}$.

Lemma 2.2. *Suppose* $\phi\circ f\in L^1(X)$ *whenever* $\phi\in\mathcal{R}_\infty(\mathcal{E})$*. Then there exists* $n\in\mathbb{N}$ *such that* $\phi\circ f\in L^1(X)$ *if* $\phi\in\mathcal{R}_n(\mathcal{E})$*. If* $k>n$ *then* $f\in L^k(X)$*, that is,*

$$\int_X|f(x)|^k\,\mathrm{d}\mu(x)<\infty\,. \tag{8}$$

There exist distributions $F\in\mathcal{E}'$ *such that*

$$\langle F(u),\phi(u)\rangle=\int_X\phi(f(x))\,\mathrm{d}\mu(x)\,,\quad\phi\in\mathcal{R}_n(\mathcal{E})\,.$$

¶In the following lemma we use the notation f for the measurable function and f for the probably not uniquely determined distribution. We will employ such notation only if there is a danger of confusion.

Proof. The proof follows from our analysis, but there is a little detail when establishing that $f \in L^k(X)$ if $k > n$. Indeed, if k is even, $k \geq n$, then x^k belongs to $\mathcal{R}_n(\mathcal{E})$ and thus (8) follows. Thus $f \in L^n(X)$ or $f \in L^{n+1}(X)$, and since if $f \in L^q(X)$ then $f \in L^k(X)$ for $k > q$, we obtain (8) whenever $k > n$. □

In the case X is $\mathbb{R}$ or $[0, \infty)$ we may also obtain a stronger result. Actually, for a general space X the Lemma 2.2 applies to Lebesgue integrals, which for $\mathbb{R}$ or $[0,\infty)$ means *absolutely* convergent integrals. However, in this case we may also consider *conditionally* convergent integrals.

Lemma 2.3. *Let $f \in L^\infty([0,\infty))$ and g measurable and non-negative in $[0,\infty)$ such that the integrals*

$$\int_0^\infty \phi(f(x))\, g(x)\, \mathrm{d}x\,, \tag{9}$$

are (maybe conditionally) convergent at the origin for any $\phi \in \mathcal{R}_\infty(\mathcal{E})$. Then for some $n \in \mathbb{N}$ the integrals are all absolutely convergent if $\phi \in \mathcal{R}_n(\mathcal{E})$. Also

$$\int_0^\infty |f(x)|^k\, g(x)\, \mathrm{d}x < \infty\,,$$

if $k > n$. There exist distributions $F \in \mathcal{E}'$ such that

$$\langle F(u), \phi(u)\rangle = \int_0^\infty \phi(f(x))\, \mathrm{d}x\,,$$

for $\phi \in \mathcal{R}_n(\mathcal{E})$.

Proof. Indeed, if we define the functional $F_\infty \in \mathcal{R}'_\infty(\mathcal{E})$ by the formula (9) then it will have a uniquely defined extension to $\mathcal{R}'_{n_0}(\mathcal{E})$ for some n_0. As the proof of the Lemma 2.2 shows, this implies that $f \in L^k([0,\infty); g(x)\, \mathrm{d}x)$ if $k > n_0$. Consequently, if $n = n_0 + 1$, then the integrals (9) will be absolutely convergent if $\phi \in \mathcal{R}_n(\mathcal{E})$. □

Example 2.1 shows that a corresponding result on absolute convergence does not hold for integrals of the type (7), in general.

The next lemma complements these results.

Lemma 2.4. *Let $f \in L^\infty([0,\infty))$ and g measurable and non-negative in $[0,\infty)$. Then $\int_0^\infty \phi(f(x))\, g(x)\, \mathrm{d}x$ admits regularizations in $\mathcal{E}$, that is, there exist distributions $F \in \mathcal{E}'$ such that $\langle F(u), \phi(u)\rangle = \int_0^\infty \phi(f(x))\, \mathrm{d}x$ for $\phi \in \mathcal{R}_\infty(\mathcal{E})$ if and only if there exists k such that $\int_0^\infty |f(x)|^k\, g(x)\, \mathrm{d}x < \infty$.*

We can also consider the extension of distributions defined by series, that is, if $X = \mathbb{N}$ and μ is a positive multiple of the counting measure. The proof of the ensuing result is identical to that of the Lemma 2.3.

Lemma 2.5. *Let $\{a_q\}_{q=0}^\infty$ be a sequence of real numbers with $\lim_{q\to\infty} a_q = 0$ and $\{\mu_q\}_{q=0}^\infty$ another sequence with $\mu_q > 0$ for all q. Suppose that the series*

$$\sum_{q=0}^\infty \phi(a_q)\, \mu_q\,, \tag{10}$$

are (maybe conditionally) convergent if $\phi \in \mathcal{R}_\infty(\mathcal{E})$. Then for some $n \in \mathbb{N}$ the series are absolutely convergent if $\phi \in \mathcal{R}_n(\mathcal{E})$. If $k > n$ then

$$\sum_{q=0}^\infty |a_q|^k\, \mu_q < \infty\,. \tag{11}$$

There are distributions $F \in \mathcal{E}'$ such that

$$\langle F(u), \phi(u)\rangle = \sum_{q=0}^\infty \phi(a_q)\, \mu_q\,, \quad \phi \in \mathcal{R}_n(\mathcal{E})\,. \tag{12}$$

Observe that a distribution F that satisfies (12) is a regularization of the series of delta functions

$$\sum_{q=0}^\infty \mu_q \delta(u - a_q)\,, \tag{13}$$

that is probably divergent in $\mathcal{E}'$. Notice also that the support of any such distribution F is the set $\operatorname{supp} F = \{a_q : q \geq 0\} \cup \{0\}$. In case $\mu_q = 1$ for all q, then the convergence of the series (10) for all $\phi \in \mathcal{R}_\infty(\mathcal{E})$ implies that $\lim_{q\to\infty} a_q = 0$. The series (13) admits a regularization in $\mathcal{E}$ if and only if (11) is satisfied for some k.

2.3. Several variables

Most of the ideas of the one variable case also apply to the multidimensional situation. In this article we would be interested mainly in the situation of functions into $\mathbb{C}$, that we shall identify with $\mathbb{R}^2$ in this analysis.

Indeed, if $\mathcal{A}$ is a space of test functions in $\mathbb{R}^d$ denote by $\mathcal{R}_n = \mathcal{R}_n(\mathcal{A})$ the subspace

$$\mathcal{R}_n = \left\{\phi \in \mathcal{A} : \nabla^{\mathbf{j}}\phi(\mathbf{0}) = 0 \, , \ 0 \le |\mathbf{j}| \le n-1\right\} ,$$

for $n = 1, 2, \ldots$, or for $n = \infty$. Here ∇ is the gradient vector operator, $\mathbf{j} = (j_1, \ldots, j_d)$ is a multi-index and $|\mathbf{j}| = \sum_{q=1}^{d} j_q$. Since $\mathcal{R}_n$ is a closed subspace of $\mathcal{A}$ for any n, it follows from the Hahn-Banach theorem that any distribution $f_n \in \mathcal{R}_n'$ has extensions $f \in \mathcal{A}'$. If f_* is an extension, then the general form of all extensions is

$$f(x) = f_*(x) + \sum_{|\mathbf{j}| \le n-1} c_{\mathbf{j}} \nabla^{\mathbf{j}} \delta(\mathbf{x}) \, , \tag{14}$$

where the $c_{\mathbf{j}}$ are arbitrary constants. (For $\mathcal{R}_\infty$ it would be (14) for an arbitrary n).

The extension from $\mathcal{R}_\infty$ can be done in two steps. First, because of the way the topology of spaces of test functions is defined, in terms of the seminorms of the type

$$\|\phi\|_{a,n} = \max_{|\mathbf{x}| \le a, \, |\mathbf{j}| \le n-1} \left|\nabla^{\mathbf{j}}\phi(\mathbf{x})\right| \, ,$$

(plus some other condition at infinity that is not important presently), it follows that if $f_\infty \in \mathcal{R}_\infty'$ then there exists $n \in \mathbb{N}$ such that f_∞ admits a continuous extension to $\mathcal{R}_n$, that is, f_∞ is continuous with respect to this seminorm:

$$|\langle f_\infty(x), \phi(x)\rangle| \le M \|\phi\|_{a,n} \, ,$$

whenever $\phi \in \mathcal{R}_\infty$ has support included in the ball $B_a = \{\mathbf{x} : |\mathbf{x}| \le a\}$. Hence f_∞ can be extended (in a unique way) to this $\mathcal{R}_n$; another extension where one actually uses the Hahn-Banach theorem gives extensions to $\mathcal{A}'$, that contain arbitrary constants $c_{\mathbf{j}}$ for $|\mathbf{j}| < n$.

Repeating the arguments of the previous section, we obtain the following result for functions from a measure space (X, μ) to $\mathbb{R}^d$.

Lemma 2.6. *Let $\mathbf{f} : X \to \mathbb{R}^d$ be a bounded measurable function. Suppose $\phi \circ \mathbf{f} \in L^1(X)$ whenever $\phi \in \mathcal{R}_\infty(\mathcal{E})$. Then there exists $n \in \mathbb{N}$ such that $\phi \circ \mathbf{f} \in L^1(X)$ if $\phi \in \mathcal{R}_n(\mathcal{E})$. If $k > n$ then*

$$\int_X \|\mathbf{f}(x)\|^k \, \mathrm{d}\mu(x) < \infty \, .$$

There exist distributions $F \in \mathcal{E}'$ such that

$$\langle F(\mathbf{u}), \phi(\mathbf{u})\rangle = \int_X \phi(\mathbf{f}(x)) \, \mathrm{d}\mu(x) \, , \quad \phi \in \mathcal{R}_n(\mathcal{E}) \, .$$

This lemma covers the case of Lebesgue integrals. When $X = [0, \infty)$ or $\mathbb{R}$ we may consider conditionally convergent integrals.

Lemma 2.7. *Let $\mathbf{f} : [0, \infty) \to \mathbb{R}^d$ be a bounded measurable function and g measurable and non-negative in $[0, \infty)$ such that the integrals*

$$\int_0^\infty \phi(\mathbf{f}(x)) \, g(x) \, \mathrm{d}x \, ,$$

are (maybe conditionally) convergent at the origin for any $\phi \in \mathcal{R}_\infty(\mathcal{E})$. Then for some $n \in \mathbb{N}$ the integrals are all absolutely convergent if $\phi \in \mathcal{R}_n(\mathcal{E})$. Also

$$\int_0^\infty \|\mathbf{f}(x)\|^k \, g(x) \, \mathrm{d}x < \infty \, , \tag{15}$$

if $k > n$. There are distributions $F \in \mathcal{E}'$ such that

$$\langle F(\mathbf{u}), \phi(\mathbf{u})\rangle = \int_0^\infty \phi(\mathbf{f}(x)) \, g(x) \, \mathrm{d}x \, , \quad \phi \in \mathcal{R}_n(\mathcal{E}) \, .$$

The existence of such regularizations F is equivalent to the existence of k for which (15) holds.

For conditionally convergent series we obtain the next result.

Lemma 2.8. *Let* $\{\mathbf{a}_q\}_{q=0}^{\infty}$ *be a sequence in* $\mathbb{R}^d$ *with* $\lim_{q\to\infty}\mathbf{a}_q = 0$ *and* $\{\mu_q\}_{q=0}^{\infty}$ *a sequence with* $\mu_q > 0$ *for all* q*. Suppose that the series*

$$\sum_{q=0}^{\infty} \phi\left(\mathbf{a}_q\right)\mu_q\,,$$

are (maybe conditionally) convergent if $\phi \in \mathcal{R}_{\infty}\left(\mathcal{E}\right)$*. Then for some* $n \in \mathbb{N}$ *the series are all absolutely convergent whenever* $\phi \in \mathcal{R}_n\left(\mathcal{E}\right)$*. If* $k > n$ *then*

$$\sum_{q=0}^{\infty} \left\|\mathbf{a}_q\right\|^k \mu_q < \infty\,. \tag{16}$$

There are distributions $F \in \mathcal{E}'$ *such that*

$$\left\langle F\left(\mathbf{u}\right), \phi\left(\mathbf{u}\right)\right\rangle = \sum_{q=0}^{\infty} \phi\left(\mathbf{a}_q\right)\mu_q\,, \quad \phi \in \mathcal{R}_n\left(\mathcal{E}\right)\,.$$

The existence of such regularizations F *is equivalent to the existence of* k *for which (16) holds.*

3. The expansion of quotients of gamma functions

A key component in the study of the asymptotic behavior of moments is the expansion of quotients of gamma functions

$$\frac{\Gamma\left(z+\alpha\right)}{\Gamma\left(z\right)}\,,$$

as $z \to \infty$. This problem has been studied for many years [12, 33], and it admits a rather simple and elegant asymptotic formula in terms of the Stirling numbers.

The *form* of the expansion can be obtained from Stirling's formula [10, (3.88)]

$$\ln\Gamma\left(z\right) \sim \left(z-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln 2\pi + \sum_{k=0}^{\infty}\frac{B_{k+2}}{\left(k+1\right)\left(k+2\right)z^{k+1}}\,,$$

where the B_k are the Bernoulli numbers. Indeed, with a little work one obtains the expansion of $\ln\left(\Gamma\left(z+\alpha\right)/\Gamma\left(z\right)\right) = \ln\Gamma\left(z+\alpha\right) - \ln\Gamma\left(z\right)$ as

$$\ln\left(\frac{\Gamma\left(z+\alpha\right)}{\Gamma\left(z\right)}\right) \sim \ln z^{\alpha} + \sum_{k=1}^{\infty}\frac{a_k\left(\alpha\right)}{z^k}\,, \tag{17}$$

where the $a_k\left(\alpha\right)$ are some polynomial expressions of α. With a little more work one can find a formula for the first few a_k; for instance [20]

$$a_1\left(\alpha\right) = \frac{\alpha\left(\alpha-1\right)}{2}\,.$$

If we take exponentials in (17) we obtain that

$$\frac{\Gamma\left(z+\alpha\right)}{\Gamma\left(z\right)} \sim z^{\alpha}\left\{1+\sum_{k=1}^{\infty}\frac{A_k\left(\alpha\right)}{z^k}\right\}\,, \tag{18}$$

where the A_k are polynomials in α. In fact, Tricomi and Erdélyi [33] give formulas for the first polynomials A_k by using binomial coefficients, namely,

$$A_1\left(\alpha\right) = \binom{\alpha}{2}\,, \quad A_2\left(\alpha\right) = \frac{3\alpha-1}{4}\binom{\alpha}{3}\,, \quad A_3\left(\alpha\right) = \binom{\alpha}{2}\binom{\alpha}{4}\,. \tag{19}$$

While the form of the expansion (18) is simple, it looks as if a formula for the A_k would be very complicated; surprisingly, they can be written rather easily in terms of the Stirling numbers of the first kind, as we now explain.

The idea is the following: since the A_k are polynomials, it is enough to find their expression for *some* values of α, in particular, it is enough to find the formula when α is a positive integer. But if $\alpha = m$, a positive integer, we obtain

$$\frac{\Gamma\left(z+m\right)}{\Gamma\left(z\right)} = z\left(z+1\right)\cdots\left(z+m-1\right)\,.$$

The polynomial $z\left(z+1\right)\cdots\left(z+m-1\right)$ has degree m and no constant term, and thus it admits a development in terms of z, $z^2, \ldots, z^m$. The coefficients of this development are *exactly* the (signless) Stirling numbers of the first kind [16], that is

$$z\left(z+1\right)\cdots\left(z+m-1\right) = \begin{bmatrix} m \\ 1\end{bmatrix}z + \begin{bmatrix} m \\ 2\end{bmatrix}z^2 + \begin{bmatrix} m \\ 3\end{bmatrix}z^3 + \cdots + \begin{bmatrix} m \\ m\end{bmatrix}z^m, \tag{20}$$

where we use the notation $\left[{m \atop k}\right]$ [19] for the Stirling numbers. We remark that Jordan notation is S_m^k, for the signed version,

$$\begin{bmatrix} m \\ k \end{bmatrix} = (-1)^{m+k} S_m^k .$$

We can rewrite (20) as

$$\frac{\Gamma(z+m)}{\Gamma(z)} = z^m \sum_{k=0}^{\infty} \begin{bmatrix} m \\ m-k \end{bmatrix} z^{-k},$$

if we set $\left[{m \atop k}\right] = 0$ for $k \le 0$. We now observe that Jordan [16, p. 149] establishes that $\left[{m \atop m-k}\right]$ is a polynomial of degree $2k$ in m. Actually,

$$\begin{bmatrix} m \\ m \end{bmatrix} = 1 ,$$

$$\begin{bmatrix} m \\ m-1 \end{bmatrix} = \binom{m}{2} = \frac{m^2-m}{2} ,$$

$$\begin{bmatrix} m \\ m-2 \end{bmatrix} = 3\binom{m}{4} + 2\binom{m}{3} ,$$

$$\begin{bmatrix} m \\ m-3 \end{bmatrix} = 15\binom{m}{6} + 20\binom{m}{5} + 6\binom{m}{4} ,$$

and more generally

$$\begin{bmatrix} m \\ m-k \end{bmatrix} = \sum_{j=0}^{k-1} C_{k,j}\binom{m}{2k-j} .$$

A table for the $C_{k,j}$ is given in [16, p. 152]. Therefore *we define* $\left[{\alpha \atop \alpha-k}\right]$ for $\alpha \in \mathbb{C}$ as this polynomial of degree $2k$ evaluated at α. Since $A_k(m) = \left[{m \atop m-k}\right]$ when m is a positive integer we immediately obtain the formula[‖]

$$A_k(\alpha) = \begin{bmatrix} \alpha \\ \alpha-k \end{bmatrix}, \quad \alpha \in \mathbb{C} .$$

Summarizing, we have the following expansion.

Proposition 3.1. *If $\alpha \in \mathbb{C}$ then*

$$\frac{\Gamma(z+\alpha)}{\Gamma(z)} \sim z^\alpha \sum_{k=0}^{\infty} \begin{bmatrix} \alpha \\ \alpha-k \end{bmatrix} z^{-k}, \tag{21}$$

as $z \to \infty$ in any sector that does not contain $\alpha - n$ for $n \in \mathbb{N}$ large.

We remark that the series (21) is always asymptotic, but sometimes it converges. Indeed, as we saw, if $\alpha \in \mathbb{N}$ then the series is actually a polynomial. On the other hand, if $\alpha = -1, -2, -3, \dots$, i.e, $\alpha = -q$ then

$$\frac{\Gamma(z-q)}{\Gamma(z)} = \frac{1}{(z-1)\cdots(z-q)} = z^{-q} \sum_{k=0}^{\infty} \begin{bmatrix} -q \\ -q-k \end{bmatrix} z^{-k}, \tag{22}$$

is a series that converges for $|z| > q$; it reduces to the geometric series if $q = 1$. In fact, if the signless Stirling numbers of the second kind $\left\{{k \atop m}\right\}$ are defined as [19]

$$\begin{Bmatrix} k \\ m \end{Bmatrix} = \begin{bmatrix} -m \\ -k \end{bmatrix},$$

then (22) can be rewritten as

$$\frac{\Gamma(z-q)}{\Gamma(z)} = z^{-q} \sum_{k=0}^{\infty} \begin{Bmatrix} q+k \\ q \end{Bmatrix} z^{-k}.$$

Interestingly, when $\alpha \notin \mathbb{Z}$ then the series (21) is a *divergent* asymptotic series. Indeed, if (21) converges for some z_0 then it would converge for $|z| > |z_0|$ and thus $F(z) = \sum_{k=0}^{\infty} \left[{\alpha \atop \alpha-k}\right] z^{-k}$ would be an analytic function in the region $|z| > |z_0|$ and thus $z^{-\alpha}\Gamma(z+\alpha)/\Gamma(z)$ would likewise be analytic in that exterior disc, and this is never true if $\alpha \notin \mathbb{Z}$.

A similar analysis yields the expansion

$$\begin{aligned} \frac{\Gamma(z+1)}{\Gamma(z+\beta+1)} &\sim z^{-\beta} \sum_{k=0}^{\infty} \begin{bmatrix} -\beta \\ -\beta-k \end{bmatrix} (-1)^k z^{-k} \\ &\sim z^{-\beta} \sum_{k=0}^{\infty} \begin{Bmatrix} \beta+k \\ \beta \end{Bmatrix} (-1)^k z^{-k} \end{aligned} \tag{23}$$

[‖] That this coincides with the Tricomi-Erdélyi formulas (19) is easy to see.

as $z\to\infty$ inside sectors that do not contain the negative integers. Notice in particular the finite development

$$z\left(z-1\right)\cdots\left(z-m+1\right)=\frac{\Gamma\left(z+1\right)}{\Gamma\left(z-m+1\right)}=z^{m}\sum_{k=0}^{m-1}\begin{bmatrix} m \\ m-k\end{bmatrix}\left(-1\right)^{k}z^{-k},$$

that Jordan [16, p. 142] employs to define the Stirling numbers of the first kind S_m^k.

4. Asymptotic behavior of the moments of distributions with compact support

Let $F\in\mathcal{E}'\left(\mathbb{R}\right)$ be a distribution of compact support in one variable, with $\operatorname{supp}F=\left[a,b\right]$. We will now show how the asymptotic behavior of the moments

$$\mu_{n}=\mu_{n}\left(F\right)=\left\langle F\left(u\right),u^{n}\right\rangle \,,$$

can be obtained from the distributional behavior of F at the endpoints. We start with a simple observation.

Lemma 4.1. *If* $\operatorname{supp}F=\left[a,b\right]$ *then there exists* $k\in\mathbb{N}$ *such that*

$$\mu_{n}=O\left(n^{k}c^{n}\right)\,, \tag{24}$$

as $n\to\infty$ *where* $c=\max\left\{\left|a\right|,b\right\}$.

Proof. Indeed, there exists $M>0$ and $k\in\mathbb{N}$ such that

$$\left|\left\langle F\left(u\right),\phi\left(u\right)\right\rangle\right|\le M\max_{a\le u\le b}\left\{\left|\phi^{(j)}\left(u\right)\right|:j\le n\right\}\,,$$

and this yields (24). □

Notice, that the lemma gives that if $c_1>c$ then $\mu_n=o\left(c_1^n\right)$ as $n\to\infty$.

In our analysis we will employ the following terminology. Let α be a real number that is not a negative integer and let $F\in\mathcal{D}'\left(\mathbb{R}\right)$ be a distribution. We say that

$$F\left(x\right)=o\left(\left|x-a\right|^{\alpha}\right)\,,\quad \text{as }\; x\to a\,,\ \text{distributionally}\,,$$

if $F\left(a+\varepsilon x\right)=o\left(\varepsilon^{\alpha}\right)$ as $\varepsilon\to0$ in $\mathcal{D}'\left(\mathbb{R}\right)$, that is, if for all test functions $\phi\in\mathcal{D}\left(\mathbb{R}\right)$ we have

$$\left\langle F\left(a+\varepsilon x\right),\phi\left(x\right)\right\rangle=o\left(\varepsilon^{\alpha}\right)\,,\quad\text{as } \varepsilon\to0\,.$$

When $\operatorname{supp}F\subset\left[a,\infty\right)$ we just write $F\left(x\right)=o\left(\left(x-a\right)^{\alpha}\right)$ as $x\to a^{+}$, while if $\operatorname{supp}F\subset\left(-\infty,a\right]$ we write $F\left(x\right)=o\left(\left(a-x\right)^{\alpha}\right)$ as $x\to a^{-}$. It is easy to see [10,27] that if φ is a smooth function with $\varphi\left(a\right)=b$ and with $\varphi'\left(a\right)\neq0$, then $F\left(x\right)=o\left(\left|x-b\right|^{\alpha}\right)$ as $x\to b$ distributionally if and only if $F\left(\varphi\left(x\right)\right)=o\left(\left|x-a\right|^{\alpha}\right)$ as $x\to a$ distributionally.

Lemma 4.2. *Let* $\alpha\in\mathbb{R}\setminus\left\{-1,-2,-3,\ldots\right\}$. *Suppose* $\operatorname{supp}F=\left[\varepsilon,b\right]$ *where* $\varepsilon>0$ *and*

$$F\left(u\right)=o\left(\left(b-u\right)^{\alpha}\right)\,,$$

distributionally as $u\to b^{-}$. *Then*

$$\mu_{n}=o\left(b^{n}n^{-\alpha-1}\right)\,,\quad\textit{as }\; n\to\infty\,. \tag{25}$$

Proof. Indeed, writing $u=be^{-t}$, $g\left(t\right)=f\left(be^{-t}\right)$ we see that $g\left(t\right)=o\left(t^{\alpha}\right)$ as $t\to0^{+}$ distributionally. Therefore if $x=\left(n+1\right)t$, we see that

$$\mu_{n}=\left\langle f\left(u\right),u^{n}\right\rangle_{u}=-b^{n+1}\left\langle f\left(be^{-t}\right),e^{-(n+1)t}\right\rangle_{t}=\frac{-b^{n+1}}{\left(n+1\right)}\left\langle g\left(\left(\frac{1}{n+1}\right)x\right),e^{-x}\right\rangle_{x}=\frac{b^{n+1}}{\left(n+1\right)}o\left(\frac{1}{\left(n+1\right)^{\alpha}}\right)\,,$$

and (25) follows. □

We will denote as A_α a (fixed) function that has the asymptotic expansion

$$A_{\alpha}\left(n\right)\sim n^{-\alpha-1}\sum_{k=0}^{\infty}\begin{bmatrix}-\alpha-2\\ -\alpha-2-k\end{bmatrix}\left(-1\right)^{k}n^{-k}.$$

as $n\to\infty$. Clearly $A_{\alpha}\left(n\right)\sim n^{-\alpha-1}$ as $n\to\infty$. Employing the asymptotic formula (23) we obtain that a possible choice for the function A_α is the normalized moment function

$$\Gamma(\alpha+1)A_{\alpha}\left(n\right)=\left\langle H\left(u\right)H\left(1-u\right)\,\left(1-u\right)^{\alpha},u^{n}\right\rangle=\int_{0}^{1}u^{n}\left(1-u\right)^{\alpha}\mathrm{d}u\,,$$

H being the Heaviside function, since

$$A_\alpha(n) = \frac{B(n+1,\alpha+1)}{\Gamma(\alpha+1)} = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+2)},$$

where $B(x,y)$ is the beta function [20].

Notice that the moments of the function $H(u)\,H(b-u)\,(b-u)^\alpha$ can then be written as

$$\left\langle H(u)\,H(1-u)\,(1-u)^\alpha, u^n\right\rangle = b^{n+\alpha+1}\Gamma(\alpha+1)\,A_\alpha(n)\,. \tag{26}$$

We thus immediately obtain the following.

Lemma 4.3. *Suppose* $\alpha_j \nearrow \infty$. *Suppose* $\operatorname{supp} F = [\varepsilon, b]$ *where* $\varepsilon > 0$ *and that for some constants* C_j

$$F(u) \sim \sum_{j=1}^{\infty} C_j\,(b-u)^{\alpha_j}\,, \tag{27}$$

distributionally as $u \to b^-$. *Then*

$$\mu_n \sim \sum_{j=1}^{\infty} b^{n+\alpha_j+1} C_j \Gamma(\alpha_j+1)\,A_{\alpha_j}(n)\,. \tag{28}$$

as $n \to \infty$.

Proof. The asymptotic expansion (27) means that as $u \to b^-$ for all m we have $F(u) = \sum_{j=1}^{m} C_j\,(b-u)^{\alpha_j} + o\left((b-u)^{\alpha_m}\right)$ distributionally. Hence

$$F(u) = \sum_{j=1}^{m} C_j H(u)\,H(b-u)\,(b-u)^{\alpha_j} + R_m(u)\,,$$

where the remainder R_m satisfies $R_m(u) = o\left((b-u)^{\alpha_m}\right)$ distributionally as $u \to b^-$. Hence (26) yields

$$\mu_n = \sum_{j=1}^{m} b^{n+\alpha_j+1} C_j \Gamma(\alpha_j+1)\,A_{\alpha_j}(n) + o\left(b^n n^{-\alpha_m-1}\right)\,,$$

or,

$$\mu_n \sim \sum_{j=1}^{m} b^{n+\alpha_j+1} C_j \Gamma(\alpha_j+1)\,A_{\alpha_j}(n) + o\left(b^n A_{\alpha_m}(n)\right)\,,$$

since $A_{\alpha_m}(n) \sim n^{-\alpha_m-1}$. □

It should be pointed out that when $\alpha_1 = \alpha$ and $\alpha_{k+1} = \alpha_k + 1$, $k \geq 1$. Then (28) becomes the expansion

$$\mu_n = \Gamma(\alpha+1)C_1 n^{-(\alpha+1)} + \sum_{k=1}^{\infty}\left(C_k\Gamma(\alpha+k+1)\sum_{j=1}^{k}(-1)^j C_{k-j}\Gamma(\alpha+k+1-j)\begin{bmatrix}-\alpha-(k-(j-1))\\ -\alpha-(k+1)\end{bmatrix}\right) n^{-k-(\alpha+1)}. \tag{29}$$

It is also easy to obtain the development of distributions of compact support in the negative half line. If $\operatorname{supp} F = [a, -\varepsilon]$ where $\varepsilon > 0$ then we consider the distribution $G(u) = F(-u)$ whose moments are related as $\mu_n(F) = (-1)^n \mu_n(G)$, and consider the behavior of G at $|a|$.

Lemma 4.4. *Suppose* $\beta_j \nearrow \infty$. *Suppose* $\operatorname{supp} F = [a, -\varepsilon]$ *where* $\varepsilon > 0$ *and*

$$F(u) \sim \sum_{j=1}^{\infty} D_j H(u-a)\,(u-a)^{\beta_j}\,,$$

distributionally as $u \to a^+$. *Then*

$$\mu_n \sim \sum_{j=1}^{\infty} (-1)^n\,|a|^{n+\beta_j+1} D_j \Gamma(\beta_j+1)\,A_{\beta_j}(n)\,. \tag{30}$$

Let us now consider the general case of a distribution F with $\operatorname{supp} F = [a,b]$ where $a < 0 < b$. We can decompose F as $F_a + F_\varepsilon + F_b$ where $\operatorname{supp} F_a = [a, -\varepsilon]$, $\operatorname{supp} F_\varepsilon = [-\varepsilon, \varepsilon]$, and $\operatorname{supp} F_b = [\varepsilon, b]$. We know how to find the asymptotic expansion of $\mu_n(F_a)$ and $\mu_n(F_b)$, while $\mu_n(F_\varepsilon) = o(c^n)$, where $c = \max\{|a|, b\}$. In case $|a| \neq b$ then the expansion of $\mu_n(F)$ will be equal to that of $\mu_n(F_a)$ if $|a| > b$ or of $\mu_n(F_b)$ if $b > |a|$. If $a = -b$ then the expansion of $\mu_n(F)$ will be the sum of the expansions of $\mu_n(F_a)$ and of $\mu_n(F_b)$; in such a case some cancellation is possible.

Naturally one can consider other types of asymptotic behavior of F at the endpoints, but our aim presently is to illustrate the possible *methods* of analysis.

5. Expansion of moments in the real case

In this section we will give the *form* of the expansion of moments sequences

$$M_n(f) = \int_{-\infty}^{\infty} (f(x))^n \, \mathrm{d}x \,,$$

as $n \to \infty$ where f is a bounded smooth function such that for some $n_0 \in \mathbb{N}$

$$\int_{-\infty}^{\infty} |f(x)|^n \, \mathrm{d}x < \infty \,, \quad n \geq n_0 \,.$$

The results of Section 2 show that there are distributions of compact support F such that for some $m \in \mathbb{N}$

$$\langle F(u), \phi(u) \rangle = \int_{-\infty}^{\infty} \phi(f(x)) \, \mathrm{d}x \,,$$

for any test function $\phi \in \mathcal{R}_m(\mathcal{E}(\mathbb{R}))$ and such that

$$\mu_n = \langle F(u), u^n \rangle = M_n(f) \,, \quad n \geq m \,.$$

We already know how to find the asymptotic expansion of the moments μ_n in terms of the behavior of the distribution F at the endpoints of its compact support. Furthermore, we can actually relate the endpoint asymptotic behavior of F to the behavior of the smooth function f at its absolute maxima and minima.

The following lemma will be useful momentarily. A corresponding result for absolute minimum values also holds.

Lemma 5.1. *Let $f \in \mathcal{C}^\infty(\mathbb{R})$. Suppose $f(0) = b$ is the absolute maximum value of f and $f''(0) < 0$. Then there exists a unique increasing smooth function ψ such that $f(x) = b - \psi^2(x)$ for all x in an interval I containing 0.*

Suppose, to fix the ideas, that we have the situation in the previous lemma, but that the smooth function f achieves its global maximum only once, say at $x = 0$. By the lemma, in a neighborhood of $x = 0$, we have $f = b - \psi^2$. Furthermore, the inverse function f^{-1} exists in a possibly smaller neighborhood of $x = 0$. Actually, the inverse function f^{-1} has two branches, f_-^{-1} and f_+^{-1}, which are well-defined to the left and right of $x = 0$, respectively. These two branches can be expanded as

$$f_+^{-1}(u) \sim \sum_{k=0}^{\infty} C_k (b-u)^{k/2}, \quad u \to b^-, \tag{31}$$

$$f_-^{-1}(u) \sim \sum_{k=0}^{\infty} (-1)^k C_k (b-u)^{k/2}, \quad u \to b^-, \tag{32}$$

where the C_j are the Taylor coefficients. It is important to observe that the coefficients of these two expansions are, except for the change in signs, basically the same.

We are now ready to consider the integral

$$\int_{-\infty}^{\infty} \phi(f(x)) \, \mathrm{d}x \,,$$

where ϕ is a test function that vanishes of enough order at the origin to assure convergence. If we make the change of variables $u = f(x)$, we obtain

$$\langle F(u), \phi(u) \rangle = \int_{-\infty}^{\infty} \left(\frac{1}{f'\left(f_+^{-1}(u)\right)} + \frac{1}{f'\left(f_-^{-1}(u)\right)} \right) \phi(u) \, \mathrm{d}u \,,$$

so that for u near b,

$$F(u) = \frac{1}{f'\left(f_+^{-1}(u)\right)} + \frac{1}{f'\left(f_-^{-1}(u)\right)} \,.$$

Inversion of (31) and of (32) yields that as $u \to b^-$ the function $f_+^{-1}(u)$ has an expansion of the form

$$f_+^{-1}(u) \sim B_1 (b-u)^{1/2} + B_2 (b-u) + B_3 (b-u)^{3/2} + \cdots \,,$$

and similarly

$$f_-^{-1}(u) \sim -B_1 (b-u)^{1/2} + B_2 (b-u) - B_3 (b-u)^{3/2} + \cdots \,.$$

Notice, again, that the coefficients of the two expansions are obtained from the other by appropriate changes of sign. Next, by composing with $f'(x)$, we obtain expansions with the form

$$f'\left(f_+^{-1}(u)\right) \sim D_{1/2}(b-u)^{1/2} + D_1(b-u) + \cdots .$$

and

$$\frac{1}{f'\left(f_+^{-1}(u)\right)} \sim E_{-1/2}(b-u)^{-1/2} + E_0 + E_{1/2}(b-u)^{1/2} + \cdots . \tag{33}$$

Similarly

$$\frac{1}{f'\left(f_-^{-1}(u)\right)} = E_{-1/2}(b-u)^{-1/2} - E_0 + E_{1/2}(b-u)^{1/2} - \cdots . \tag{34}$$

Thus, if we add (33) and (34), we can see that all integral powers of $(b-u)$ cancel. Therefore

$$F(u) \sim 2E_{-1/2}(b-u)^{-1/2} + 2E_{1/2}(b-u)^{1/2} + 2E_{1/2}(b-u)^{3/2} + \cdots , \tag{35}$$

as $u \to b^-$. Naturally in any particular example one needs to find these coefficients $E_{-k/2}$, perhaps numerically.

We then obtain that the moments $M_n(f)$, that equal the moments μ_n of F, will have a development of the type given by (29), where we replace C_j by $2E_{1/2+j}$. Clearly, the first term has the form

$$M_n(f) \sim C\frac{b^{n+1/2}}{\sqrt{n}} ,$$

where

$$C = \sqrt{\frac{-2\pi}{f''(0)}} .$$

It should be clear how one can obtain the expansion of the moments $M_n(f)$ if $|f|$ has a finite number of absolute maxima, by adding the expansions corresponding to (35) at each of the maxima of f and those of the type (30) at the absolute minima of f.

Interestingly, if one considers integrals over a finite interval,

$$\int_\sigma^\tau (f(x))^n \, \mathrm{d}x ,$$

and the absolute maximum value b, or the absolute minimum value $-b$, is attained at one of the endpoints then the form of the expansion of the $M_n(f)$ would be different. Indeed, the reason that the distribution $F(u)$ has an expansion in terms of powers of the form $(b-u)^{-1/2}, (b-u)^{1/2}, (b-u)^{3/2}, \ldots$, is the cancellation in the expansions of functions of f_+^{-1} and f_-^{-1}. In the endpoint case this cancellation would not occur since we would not need to consider the branch f_-^{-1}. Consequently, the expansion of $F(u)$ will be given in terms of powers of the form $(b-u)^{-1/2}, (b-u)^0, (b-u)^{1/2}, (b-u), (b-u)^{3/2}, \ldots$. This, in turn, means that the form of the expansion of M_n will be given by (28) where the exponents are $\alpha_j = -\frac{1}{2} + j\frac{1}{2}$, $j = 0, 1, 2, \ldots$.

6. Expansions in the complex case

In this section we shall consider several questions on the behavior of moment *series* in the complex plane. We refer to [9] for the use of generalized functions in the study of integrals with complex values.

We will show that the moment series of $\sum_{q=0}^\infty \xi_q^p$ of a sequence of complex numbers $\{\xi_q\}_{q=0}^\infty$ can be completely arbitrary, that is, given any arbitrary sequence of complex numbers $\{\mu_p\}_{p=1}^\infty$ there is a sequence $\{\xi_q\}_{q=0}^\infty$ such that

$$\sum_{q=0}^\infty \xi_q^p = \mu_p , \quad p = 1, 2, 3, \ldots .$$

In general the series $\sum_{q=0}^\infty \xi_q^p$ will be conditionally convergent but no series is absolutely convergent. Our main tool to show this is a rearrangement result for vector series [18]. We also establish that the moments $\int_0^\infty (f(x))^p \, \mathrm{d}x$ of smooth functions with complex values can also be arbitrary.

We will also show that in case *one* of the moment series $\sum_{q=0}^\infty \xi_q^p$ is absolutely convergent then the moments μ_p satisfy an asymptotic relation, generalizing the results of Boudabra and Markowsky [3] who showed that $\lim_{p\to\infty} |\mu_p| > 0$.

6.1. Rearrangements

Let $\{\xi_q\}_{q=0}^{\infty}$ be a sequence of complex numbers such that the moments series

$$M_p = M_p(\{\xi_q\}) = \sum_{q=0}^{\infty} \xi_q^p ,$$

converge for all $p \in \mathbb{N}$ with $p \geq 1$. Clearly, if the series is absolutely convergent for p_0 then it will also be absolutely convergent for any $p \geq p_0$. We shall see that if no series is absolutely convergent, then for each arbitrary sequence of complex numbers $\{\mu_p\}_{p=1}^{\infty}$ there is a rearrangement $\{\rho_q\}_{q=0}^{\infty}$ of the series $\{\xi_q\}_{q=0}^{\infty}$ such that

$$M_p(\{\rho_q\}) = \mu_p , \quad p = 1, 2, 3, \dots . \tag{36}$$

As there are series such that $M_p(\{\xi_q\})$ is conditionally convergent for all p, it follows that given an arbitrary sequence $\{\mu_p\}_{p=1}^{\infty}$ of complex numbers, then (36) always has solutions $\{\rho_q\}_{q=0}^{\infty}$.

We will employ a rearrangement result of Katznelson and McGehee [18], a generalization of the Levy-Steinitz theorem [29] to $\mathbb{R}^{\mathbb{N}}$, according to which if $\{\mathbf{x}_q\}_{q=0}^{\infty}$ is a sequence in $\mathbb{R}^{\mathbb{N}}$ such that the series

$$\sum_{q=0}^{\infty} \mathbf{x}_q ,$$

converges, then the set $S = S(\{\mathbf{x}_q\})$ of all the possible sums of convergent rearrangements of $\{\mathbf{x}_q\}_{q=0}^{\infty}$ is an affine manifold of $\mathbb{R}^{\mathbb{N}}$, that is,

$$S = \mathbf{z} + N ,$$

where N is a vector subspace of $\mathbb{R}^{\mathbb{N}}$ and $\mathbf{z}$ is any element of S. In fact, as Halperin [14] shows, N is the polar set

$$N = \left\{ \mathbf{y} \in \mathbb{R}^{(\mathbb{N})} : \sum_{q=0}^{\infty} \langle \mathbf{y}, \mathbf{x}_q \rangle \text{ converges absolutely} \right\}^0 .$$

Here we identify $\mathbb{R}^{(\mathbb{N})}$, the space of sequences with only a finite number of non-zero terms, with the dual space $(\mathbb{R}^{\mathbb{N}})'$ [32]. Therefore, we have that $S = \mathbb{R}^{\mathbb{N}}$ precisely when $\sum_{q=0}^{\infty} \langle \mathbf{y}, \mathbf{x}_q \rangle$ does not converge absolutely for any non zero $\mathbf{y} \in \mathbb{R}^{(\mathbb{N})}$.

We start with a simple result on complex series.

Lemma 6.1. *Let $\{\xi_q\}_{q=0}^{\infty}$ be a sequence of complex numbers such that $\sum_{q=0}^{\infty} \xi_q$ and $\sum_{q=0}^{\infty} \xi_q^2$ are convergent. If $\sum_{q=0}^{\infty} \Re e(\xi_q)$ converges absolutely, then $\sum_{q=0}^{\infty} \xi_q^2$ converges absolutely, too.*

Proof. Write $\xi_q = x_q + i y_q$. If $\sum_{q=0}^{\infty} |x_q|$ converges, so will $\sum_{q=0}^{\infty} x_q^2$. Hence

$$\sum_{q=0}^{\infty} y_q^2 = \sum_{q=0}^{\infty} x_q^2 - \sum_{q=0}^{\infty} \Im m\left(\xi_q^2\right) ,$$

converges as well, that is $\{x_q\}$ and $\{y_q\}$ both belong to l^2. Consequently, $\{\xi_q\} = \{x_q\} + i\{y_q\}$ belongs to l^2. □

We also obtain that if $\sum_{q=0}^{\infty} \xi_q$ and $\sum_{q=0}^{\infty} \xi_q^2$ are convergent and $\sum_{q=0}^{\infty} \alpha \Re e(\xi_q) + \beta \Im m(\xi_q)$ converges absolutely for some $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, then $\sum_{q=0}^{\infty} \xi_q^2$ converges absolutely. Our aim is to obtain an analog in higher dimension.

Observe that if all the moments series of $\{\xi_q\}_{q=0}^{\infty}$ converge, and if P is a polynomial without constant term, then all the moment series of $\{P(\xi_q)\}_{q=0}^{\infty}$ converge as well.

Lemma 6.2. *Suppose all the moments series of $\{\xi_q\}_{q=0}^{\infty}$ converge. If there are real constants $\alpha_j, \beta_j \in \mathbb{R}$ for $1 \leq j \leq d$ such that*

$$\sum_{q=0}^{\infty} \sum_{j=1}^{d} (\alpha_j \Re e(\xi_q^j) + \beta_j \Im m(\xi_q^j)) ,$$

converges absolutely, then $\sum_{q=0}^{\infty} |\xi_q|^{2d}$ converges.

Proof. Indeed, let $P(z) = \sum_{j=1}^{d} (\alpha_j + i\beta_j) z^j$. It follows from the Lemma 6.1 that the series $\sum_{q=0}^{\infty} P(\xi_q)^2$ converges absolutely. If r is the degree of the term with smallest order in the polynomial $P^2(z)$ then since $|\xi_q| \to 0$ it follows that $\lim_{q\to\infty} |P^2(\xi_q)| / |\xi_q|^r = C$, with $0 < C < \infty$. Thus $\sum_{q=0}^{\infty} |\xi_q|^r$ converges, and since $r \leq 2d$ we obtain that $\sum_{q=0}^{\infty} |\xi_q|^{2d}$ converges. □

We can now prove our result.

Theorem 6.1. *Let $\{\xi_q\}_{q=0}^{\infty}$ be a non-zero sequence of complex numbers such that the moment series for $M_p(\{\xi_q\})$,*

$$\sum_{q=0}^{\infty} \xi_q^p, \quad p = 1, 2, 3, \ldots ,$$

all converge but never absolutely. Then for each arbitrary sequence of complex numbers $\{\mu_p\}_{p=1}^{\infty}$ there is a rearrangement $\{\rho_q\}_{q=0}^{\infty}$ of the series $\{\xi_q\}_{q=0}^{\infty}$ such that

$$\sum_{q=0}^{\infty} \rho_q^p = \mu_p, \quad p = 1, 2, 3, \ldots .$$

Proof. Consider the sequence $\{\mathbf{w}_k\}_{k=0}^{\infty}$ of $\mathbb{R}^{\mathbb{N}}$ where

$$\mathbf{w}_k = \left(\Re e(\xi_k), \Im m(\xi_k), \Re e\left(\xi_k^2\right), \Im m\left(\xi_k^2\right), \Re e\left(\xi_k^3\right), \Im m\left(\xi_k^3\right), \ldots\right) .$$

If there is a non zero $\mathbf{y} \in \mathbb{R}^{(\mathbb{N})}$, with $\mathbf{y} = \{y_q\}_{q=0}^{\infty}$ and $y_q = 0$ for $q > 2d$, such that $\sum_{k=0}^{\infty} \langle \mathbf{y}, \mathbf{w}_k \rangle$ converges absolutely, then we may use Lemma 6.2 to conclude that $\sum_{q=0}^{\infty} \xi_q^p$ converges absolutely for $p \geq 2d$, contrary to our hypotheses. Hence $\sum_{k=0}^{\infty} \langle \mathbf{y}, \mathbf{w}_k \rangle$ converges absolutely only if $\mathbf{y} = \mathbf{0}$, and thus the results of Katznelson-McGehee and of Halperin yield that the set of all possible sums of convergent rearrangements of the series $\sum_{k=0}^{\infty} \mathbf{w}_k$ is the whole $\mathbb{R}^{\mathbb{N}}$. □

In particular, we may start with a sequence like the one constructed by Lenard [21], whose moments all vanish.

Corollary 6.1. *Let $\{\xi_q\}_{q=0}^{\infty}$ be a non-zero sequence of complex numbers such that*

$$\sum_{q=0}^{\infty} \xi_q^p = 0, \quad p = 1, 2, 3, \ldots . \tag{37}$$

Then for each arbitrary sequence of complex numbers $\{\mu_p\}_{p=1}^{\infty}$ there is a rearrangement $\{\rho_q\}_{q=0}^{\infty}$ of the series $\{\xi_q\}_{q=0}^{\infty}$ such that

$$\sum_{q=0}^{\infty} \rho_q^p = \mu_p, \tag{38}$$

for $p = 1, 2, 3, \ldots$. In particular, for any sequence of complex numbers $\{\mu_p\}_{p=1}^{\infty}$ the moment problem (38) always has solutions $\{\rho_q\}_{q=0}^{\infty}$.

Proof. Indeed, Boudabra and Markowsky [3] proved that when (37) holds for each p then no such series can be absolutely convergent; this will also follow from the Proposition 6.1. □

It was proved by Priestley [28] that for the sequence $\{\xi_q\}_{q=0}^{\infty}$ constructed by Lenard [21] the series

$$\sum_{q=0}^{\infty} z\delta(z - \xi_q), \tag{39}$$

converges as an analytic functional; it actually converges to zero in $\mathfrak{O}'(\mathbb{C})$, in other words, for all entire functions g, $g \in \mathfrak{O}(\mathbb{C})$, we have

$$\left\langle \sum_{q=0}^{\infty} \delta(z - \xi_q), zg(z) \right\rangle = \sum_{q=0}^{\infty} \xi_q g(\xi_q) = 0 .$$

On the other hand our results show that the series (39) is *divergent* in the space $\mathcal{D}'(\mathbb{C})$. In fact it follows from the Lemma 2.8 that a series of delta functions of the form $\sum_{q=0}^{\infty} z^m \delta(z - \xi_q)$ converges distributionally for some m if and only if $\{\xi_q\}$ belongs to l^p for some p.

A simple modification of the series (39) gives a series of delta functions $\sum_{q=0}^{\infty} z\delta(z - \rho_q)$ with $|\rho_q| \leq M$ for all q, that converges in the space $\mathfrak{O}'(\mathbb{C})$ of analytic functionals with compact support but whose moments do not satisfy $\mu_p = O(M^p)$ as $p \to \infty$. This means that the analytic functional $F(z) = \sum_{q=0}^{\infty} z\delta(z - \rho_q)$ does not have a carrier contained in the disc $|z| \leq M$ although all of its terms have carriers, $\{\rho_q\} \subset \mathbb{D}_M = \{z : |z| \leq M\}$. The minimal convex carrier [11] of F will not be contained in $\mathbb{D}_M$. Such series of delta functions cannot converge distributionally.

6.2. Smooth complex functions

We also obtain that the moments of smooth complex valued functions $f \in C^{\infty}(\mathbb{R}, \mathbb{C})$ can be arbitrary.

Corollary 6.2. *For each arbitrary sequence of complex numbers* $\{\mu_p\}_{p=1}^{\infty}$ *there are* C^{∞} *functions from* $[0,\infty)$ *to* $\mathbb{C}$ *such that*

$$\int_0^{\infty} (f(x))^p \, \mathrm{d}x = \mu_p\,, \quad p = 1, 2, 3, \ldots\,. \tag{40}$$

Proof. Let $\phi \in \mathcal{D}(\mathbb{R})$ be a non zero positive function with $\operatorname{supp}\phi \subset (0,1)$. Let

$$\lambda_p = \int_0^1 (\phi(x))^p \, \mathrm{d}x\,.$$

Since $\lambda_p > 0$ for all p, we can find sequences $\{\rho_q\}_{q=0}^{\infty}$ such that

$$\sum_{q=0}^{\infty} \rho_q^p = \frac{\mu_p}{\lambda_p}\,, \quad p \geq 1\,.$$

We may then choose f as

$$f(x) = \sum_{q=0}^{\infty} \rho_q \phi(x-q)\,.$$

Clearly f is well defined and smooth. □

It is known [7] that the moment problem

$$\int_0^{\infty} \varphi(x)\, x^p \mathrm{d}x = \mu_p\,, \quad p \geq 0\,,$$

has solutions $\varphi \in \mathcal{S}(\mathbb{R})$ for all sequences of complex numbers $\{\mu_p\}_{p=0}^{\infty}$. However, the problem (40) may not have solutions of rapid decay at infinity, since, in general, the integrals constructed in the Corollary 6.2, $\int_0^{\infty} (f(x))^p \, \mathrm{d}x$, are *conditionally* convergent at infinity.

6.3. Asymptotic expansion of moment series

In the previous subsection we saw that the sequence $\{\mu_p\}_{p=1}^{\infty}$ of moment series $\mu_p = \sum_{q=0}^{\infty} \xi_q^p$ of a series of complex numbers that converges to zero is *completely arbitrary*. However, as we shall now see, when the series $\sum_{q=0}^{\infty} \phi(a_q)$ converges for all test functions $\phi \in \mathcal{R}_{\infty}(\mathcal{E}(\mathbb{C}))$, then the moments satisfy a rather simple asymptotic expansion. We start with the following reformulation of the Lemma 2.8.

Lemma 6.3. *Let* $\{\xi_q\}_{q=0}^{\infty}$ *be a sequence of complex numbers with* $\xi_q \to 0$. *Suppose that the series*

$$\sum_{q=0}^{\infty} \delta(z - \xi_q)\,,$$

is convergent in $\mathcal{R}'_{\infty}(\mathcal{E}(\mathbb{C}))$. *Then for some* $n \in \mathbb{N}$

$$\sum_{q=0}^{\infty} |\phi(\xi_q)| < \infty\,,$$

whenever $\phi \in \mathcal{R}_n(\mathcal{E}(\mathbb{C}))$. *If* $k > n$ *then*

$$\sum_{q=0}^{\infty} |\xi_q|^k < \infty\,.$$

There are distributions $F \in \mathcal{E}'(\mathbb{C})$ *such that*

$$\langle F(u), \phi(u)\rangle = \sum_{q=0}^{\infty} \phi(\xi_q)\,, \quad \phi \in \mathcal{R}_n(\mathcal{E}(\mathbb{C}))\,.$$

A distribution $F \in \mathcal{E}'(\mathbb{C})$ that satisfies (12) is a regularization of the series of delta functions $\sum_{q=0}^{\infty} \delta(z - \xi_q)$, which may be divergent in $\mathcal{E}'(\mathbb{C})$, but that converges in $\mathcal{R}_n(\mathcal{E}(\mathbb{C}))$. Since $\xi_q \to 0$ we may suppose that $\{|\xi_q|\}$ is a decreasing sequence. If $|\xi_0| = b$ then its support is contained in a disc of radius b and center at the origin,

$$\operatorname{supp} F \subset \{z : |z| \leq b\}\,,$$

and in fact
$$\left|\sum_{q=0}^{\infty}\xi_q^p\right| \le \sum_{q=0}^{\infty}\left|\xi_q^p\right| \le db^p + o\left(b^p\right), \quad p\to\infty,$$
where d is the number of terms of the sequence $\{\xi_q\}$ with $|\xi_q| = b$. The moment sums cannot be arbitrary in this case. In fact, we have the following asymptotic expansion.

Proposition 6.1. *Let $\{\xi_q\}_{q=0}^{\infty}$ be a sequence of complex numbers with $|\xi_q| \searrow 0$. If the series $\sum_{q=0}^{\infty}\delta\left(z-\xi_q\right)$ converges in $\mathcal{R}_\infty\left(\mathcal{E}\left(\mathbb{C}\right)\right)$ then as $p\to\infty$*
$$\sum_{q=0}^{\infty}\xi_q^p = c_p b^p + O\left(b_1^p\right), \tag{41}$$
where $b = \max|\xi_q|$ is attained d times and $b_1 = \max\{|\xi_q| : q \ge d\}$. The sequence c_p satisfies
$$0 \le |c_p| \le d, \tag{42}$$
and
$$\limsup_{p\to\infty}|c_p|^{1/p} = 1. \tag{43}$$

Proof. Let $\omega_q = \xi_q/b$ and
$$c_p = \sum_{q=0}^{d-1}\omega_q^p.$$
If p is large enough for the series to be absolutely convergent, then we can write
$$\sum_{q=0}^{\infty}\xi_q^p = c_p b^p + \sum_{q=d}^{\infty}\xi_q^p,$$
and (41) follows since $\sum_{q=d}^{\infty}\xi_q^p = O\left(b_1^p\right)$.

Notice that $|\omega_q| = 1$ for $0 \le q \le d-1$ and hence (42) holds. To show that (43) is true observe that the meromorphic function
$$g\left(z\right) = \sum_{q=0}^{d-1}\frac{1}{z-\omega_q},$$
has all its poles in the unit circle. At infinity it has the Laurent series
$$g\left(z\right) = \sum_{p=0}^{\infty}\frac{c_p}{z^{p+1}},$$
and thus the radius of convergence of the power series $\sum_{p=0}^{\infty}c_p\zeta^p$ is exactly 1. □

Observe that in the case when the series $\sum_{q=0}^{\infty}\delta\left(z-\xi_q\right)$ converges in $\mathcal{R}_\infty\left(\mathcal{E}\left(\mathbb{C}\right)\right)$ then
$$\limsup_{p\to\infty}\left|\sum_{q=0}^{\infty}\xi_q^p\right|^{1/p} = \max_{q\ge0}|\xi_q|,$$
a result already given in [3].

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