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Research Article

The concept of dual of Γ-ideals in a co-ordered Γ-semigroup with apartness

Daniel A. Romano*,†

International Mathematical Virtual Institute, Kordunaška Street 6, 78000 Banja Luka, Bosnia and Herzegovina

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Abstract

In 1981, Sen introduced the notion of Γ-semigroups as a generalization of semigroups. The idea of Γ-semigroups with apartness in the Bishop's constructive framework was introduced and analyzed in 2019 by the present author. The concept of co-filters as well as some other related concepts (and their interrelationships) in a co-ordered Γ-semigroup with apartness is the focus of the present author's interest. In this paper, as a continuation of the previous research, the author discusses the design of the concept of (left, right, both-sided, prime) co-ideals as duals of corresponding ideals in the before-mentioned algebraic systems.

Keywords: Bishop's constructive algebra; co-ordered Γ-semigroup with apartness; co-ideal; ordered co-ideal.

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1. Introduction

Suppose that the logical environment, in which the algebraic concepts and processes with them are analyzed in this paper, is the intuitionistic logic **IL** [10, 44]. This assumption implies that the logical axiom 'Principe TND' (tertium non datur – the logical principle of 'the exclusion of the third') is not valid in this setting. In this logic, the diversity relation is a fundamental concept equally important as the concept of equality in the classical logic. The concepts discussed in this article are based on the Bishop's principled-philosophical orientation **BISH** (for example, see [2, 3, 18, 31]). Therefore, for a carrier S of an algebraic structure, $(S, =, \neq)$ is a relational system, where '$=$' is the standard equality and '$\neq$' is an apartness [24]:

$(\forall\, x, y \in S)\ (x \neq y \implies \neg(x = y))$ (consistency);

$(\forall\, x, y \in S)\ (x \neq y \implies y \neq x)$ (symmetry);

$(\forall\, x, y, z \in S)\ (x \neq z \implies (x \neq y \vee y \neq z))$ (co-transitivity).

This relation is extensional with respect to the equality in the standard way

$$= \circ \neq \subseteq \neq \text{ and } \neq \circ = \subseteq \neq$$

where '$\circ$' is the standard composition of relations. In addition, any relation R on S, any function f between such sets and any operation w in S appearing in this article are strongly extensional relative to the apartness (for example, see [24, 31]). In what follows, for a strongly extensional mapping, 'se-mapping' will be briefly written. Also, a predicate P on S is strongly extensional if the following is valid:

$$(\forall\, x, y \in S)\ (P(y) \implies (P(x) \vee x \neq y)).$$

In this sense, the subset P of the set S is said to be strongly extensional subset of S if the following holds

$$(\forall\, x, y \in S)\ (y \in P \implies (x \in P \vee x \neq y)).$$

Accepting the existence of previous concepts allows to understand the concept of strong compliment $B^{\lhd}$ of a subset B in a set $(S, =, \neq)$ in the following sense $B^{\lhd} = \{u \in S : (\forall\, x \in B)\ (x \neq u)\}$. The following terminology is used: The subset B of the set $(S, =, \neq)$ is a constructive dual of a subset A in the set $(S, =)$ with property P on $(S, =)$ if the set $B^{\lhd}$ has the property P also.

*E-mail address: bato49@hotmail.com
†Google Scholar profile: https://scholar.google.com/citations?user=bM0JVwsAAAAJ&hl=en

Constructive mathematics is not a unique notion. Various forms of constructivism have been developed over time [3]. It has to be emphasized that Errett Bishop's constructive mathematics **BISH** forms the framework for this work [2]. Constructive algebra is a relatively old discipline developed among others by L. Kronecker, van der Waerden, and A. Heyting. One of the main topics in constructive algebra is the topic of constructive algebraic structures with relation of (tight) apartness '$\neq$', that is the second most important relation in constructive mathematics. Thus, principal novelty in treating algebraic structures constructively is that (tight) apartness becomes a fundamental notion.

Examples of the standard references on constructive algebra are the defended doctoral dissertations written by Tennenbaum [43], Ruitenburg [38] and Romano [22]; articles written by Richman [21], Julian, Mines and Richman [14], van Dalen [9] and Romano [23]; and books [3,18]. In the review paper [31], one can find enough information to get an impression of the achieved level of research of algebraic structures with apartness.

Although the classical theory of semigroups was considerably developed in the last decades, researchers working on constructive mathematics have not paid much attention to the semigroup theory. The initial step towards grounding this theory is done through the present author's works (for example, see [27–29]) and his collaborators' contributing papers [6,7,19].

In this paper, as a continuation of his earlier research on the Γ-semigroup with apartness, the author presents and discuss the (co-ordered) ideals and co-ideals in ordered Γ-semigroups with apartness under a co-order. Apart from the fact that the concept of co-order relation '$\nleqslant$' on the set $(S, =, \neq)$ is a specific ordered relation, its strong complement $\nleqslant^{\lhd}$ is a partial order relation on the set $(S, \neq^{\lhd}, \neq)$.

The paper is organized as follows. In Section 2, the concept of (co-ordered) Γ-semigroup with apartness is given. Section 3 is the central part of this paper where the concept of co-ordered (left, right) co-ideals in the before-said semigroups is introduced as a dual of notion of (left, right) ideals (see Definition 3.3). Also, some important properties of these substructures in such algebraic structures are proven in Section 3. Moreover, in addition to the described existential specificities of duals, in Section 3 the author also exposes specific technologies for proving processes with these duals in the observed algebraic structure within the Bishop's principled-logical framework. As an example in which such specificities are explicitly illustrated, see Lemma 3.1. At the end of Section 3, a constructive dual of the classical notions of quasi-Γ-ideal (Theorem 3.7) and a constructive dual of a s-prime Γ-ideal (Note 3.1) in such algebraic structures are designed. In addition, Note 3.2, which concludes this paper, provides some specificities of co-ordered left and right Γ-coideals in a co-ordered Γ-semigroup with apartness S if S is a regular Γ-semigroup.

We point out here that in the techniques of proving results of this paper we often rely, among others, on the following fact

$$\frac{A \vee B, \neg B}{A}$$

that can be deduced in the intuitionistic logic.

2. Preliminaries: background and known results

To understand the notation and terminology, and to relate them to concepts in classical Γ-semigroup theory used in this article, which not previously described, the reader may look at the articles [11,12,16,17,39,40,42]. The idea of designing the concept of Γ-semigroups and some basic properties of this class of algebraic structures can be found in [17,39,40]. Fundamental information about the ordered Γ-semigroups can be found in papers [16,17]. Ideals in the ordered Γ-semigroups are discussed in the articles [12,42].

We draw the reader's attention to the fact that logical notations are used in a standard way. For example, $\wedge$ is a conjunction, $\vee$ is a disjunction, and so on. The symbol ':=', in the sense of $A := F(x, y, ...)$ is used when the formula $F(x, y, ...)$ is to be replaced by the abbreviation A.

2.1. Γ-semigroups within classical case

Within the classical mathematics, the theory of Γ-semigroups has been around for more than three decades and counts hundreds of research papers and many PhD theses. It was Sen [39] who defined in 1981 the concept of a Γ-semigroup as a generalization of a semigroup. Besides Sen's definition, going through literature on the topic, there is another definition given by Sen and Saha [40] in 1986; latter modified by Saha in 1994, which is, in some sense, more used.

Definition 2.1. *Let S and Γ be two non-empty sets. $S := (S, \Gamma)$ is called a Γ-semigroup if there is a mapping*

$$w_S : S \times \Gamma \times S \longrightarrow S$$

such that

$$(\forall\, x, y, z \in S)\ (\forall\, a, b \in \Gamma)\ (w_S(w_S(x, a, y), b, z) = w_S(x, a, w_S(y, b, z))).$$

Let $xay := w_s(x, a, y)$ for any $x, y \in S$ and $a \in \Gamma$. Now, the previous equation has a form

$$(\forall\, x, y, z \in S)\ (\forall\, a, b \in \Gamma)\ (xa(ybz) = (xay)bz).$$

An equivalent version of Definition 2.1 was appeared in the Kehayopulu's papers [15, 16].

Remark 2.1. *Any semigroup* $(S, \cdot)$ *is a* Γ*-semigroup. Indeed, let* Γ *be any nonempty set. If we define a mapping* $S \times \Gamma \times S \longrightarrow S$ *by* $xay := x \cdot y$ *for all* $x, y \in S$ *and* $a \in \Gamma$*, then* S *is a* Γ*-semigroup.*

Example 2.1. *Let* $S := \{a, b, c\}$ *be a set,* $\mathcal{P}(S) = \{Y : Y \subseteq S\}$ *be the power-set of* S *and* $\Gamma := \{\emptyset, \{a\}, S\}$*. If the function* $w_{\mathcal{P}(S)} : \mathcal{P}(S) \times \Gamma \times \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$ *is defined as follows*

$$(\forall\, X, Y \in \mathcal{P}(S))\ (\forall\, A \in \Gamma)\ (w_{\mathcal{P}(S)}(X, A, Y) := X \cap A \cap Y),$$

then $\mathcal{P}(S)$ *is a* Γ*-semigroup.*

Example 2.2 ([12], Example 3)**.** *Let* S *be the set of all negative rational numbers. Let* $\Gamma := \{-\frac{1}{p} : p \text{ is prime}\}$*. Let* $x, y, z \in S$ *and* $a \in \Gamma$*. Now if the function* w_S *is defined by* $w_S(x, a, y)$ *is equal to the usual product* axy *of rational numbers* x, a, y*, then* S *is a* Γ*-semigroup.*

Other examples of Γ-semigroups can be found in [13, 40]. Along with Γ-semigroups, other structures such as ordered Γ-semigroups (and fuzzy Γ-semigroups) have been studied in recent years. In the view of the present paper's topic, the ordered Γ-semigroups, also known as po-Γ-semigroups, are of special interest. Following is the definition of such structures given by Sen and Saha [41] in 1993.

Definition 2.2. *Po-*Γ*-semigroup* (S, Γ) *is a* Γ*-semigroup together with a partial order '*$\leqslant$*' which is (left and right)* Γ*-compatible, i.e. if the following holds*

$$(\forall\, x, y, z \in S)\ (\forall\, a \in \Gamma)\ (x \leqslant y \implies (xaz \leqslant yaz\ \wedge\ zax \leqslant zay)).$$

As already stated, the theory of Γ-semigroups has become an enormously broad topic and has advanced on a very broad front. As it is mentioned in [8, 20],

- the majority of the results proved so far are Γ-analogues of the well-known results of ordinary semigroup theory;
- there is a striking similarity between the proofs of the original semigroup theorems and their Γ-semigroup analogues; and
- this similarity is one of the main reasons for a growing concern that many of the results in Γ-semigroup theory are logically equivalent with their counterparts in ordinary semigroups.

For nonempty subsets X and Y of S and a nonempty subset Λ of Γ, let

$$X\Lambda Y := \{xay : x \in X\ \wedge\ a \in \Lambda\ \wedge\ y \in Y\}.$$

For the sake of completeness, we include here some existing basic concepts of the classical theory of Γ-semigroups (for example, see [11–13, 15–17]):

- A nonempty subset T of a Γ-semigroup S is called a sub-Γ-semigroup of S if for all $x, y \in T$ and $a \in \Gamma$, it holds that $xay \in T$.
- Let S be a po-Γ-semigroup and J be a nonempty subset of S. Then J is called a right (respectively, left) ideal of S if the following holds
  (J1). $J\Gamma S \subseteq J$ (respectively, $S\Gamma J \subseteq J$), and
  (J2). $(\forall\, x, y \in S)\ ((y \in J\ \wedge\ x \leqslant y) \implies x \in J)$.
  J is called an ideal of S if it is right and left ideal of S.
- Let S be a Γ-semigroup and T be a Λ-semigroup. Then $(h, \varphi) : (S, \Gamma) \longrightarrow (T, \Lambda)$ is called a homomorphism if $h : S \longrightarrow T$ and $\varphi : \Gamma \longrightarrow \Lambda$ are the functions that satisfy

$$(\forall\, x, y \in S)\ (\forall\, a \in \Gamma)\ ((h, \varphi)\ (w_S(x, a, y) = w_T(h(x), \varphi(a), h(y))).$$

2.2. Γ-semigroup with apartness

The concept of Γ-semigroups with apartness was developed by the present author in the article [32] and the concept of ordered Γ-semigroup with apartness under a co-order was exposed in the papers [33, 35]. Before embarking on the development of ideas on the Γ-semigroups with apartness ordered under a co-order relation (the concept discussed in Section 3), let us recall what is the principled-philosophical approach of Bishop's constructive orientation to the concept of a set and the algebraic structures built on it.

2.2.1. Set with apartness into the Bishop's constructive approach

Contrary to the classical case, within the principled-philosophical approach of Bishop's constructive orientation, a set exists only when it is defined. The determinations of the two fundamental notions, the notion of a set and the notion of the apartness relation, which we will use in this paper, are taken from the famous book of Errett Bishop ([2], page 13). A set $(S, =)$ is considered to be defined by describing what must be done to construct an element of the set, and what must be done to show that two elements of the set are equal, with a given proof that such an equality '$=$' on S is an equivalence relation. A set $(S, =)$ is inhabited if we can construct an element of S. Furthermore, we accept in advance, in accordance with the attitude of both classical and intuitionistic logic, that the following holds for any predicate symbol P:

$$(\forall\, x)\ (\forall\, y)\ ((P(x) \wedge x = y) \implies P(y)).$$

Following Mines, Richman, and Ruitenburg's definition ([18], page 8), by an apartness on an inhabited set $(S; =)$, we mean a binary relation '$\neq$' on S which satisfies the axioms of consistency, symmetry and co-transitivity:

$(\forall\, x, y \in S)\ (x \neq y \implies \neg(x = y))$ (consistency);

$(\forall\, x, y \in S)\ (x \neq y \implies y \neq x)$ (symmetry);

$(\forall\, x, y, z \in S)\ (x \neq z \implies (x \neq y \vee y \neq z))$ (co-transitivity).

This relation is extensional with respect to the equality in the standard way

$$= \circ \neq \subseteq \neq \ \text{ and } \neq \circ = \subseteq \neq$$

where '$\circ$' is the standard composition of relations. In addition, any relation R on S, any functions f between such sets and any operation w in S appearing in this article are strongly extensional relative to the apartness (for example, see [24, 31]). In what follows, for a strongly extensional mapping, 'se-mapping' will be briefly written. Also, a predicate P on S is strongly extensive if the following

$$(\forall\, x, y \in S)\ (P(y) \implies (P(x) \vee x \neq y))$$

is valid. In this sense, the subset P of the set S is said to be strongly extensional subset of S if the following holds

$$(\forall\, x, y \in S)\ (y \in P \implies (x \in P \vee x \neq y)).$$

However, the assumption that if any subset of a set with apartness is strongly extensional entails **LPO** (Limited Principle of Omniscience) which is not acceptable in **BISH**. This implies that in each particular case when designing a concept on the set $(S, =)$ one should check whether it is strictly extensional in S.

Let B be a subset of a set S with apartness. A subset of S defined by $B^{\lhd} := \{x \in S : (\forall\, y \in B)\ (x \neq y)\}$ is called the strong complement of S. In the subsequent sections we are going to see that for a certain types of subsets we have (some kind of) constructive duality in the sense that $B^{\lhd}$ has certain property P when B has a property which is constructive version of the property P. A subset Y of S is a detachable subset in S if the following holds

$$(\forall\, x \in S)\ (x \in Y \vee x \lhd Y).$$

An inhabited subset of $S \times S$ is called a binary relation on S. A relation α defined on a set with apartness S is:

- consistent if $(\forall\, x, y \in S)\ ((x, y) \in \alpha \implies x \neq y)$;
- symmetric if $(\forall\, x, y \in S)\ ((x, y) \in \alpha \implies (y, x) \in \alpha)$;
- co-transitive if $(\forall\, x, y, z \in S)\ ((x, z) \in \alpha \implies ((x, y) \in \alpha \vee (y, z) \in \alpha)$;
- linear if $(\forall\, x, y \in S)\ (x \neq y \implies ((x, y) \in \alpha \vee (y, x) \in \alpha))$.

A relation α defined on a set with apartness S is:

- co-quasiorder if it is consistent and co-transitive;
- co-order if it is linear co-quasiorder;
- co-equivalence if it is symmetric co-quasiorder.

Let α be a co-order (co-quasiorder) on a set with apartness S. A tuple $(S, =, \neq, \alpha)$ is called co-quasiordered (co-ordered) set with apartness.

The concept of co-order relation on a set with apartness was first discussed in the paper [25] while the concept of co-quasiorder relation on set with apartness was first introduced in the paper [24]. Some of the properties of this relation in semigroups with separation can be found in [28, 29]. The concept of co-equivalence was introduced in 1985 by the present author in his dissertation [22]. The compatibility of this relation with the commutative ring with apartness is discussed in [23].

Generally speaking, for a co-quasiorder defined on a set with apartness, we can not prove that its left and/or right classes are detachable subsets – such assumption entails **LPO**. On the other side, for a given co-quasiorder α, we have the following result.

Lemma 2.1 ([7])**.** *Let α be a co-quasiorder on a set with apartness S. Then $a\alpha$ (respectively αa) is a strongly extensional subset of S such that $a \lhd a\alpha$ (respectively $a \lhd \alpha a$) for any $a \in S$. Moreover, if $(a, b) \in \alpha$ then $a\alpha \cup \alpha b = S$ is true for all $a, b \in S$.*

2.2.2. Γ-semigroup with apartness

This subsection relate to the design of Γ-semigroups with separation and describing the properties of notions related to it as well as some of the basic processes with them. This class of semigroups with apartness is in the focus of interest of the present author in the last few years (for example, see [32–35, 37]).

Before going further, it is recalled that an apartness relation on $S \times \Gamma \times S$ is determined by the following way

$$(\forall\, x, y, u, v \in S)\ (\forall\, a, b \in \Gamma)\ ((x, a, y) \neq (u, b, v) \iff (x \neq_S u \vee a \neq_\Gamma b \vee y \neq_S v)).$$

Definition 2.3 ([32], Definition 2.1)**.** *Let $(S, =_S, \neq_S)$ and $(\Gamma, =_\Gamma, \neq_\Gamma)$ be two non-empty sets with apartness. Then S is called a* Γ-semigroup with apartness *if there exists a strongly extensional mapping*

$$w_S : S \times \Gamma \times S \ni (x, a, y) \longmapsto w_S(x, a, y) := xay \in S$$

satisfying the condition

$$(\forall\, x, y, z \in S)\ (\forall\, a, b \in \Gamma)\ ((xay)bz =_S xa(ybz)).$$

We recognize immediately that the following implications

$$(\forall\, x, y, u, v \in S)\ (\forall\, a, b \in \Gamma)\ (xay \neq_S ubv \implies (x \neq_S u \vee a \neq_\Gamma b \vee y \neq_S v)),$$

$$(\forall\, x, y \in S)\ (\forall\, a, b \in \Gamma)\ (xay \neq_S xby \implies a \neq_\Gamma b)$$

are valid because w_S is a strongly extensional function.

Proposition 2.1. *Any semigroup with apartness is a Γ-semigroup with apartness.*

Proof. Let $(S, =, \neq, \cdot)$ be a semigroup with apartness and Γ be an inhabited set. Define a mapping w_S from $S \times \Gamma \times S$ to S as $w_S(x, a, y) = xay := xy$ for all $x, y \in S$ and $a \in \Gamma$. In order to confirm that w_S is correctly defined, it should be verified that w_S is an extensive and strongly extensive function.

- Let $x, y, u, v \in S$ and $a, b \in \Gamma$ be such $(x, a, y) = (u, b, v)$. Then $x = u$ and $y = v$. Thus $w_S(x, a, y) = xy = uv = w_S(u, b, v)$.
- Let $x, y, u, v \in S$ and $a, b \in \Gamma$ be such $xyw_S(x, a, y) \neq w_S(u, b, v) = uv$. Then $x \neq u \vee y \neq v$. Thus $(x, a, y) \neq (u, b, v)$. Then S is a Γ-semigroup with apartness.
- Let $x, y, z \in S$ and $a, b \in \Gamma$. Then $(xay)bz = (xy)z = x(yz) = xa(ybz)$.

Therefore, S is a Γ-semigroup with apartness. □

Every semigroup with apartness can be considered as a Γ-semigroup with apartness. Thus, the class of all Γ-semigroups with apartness includes the class of all semigroups with apartness. However, the class of Γ-semigroups with apartness is wider than the class of semigroups with apartness, i.e. there is a Γ-semigroup with apartness which is not a semigroup with apartness, as it is shown in Example 2.3 of [34].

Semigroups with apartness and some substructures in them are discussed in several published articles (for example, see [4,6,7,26,28,29,36]). While in [6,7] the authors discussed the general concept of semigroup with apartness, in [4] the concept of inverse semigroup with apartness was considered. An insight into some algebraic structures with apartness including, among other things, some classes of semigroups with apartness, can be found in the present author's review paper [32].

Example 2.3. *Let $\mathbb{N}$ be a semi-ring of natural numbers, $\mathbb{R}$ be the filed of real number, where the apartness is determined as follows*

$$(\forall\, x, y \in \mathbb{R}) \left(x \neq y \iff (\exists\, k \in \mathbb{N}) \left(|x - y| > \frac{1}{k} \right) \right),$$

$S := [0,1] \subseteq \mathbb{R}$ *and* $\Gamma := \{\frac{1}{n} : n \in \mathbb{N}\}$. *Then S is a commutative Γ-semigroup under the usual multiplication.*

Definition 2.4 ([32], Definition 2.2)**.** *Let S be a Γ-semigroup with apartness. A subset T of S is said to be a* Γ- cosubsemigroup *of S if the following holds*

$$(\forall\, x, y \in S)\; (\forall\, a \in \Gamma)\; (xay \in T \implies (x \in T \,\vee\, y \in T)).$$

We will assume that the empty set $\emptyset$ is a Γ-cosubsemigroup of a Γ-semigroup S by definition.

The family $\mathcal{K}_S$ of all Γ-cosubsemigroups of S is not empty and forms a complete lattice (see Theorem 2.1 in [32]).

Proposition 2.2 ([32], Proposition 2.1)**.** *If T is a Γ-cosubsemigroup of a Γ-semigroup with apartness S, then the set $T^{\lhd}$ is a Γ-subsemigroup of S.*

The concept of (left, right) co-ideals in a semigroup with apartness as dual notion of ideals, is discussed in the article [26]. Now, we introduce the concept of a left, right and two-side Γ-coideal in a Γ-semigroup.

Definition 2.5 ([32])**.** *A strongly extensional subset B of a Γ-semigroup with apartness S is said to be*

- *a* left Γ-coideal *of S if the following implication holds*

$$(\forall\, x, y \in S)\; (\forall\, a \in \Gamma)\; (xay \in B \implies y \in B);$$

- *a* right Γ-coideal *of S if the following implication is valid*

$$(\forall\, x, y \in S)\; (\forall\, a \in B)\; (xay \in B \implies x \in B);$$

- *a (two sided)* Γ-coideal *of S if the following implication is valid*

$$(\forall\, x, y \in S)\; (\forall\, a \in B)\; (xay \in B \implies (x \in B \,\wedge\, y \in B)).$$

From Definition 2.5, it immediately follows that if B is a left (right, two-sided) Γ-coideal of a Γ-semigroup with apartness S, then B is a Γ-cosubsemigroup of S. In addition, it is clear that Γ-coideal in S is left and right Γ-coideal in S at the same time.

Proposition 2.3. *Let B be a subset of a Γ-semigroup with apartness S. Then, B is a left (right) Γ-coideal if and only if the following implication holds*

$$S\Gamma A \subseteq B \implies A \subseteq B \qquad (\textit{res. } A\Gamma S \subseteq B \implies A \subseteq B)$$

for any subset A of S.

Proposition 2.4 ([32])**.** *If B is a left (right, two-sided) Γ-coideal of a Γ-semigriyop with apartness S, then the set $B^{\lhd}$ is a left (right, two-sided) Γ-ideal of S.*

Now, we repeat the notion of co-congruence in Γ-semigroup with apartness introduced in [32]. Let $(S, =_S, \neq_S, w_S)$ be a Γ-semigroup with apartness. A relation $q \subseteq S \times S$ is a co-equivalence (or a co-equality relation) on S if it is a consistent, co-transitive and symmetric relation on S (see [24]). The family $\{xq : x \in S\}$ of classes of this relation is denoted by

$$[S : q] := \{xq : x \in S\},$$

where $xq = \{y \in S : (x, y) \in q\}$ and

$$(\forall\, x, y \in S)\ (xq =_2 yq \iff (x, y) \lhd q\ \wedge\ xq \neq_2 yq \iff (x, y) \in q).$$

A set designed in this way has no counterpart in the classical theory of Γ-semigroups. Without much difficulty it can be shown that $q^{\lhd}$ is an equivalence on S (for example, see Proposition 1.1 of [31]). Thus, a factor set $S/(q^{\lhd}, q) := \{[x] : x \in S\}$ can be designed, where $[x] := xq^{\lhd}$ and

$$(\forall\, x, y \in S)\ ([x] =_1 [y] \iff (x, y) \lhd q\ \wedge\ [x] \neq_1 [y] \iff (x, y) \in q).$$

Definition 2.6 ([32], Definition 2.6)**.** *An co-equivalence q on S is said to be a* co-congruence *on S if it compatible with the operation in S in the following sense*

$$(\forall\, x, y, u, v \in S)\ (\forall\, a, b \in \Gamma)\ ((xay, ubv) \in q \implies ((x, u) \in q \vee a \neq_\Gamma b \vee (y, v) \in q)).$$

It is obvious that the following implication

$$(\forall\, x, y \in S)\ (\forall\, a, b \in \Gamma)\ ((xay, xby) \in q \implies a \neq b)$$

is valid.

Proposition 2.5 ([33], Lemma 3.2)**.** *The previous implication is equivalent to the following two implications*

$$(\forall\, x, y, u \in S)\ (\forall\, a \in \Gamma)\ ((xay, uay) \in q \implies (x, u) \in q)$$

and

$$(\forall\, x, y, v \in S)\ (\forall\, a \in \Gamma)\ ((xay, xav) \in q \implies (y, v) \in q).$$

Proposition 2.6 ([32], Proposition 2.8)**.** *If q is a Γ-cocongruence on a Γ-semigroup with apartness S, then the relation $q^{\lhd}$ is a Γ-congruence on S.*

Remark 2.2. *Let q be a Γ-cocongruence on a Γ-semigroup S with apartness. Then any class xq, generated by the element $x \in S$, is a strongly extensional subset of S.*

Theorem 2.1 ([32], Theorem 2.5)**.** *If q is a Γ-cocongruence on a Γ-semigroup with apartness S, then the family $[S : q]$ of all classes of q is a Γ-semigroup with $(xq)a(yq) = (xay)q$ for any $x, y \in S$ and $a \in \Gamma$.*

Definition 2.7 ([32], Definition 2.7)**.** *Let $(S, =_S, \neq_S, w_S)$ be a Γ-semigroup and $(T, =_T, \neq_T, w_T)$ be a Λ-semigroups with apartness. A pair (h, φ) of strongly extensional functions $h : S \longrightarrow T$ and $\varphi : \Gamma \longrightarrow \Lambda$ is called a* se-homomorphism *from Γ-semigroup S to Λ-semigroup T if the following holds*

$$(\forall\, x, y \in S)\ (\forall\, a \in \Gamma)\ ((h, \varphi)\ (xay) =_T h(x)\varphi(a)h(y)).$$

The following result can be easily verified.

Lemma 2.2. *Let (h, φ) be a se-homomorphism from Γ-semigroup with apartness S to a Λ-semigroup with apartness T. Then*

$$(h, \varphi) \circ w_S = w_T \circ (h, \varphi, h)$$

is valid, where (h, φ, h) satisfies

$$(\forall\, x, y \in S)\ (\forall\, a \in \Gamma)\ ((h, \varphi, h)(x, a, y) := (h(x), \varphi(a), h(y))).$$

The next result can also be proved easily.

Proposition 2.7. *Let $(h, \varphi) : S \longrightarrow T$ be a se-homomorphism. Then the relation*

$$\rho := Ker(h, \varphi) := \{(x, y) \in S \times S : (h, \varphi)\ (x) =_T (h, \varphi)(y)\}$$

is a Γ-congruence on the Γ-semigroup S, and the relation

$$q := Coker(h, \varphi) := \{(x, y) : (h, \varphi)(x) \neq_T (h, \varphi)(y)\}$$

is a Γ-cocongruence on S such that

$$\rho \circ q \subseteq q \text{ and } q \circ \rho \subseteq q.$$

Remark 2.3. *Theorems on isomorphism both between Γ-semigroup with apartness and between co-ordered Γ-semigroup with apartness can be found in [34, 37]. In both cases, a special Γ-semigroup with apartness appears, which has no counterpart in the classical theory of Γ-semigroups.*

3. The main results: On co-ordered Γ-semigroups with apartness

The compatibility of the relation α on the Γ-semigroup S with an operation in S covers the validity of the following formula

$$(\forall\ x, y, z \in S)\ (\forall\ a \in \Gamma)\ ((x, y) \in \alpha \implies ((xaz, yaz) \in \alpha\ \wedge\ (zax, zay) \in \alpha)).$$

In the Γ-semigroup with apartness theory we use the terminology specified in the next definition.

Definition 3.1. *A relation α defined on a Γ-semigroup with apartness S is a Γ-co-compatible relation on S if*

$$(\forall\ x, y, z \in S)\ (\forall\ a \in \Gamma)\ (((xaz, yaz) \in \alpha\ \vee\ (zax, zay) \in \alpha) \implies (x, y) \in \alpha).$$

In the following definition we recall the concept of co-order relations in Γ-semigroup with apartness.

Definition 3.2 ([35], Definition 3.1)**.** *Let S be a Γ-semigroup with apartness. If $\nleqslant_S$ is a co-order defined on S which is Γ-co-compatible, then S is a co-ordered Γ-semigroup under co-order $\nleqslant_S$, or, shortly, a co-ordered Γ-semigroup.*

Speaking by the language of the classical algebra, relation $\nleqslant_S$ is Γ-co-compatible with the internal operation in S if the operation is right and left cancellative with respect to the co-order relation $\nleqslant_S$.

Examples of co-ordered semigroups can be found in [33] (see Examples 2.9 and 2.10 in [33]) and in [34] (see Examples 2.6, 2.7 and 2.8 in [34]).

An apartness relation '$\neq$' is said to be a tight relation in a set S if

$$(\forall\ x, y \in S)(\neg(x \neq y) \implies x = y).$$

Considering the assumption that, in the general case, the apartness does not have to be a tight relation on S, in what follows we need the following lemmas.

Lemma 3.1. *If $(S, =_S, \neq_S, w_S)$ be an ordered Γ-semigroup with apartness under a co-order $\nleqslant_S$, then $(S, \neq_S^{\lhd}, \neq_S, w_S)$ is an ordered Γ-semigroup with apartness under the order relation $\nleqslant_S^{\lhd}$.*

Proof. Firstly, we show that $\nleqslant_S^{\lhd}$ is an order relation on $(S, \neq_S^{\lhd}, \neq_S)$ and that w_S is an internal operation in $(S, \neq_S^{\lhd}, \neq_S)$.

Take $x, u, v \in S$ such that $u \nleqslant_S v$. Then $u \nleqslant x\ \vee\ x \nleqslant v$. Thus $(x, x) \neq (u, v) \in \nleqslant_S$. This means that $x \nleqslant_S^{\lhd} x$.

Take $x, y, z, u, v \in S$ such that $x \nleqslant_S^{\lhd} y\ \wedge\ y \nleqslant_S^{\lhd} z$ and $u \nleqslant_S v$. Then

$$u \nleqslant_S x\ \vee\ x \nleqslant_S y\ \vee\ y \nleqslant_S z\ \vee\ z \nleqslant_S v$$

by co-transitivity of $\nleqslant_S$. Thus $u \neq_S x\ \vee\ z \neq_S v$. Hence $(x, z) \neq (u, v) \in \nleqslant_S$. So, $x \nleqslant_S^{\lhd} z$. This shows that $\nleqslant_S^{\lhd}$ is a transitive relation.

Take $x, y, u, v \in S$ such that $x \nleqslant_S^{\lhd} y\ \wedge\ y \nleqslant_S^{\lhd} x$ and $u \neq_S v$. Then

$$\begin{aligned} u \neq_S v &\implies (u \neq_S x\ \vee\ x \neq_S y\ \vee\ y \neq_S v) \\ &\implies u \neq_S x\ \vee\ x \nleqslant_S y\ \vee\ y \nleqslant_S x\ \vee\ y \neq_S v \\ &\implies u \neq_S x\ \vee\ y \neq_S v. \end{aligned}$$

This means that $(x, y) \neq (u, v) \in \neq_S$. Therefore, $x \neq_S^{\lhd} y$. This shows that $\nleqslant_S^{\lhd}$ is an antisymmetric relation.

Since $\nleqslant_S^{\triangleleft}$ is a reflexive, transitive, and antisymmetric relation on $(S, \neq_S^{\triangleleft}, \neq_S)$, thus $\nleqslant_S^{\triangleleft}$ is an order relation on $(S, \neq_S^{\triangleleft}, \neq_S)$.

It remains to prove that w_S is an internal operation in $(S, \neq_S^{\triangleleft}, \neq_S)$. Take $x, y, z, u, v \in S$ and $a, b \in \Gamma$ such that $u \neq_S v$. Then

$$u \neq_S xa(ybz) \vee xa(ybz) \neq_S (xay)bz \vee (zay)bz \neq v.$$

Thus, $x \neq_S xa(ybz) \vee (xay)bz \neq_S v$. This means that

$$(xa(ybz), (xay)bz) \neq (u, v) \in \neq_S .$$

So,

$$xa(ybz) \neq_S^{\triangleleft} (xay)bz.$$

Finally, let us prove that the following implications

$$(\forall\, x, y, z \in S)\ (\forall\, a \in \Gamma)\ (x \nleqslant_S^{\triangleleft} y \implies (xaz \nleqslant_S^{\triangleleft} yaz \wedge zax \nleqslant_S^{\triangleleft} zay))$$

are valid. Take $x, y, z, u, v \in S$ and $a \in \Gamma$ such that $x \nleqslant_S^{\triangleleft} y$ and $u \nleqslant_S v$. Then

$$\begin{aligned} u \nleqslant_S v &\implies (u \nleqslant_S xaz \vee xaz \nleqslant_S yaz \vee yaz \nleqslant_S v) \\ &\implies u \neq_S xaz \vee yza \neq_S v \\ &\implies (xaz, yaz) \neq (u, v) \in \nleqslant. \end{aligned}$$

This means that $xaz \nleqslant_S^{\triangleleft} yaz$. The second part of the implication can be proved analogously to the first one. □

3.1. Γ-coideals in a co-ordered Γ-semigroup with apartness

The concepts of co-ideals and co-filters in an ordered semigroup with apartness under a quasi-order as dual notions of ideals and filters are discussed, for example, in the papers [30, 36]. Here, we introduce the notion of left, right and both sided co-ordered co-ideals in a co-ordered Γ-semigroup with apartness $(S, =_S, \neq_S, \nleqslant_S)$.

Definition 3.3. *Let S be an ordered Γ-semigroup with apartness ordered under co-order $\nleqslant_S$.*

(a) A subset K of S is called a co-ordered left Γ-coideal *of S if*

(1) $(\forall\, x, y \in S)\ (\forall\, a \in \Gamma)\ (xay \in K \implies y \in K)$*, and*

(2) $(\forall\, x, y \in S)\ (y \in K \implies (x \in K \vee y \nleqslant_S x))$.

(b) A subset K of S is called a co-ordered right Γ-coideal *of S if*

(3) $(\forall\, x, y \in S)\ (\forall\, a \in \Gamma)\ (xay \in K \implies x \in K)$*, and*

(2) $(\forall\, x, y \in S)\ (y \in K \implies (x \in K \vee y \nleqslant_S x))$.

(c) ([33], Definition 2.11) *A subset K of S is called a* co-ordered Γ-coideal *of S if the following holds*

(4) $(\forall\, x, y \in S)\ (\forall\, a \in \Gamma)\ (xay \in K \implies (x \in K \wedge y \in K))$ *and*

(2) $(\forall\, x, y \in S)\ (y \in K \implies (x \in K \vee y \nleqslant_S x))$.

By definition, we assume that $\emptyset$ and S are co-ordered (left, right, two-sided) co-ideals in S. From the previous definition it is immediately clear that a (left, right, bilateral) co-ideal in S is Γ-cosubsemigroup of S.

Example 3.1. *Let $\mathbb{N}$ be the semiring of natural numbers and $\Gamma = 2\mathbb{N}$. Then $\mathbb{N}$ is a co-ordered Γ-semigroup under standard co-order $\nleqslant$ and $K = (3\mathbb{N} + 1) \cup (3\mathbb{N} + 2) = \{n \in \mathbb{N} : n \notin 3\mathbb{N}\}$ is a co-ideal of the Γ-semigroup $\mathbb{N}$.*

Let $(S, =_S, \neq_S, \nleqslant_S)$ be a co-ordered Γ-semigroup with apartness and let $u, v \in S$ be its two elements. Let us put

$$[u\rangle_{\nleqslant_S} := \{v \in S : u \nleqslant_S v\} \quad \text{and} \quad \langle v]_{\nleqslant_S} := \{u \in S : u \nleqslant_S v\}.$$

It is obvious that the set set $\langle v]_{\nleqslant_S}$ satisfies the condition (2). Indeed, if $x, y \in S$ such that $y \in \langle v]_{\nleqslant_S}$, we have $y \nleqslant_S v$. Then $y \nleqslant_S x \vee x \nleqslant_S v$ by co-transitivity of $\nleqslant_S$. Thus $x \in \langle v]_{\nleqslant_S} \vee y \nleqslant_S x$.

Statement 3.1. *Condition* (2) *is equivalent to the condition*

(2') $(\forall\, y \in S)\ (y \in K \implies S \subseteq K \cup [y\rangle_{\nleqslant_S})$.

Proposition 3.1. *Let K be a co-ordered (left, right) Γ-coideal of an ordered Γ-semigroup with apartness S under co-order $\nleqslant_S$. Then K is a strongly extensional subset in S.*

Proof. The claim of this proposition follows from statement (2) of Definition 3.3 and consistency of the relation $\nleqslant_S$. □

Proposition 3.2. *Let L be a left and R be a right co-ordered Γ-coideals of a co-ordered Γ-semigroup with apartness S. Then $L \cup R$ is a co-orderd Γ-cosubsemigroup in S.*

Proof. Take $u, v \in S$ and $a \in \Gamma$ such that $uab \in L \cup R$. Then $uav \in L$ or $uav \in R$. Thus $u \in L \subseteq L \cup R$ or $v \in R \subseteq L \cup R$. So, $L \cup R$ is a Γ-cosubsemigroup in S.

Take $x, y \in S$ such that $y \in L \cup R$. Then $y \in L \vee y \in R$. In both cases, then we have $x \in L \subseteq L \cup R \vee y \nleqslant_S x$ and $x \in R \subseteq L \cup R \vee y \nleqslant_S x$, according to (2). □

First, we show that the concept of (left, right, bilateral) co-ideal defined in the above way is well-defined in the following sense:

Commitment. *A dual D of a substructure H in a set (in an algebraic structure) $(X, =_X, \neq_X)$ is well-defined if the set $D^{\lhd}$ has the properties of substructure H in the set (in the algebraic structure) $(X, \neq_X^{\lhd}, \neq)$.*

In this case, we say that the substructures H and D are mutually associated. Let us note that in Section 2, the Proposition 2.2 and Proposition 2.4 are designed in accordance with this commitment.

The next result is related to the concept of co-ordered Γ-coideals in co-ordered Γ-semigroup with apartness.

Theorem 3.1. *Let K be a co-ordered left Γ-coideal in an ordered Γ-semigroup with apartness $(S, =_S, \neq_S)$ under co-order $\nleqslant_S$. Then $K^{\lhd}$ is an ordered left Γ-ideal in ordered Γ-semigroup with apartness $(S, \neq_S^{\lhd}, \neq_S)$ under the order relation $\nleqslant_S^{\lhd}$.*

Proof. Since K is a Γ-cosubsemigroup of S, $K^{\lhd}$ is a Γ-subsemigroup in S according to Proposition 2.2.
(i) Since K is a Γ-coideal of S, $K^{\lhd}$ is a Γ-ideal in S according to Proposition 2.2. Thus, the set $K^{\lhd}$ satisfies the following condition

$$(\forall\, x, y \in S)\ (\forall\, a \in \Gamma)\ (y \in K^{\lhd} \implies xay \in K^{\lhd}).$$

(ii) Let us now prove that

$$(\forall\, x, y \in S)\ ((x \nleqslant^{\lhd} y \wedge y \in K^{\lhd}) \implies x \in K^{\lhd}).$$

Let $x, y, u \in S$ be arbitrary elements such that $x \nleqslant^{\lhd} y$, $y \in K^{\lhd}$ and $u \in K$. Since $u \in K$, it follows that $u \nleqslant_S y$ or $y \in K$. Since the second option is impossible due to the hypothesis $y \in K^{\lhd}$, we must have $u \nleqslant_S y$. Now, from $u \nleqslant_S y$, it follows that $u \nleqslant_S x \vee x \nleqslant_S y$ by co-transitivity of $\nleqslant_S$. The second option is impossible. So, $u \nleqslant_S x$ and hence $x \neq_S u \in K$ by consistency of $\nleqslant_S$. This means that $x \in K^{\lhd}$.

Because $K^{\lhd}$ satisfies conditions (1) and (2) of Definition 3.1 of the article [42], we have proved that $K^{\lhd}$ is an ordered left Γ-ideal in S. □

The following two theorems can be proved analogously to the previous one.

Theorem 3.2. *Let K be a co-ordered right Γ-coideal in an ordered Γ-semigroup with apartness $(S, =_S, \neq_S)$ under co-order $\nleqslant_S$. Then $K^{\lhd}$ is an ordered right Γ-ideal in an ordered Γ-semigroup with apartnbewss $(S, \neq_S^{\lhd}, \neq)$ under the order relation $\nleqslant_S^{\lhd}$.*

Theorem 3.3. *Let K be a co-ordered Γ-coideal in an ordered Γ-semigroup with apartness $(S, =_S, \neq_S)$ under co-order $\nleqslant_S$. Then $K^{\lhd}$ is an ordered Γ-ideal in an ordered Γ-semigroup with apartness $(S, \neq_S^{\lhd}, \neq_S)$ under the order relation $\nleqslant_S^{\lhd}$.*

The following proposition shows how a new (left, right) Γ-coideal can be constructed from a given (left, right) Γ-coideal.

Proposition 3.3. *Let K be a co-ordered (left, right) Γ-coideal in a co-ordered Γ-semigroup with apartness $(S, =_S, \neq_S, \nleqslant_S, w_S)$ and let $\nleqslant_S \cap \nleqslant_S^{-1} = \emptyset$. Then the set*

$$\langle K]_{\nleqslant_S} := \{u \in S : (\exists\, v \in K)\ (u \nleqslant_S v)\}$$

is a co-ordered (left, right) Γ-coideal in S.

Proof. Take $x, y \in S$ and $a \in \Gamma$ such that $xay \in \langle K]_{\nleqslant_S}$. Then there exists an element $v \in K$ such that $xay \nleqslant_S v$. On the other hand, from $v \in K$ it follows that $xay \in K \vee v \nleqslant_S xay$ according to (2). Since the second option is impossible because of the hypothesis, we have $xay \in K$ and from here we get $y \in K$ because K is a left Γ-coideal (or $x \in K$ because K is a right Γ-coideal) according to (1) (or (3), respectively).

Take $x, y \in S$ such that $y \in \langle K]_{\nleqslant_S}$. Then there is an element $v \in K$ such that $y \nleqslant_S v$. Then $y \nleqslant_S x$ or $x \nleqslant_S v \in K$ by co-transitivity of $\nleqslant_S$. This means that $x \in \langle K]_{\nleqslant_S}$ or $y \nleqslant_S x$. Thus, the set $\langle K]_{\nleqslant_S}$ satisfies condition (2). □

The following theorem shows that the family of all co-ordered left Γ-coidals of a co-ordered Γ-semigroup with apartness forms a complete lattice.

Theorem 3.4. *The family $\mathfrak{L}(S)$ of all co-ordered left Γ-coideals in a co-ordered Γ-semigroup with apartness S forms a complete lattice and it holds that $\mathfrak{L}(S) \subseteq \mathcal{K}_S$.*

Proof. Let $\{K_i\}_{i\in I}$ be a family of co-ordered left co-ideals in a co-ordered Γ-semigroup with apartness S.

(1) Suppose that $x, y \in S$ and $a \in \Gamma$ such that $xay \in \bigcup_{i\in I} K_i$. Then there is an index $i \in I$ such that $xay \in K_i$. Thus $y \in K_i$ because K_i is a left Γ-coideal in S. Hence $y \in \bigcup_{i\in I} K_i$.

Take $x, y \in S$ such that $y \in \bigcup_{i\in I} K_i$. Then there is an index $i \in I$ such that $y \in K_i$. Thus $x \in K_i$ or $y \nleqslant_S x$ since K_i is a co-ordered left Γ-coideal in S. Hence $x \in \bigcup_{i\in I} K_i$ or $y \nleqslant_S x$.

(2) Let $\mathfrak{X}$ be the family of all co-ordered left Γ-coideals in S contained in $\bigcap_{i\in I} K_i$. Then $\cup\mathfrak{X}$ is the maximal co-ordered left Γ-coideal included in $\bigcap_{i\in I} K_i$ according to the first part of this proof.

(3) If we put $\sqcup_{i\in I} K_i = \bigcup_{i\in I} K_i$ and $\sqcap_{i\in I} K_i = \cup\mathfrak{X}$, then $(\mathfrak{L}(S), \sqcup, \sqcap)$ is a complete lattice. □

Corollary 3.1. *For any element x in a co-ordered Γ-semigroup with apartness S, there is the unique maximal co-ordered left Γ-coideal $L(x)$ in S such that $x \lhd L(x)$.*

Proof. For any element $x \in S$ there is a co-ordered left Γ-coideal $L(x)$ in S contained in the set $\{u \in S : u \neq x\}$, according to the second part of the proof of Theorem 3.4. It is obvious that $x \lhd L(x)$. Let M be a co-ordered left Γ-coideal in S such that $x \lhd M$ and suppose that $L(x) \subseteq M$ is valid. Then $M \subseteq L(x)$, because $L(x)$ is the union of all co-ordered left Γ-coideals contained in $\{u \in S : u \neq x\}$. Thus, $M = L(x)$. □

The following two theorems can be proved in an analogous way as in the previous theorem

Theorem 3.5. *The family $\mathfrak{R}(S)$ of all co-ordered right Γ-coideals in a co-ordered Γ-semigroup with apartness S forms a complete lattice and $\mathfrak{R}(S) \subseteq \mathcal{K}_S$.*

Corollary 3.2. *For any element x in a co-ordered Γ-semigroup with apartness S, there is the unique maximal co-ordered right Γ-coideal $R(x)$ in S such that $x \lhd R(x)$.*

Theorem 3.6. *The family $\mathfrak{K}(S)$ of all co-ordered Γ-coideals in a co-ordered Γ-semigroup with apartness S forms a complete lattice.*

Corollary 3.3. *For any element x in a co-ordered Γ-semigroup with apartness S, there is the unique maximal co-ordered Γ-coideal $K(x)$ in S such that $x \lhd K(x)$.*

Let L be a co-ordered Γ-coideal and R be a right co-ordered Γ-coideal in a co-ordered Γ-semigroup with apartness $(S, =_S, \neq_S, w_S, \nleqslant_S)$. Since it holds that $(L \cup R)^{\lhd} = L^{\lhd} \cap R^{\lhd}$, we have the next result.

Theorem 3.7. *The union of a co-ordered left Γ-coideal L and a co-ordered right Γ-coideal R in a co-ordered Γ-semigroup with apartness $(S, =_S, \neq_S, w_S, \nleqslant_S)$ is a dual of the notion of quasi-Γ-ideal in the ordered Γ-semigroup with apartness $(S, \neq_S^{\lhd}, \neq_S, w_S)$ under the order relation $\nleqslant_S^{\lhd}$.*

Proof. The set $L^{\lhd}$ is an ordered left Γ-ideal (Theorem 3.1) and the set $R^{\lhd}$ is an ordered right Γ-ideal (Theorem 3.2) in the ordered Γ-semigroup with apartness $(S, \neq_S^{\lhd}, \neq_S, w_S)$ under the order relation $\nleqslant_S^{\lhd}$. The proof of this meta-theorem follows directly from Theorem 2.4 of [5] since every quasi-Γ-ideal Q of a Γ–semigroup S is the intersection of a left Γ-ideal and a right Γ-ideal of S. □

3.2. Se-homomorphisms between co-ordered Γ-semigroups

Although some of the results about the concepts and processes with them related to the topic of the present paper, presented in this subsection, are proven in article [37], they are included here in order to give the reader an impression of the integrity of the presented material. In this subsection, we prove some results that refer to processes with co-ordered Γ-coideals in a co-ordered Γ-semigroup with apartness relying on the specificity of se-homomorphisms between co-ordered Γ-semigroups with apartness.

Let $(h, \varphi) : S \longrightarrow T$ be a reverse isotone se-homomorphism from a co-ordered Γ-semigroup with apartness $(S, =_S, \neq_S, w_S, \nleqslant_S)$ to a co-ordered Λ-semigroup with apartness $(T, =_T, \neq_T, w_T, \nleqslant_T)$. We say that (h, φ) is *reverse isotone* [37] if the following holds

$$(\forall\, x, y, u, v \in S)\ (\forall\, a, b \in \Gamma)\ ((h, \varphi)\ (xay) \nleqslant_T (h, \varphi)(ubv) \implies xay \nleqslant_S ubv).$$

Proposition 3.4. *Let $(h,\varphi) : S \longrightarrow T$ be a reverse isotone se-homomorphism from a co-ordered Γ-semigroup with apartness $(S, =_S, \neq_S, w_S, \nleqslant_S)$ to a co-ordered Λ-semigroup with apartness $(T, =_T, \neq_T, w_T, \nleqslant_T)$ and let B be a left (right, two sided) Λ-coideal of T. Then $(h,\varphi)^{-1}(B)$ is a left (right, both sided) Γ-coideal in S.*

Proof. Take $x, y \in S$ and $a \in \Gamma$ such that $xay \in (h,\varphi)^{-1}(B)$. Then $h(x)\varphi(a)h(y) \in B$. Thus $h(y) \in B$ since B is a left Λ-coideal in T. Also, for $y \in (h,\varphi)^{-1}(B)$, we have $h(y) \in B$. Then $h(x) \in B$ or $h(y) \nleqslant_T h(x)$. Thus $x \in (h,\varphi)^{-1}(B)$ or $y \nleqslant_S x$ because (h,φ) is a reverse isotone se-mapping. This shows that $(h,\varphi)^{-1}(B)$ is a co-ordered left Γ-coideal in S.

Analogous to the previous one, it can be shown that if B is a co-ordered right Λ-coideal, then $(h,\varphi)^{-1}(B)$ is a co-ordered right Γ-coideal. □

To prove the next theorem, we need some lemmas. Proposition 2.7 allows us to construct the structure

$$(S/(\rho,q), =_1, \neq_1) := \{[x]_\rho : x \in S\},$$

where $[x]_\rho := \{y \in S : (h,\varphi)(x) =_T (h,\varphi)(y)\}$ with

$$x\rho =_1 y\rho \iff (x,y) \in \rho \quad \text{and} \quad x\rho \neq_1 y\rho \iff (x,y) \in q$$

where $x, y \in S$. For the function

$$w_{S/(\rho,q)} : S/(\rho,q) \times \Gamma \times S/(\rho,q) \longrightarrow S/(\rho,q)$$

we have

$$(\forall\, x\rho, y\rho \in S/(\rho,q))\ (\forall\, a \in \Gamma)\ (w_{S/(\rho,q)}(x\rho, a, y\rho) := (xay)\rho).$$

Lemma 3.2 ([37]). *Let $(h,\varphi) : S \longrightarrow T$ be a reverse isotone homomorphism. Then $\not\prec := (h,\varphi)^{-1}(\nleqslant_T)$ is a co-quasiorder relation on S compatible with the operation on S such that $\not\prec \subseteq \nleqslant_S$.*

Proof. It suffices to prove that $\not\preceq$ is a co-order on $(S/(\rho,q), =_1, \neq_1)$ and that (π, i) is a reverse isotone se-epimorphism.

Let $x, y \in S$ be arbitrary elements such that $x\rho \not\preceq y\rho$. Then $x \not\prec y$. This means that $(h,\varphi)(x) \nleqslant_T (h,\varphi)(y)$. Thus $x \nleqslant_S y$ because (h,φ) is a reverse isotone mapping. Thus, $\not\preceq$ is a consistent relation on $S/(\rho,q)$.

Let $x, y, z \in S$ be arbitrary elements such that $x\rho \not\preceq z\rho$. Then $x \not\prec z$. Thus $x \not\prec y \vee y \not\prec z$ by co-transitivity of $\not\prec$. Hence, $x\rho \not\preceq y\rho \vee y\rho \not\preceq z\rho$. So, the relation $\not\preceq$ is a co-transitive relation on $S/(\rho,q)$.

Let $x, y \in S$ be arbitrary elements such that $x\rho \neq_1 y\rho$. Then $(x,y) \in q$. This means that $(h,\varphi)(x) \neq_T (h,\varphi)(y)$. From here, we have

$$(h,\varphi)(x) \nleqslant_T (h,\varphi)(y) \ \vee\ (h,\varphi)(y) \nleqslant_T (h,\varphi)(x)$$

since $\nleqslant_T$ is a co-order relation. Therefore, $x\rho \not\preceq y\rho \vee y\rho \not\preceq x\rho$ which shows that $\not\preceq$ satisfies the condition of linearity.

Let $x, y, z \in S$ and $a \in \Gamma$ be arbitrary elements such that $(xaz)\rho \not\preceq (ybz)\rho$. Then $(xaz) \not\prec (yaz)$. This means that

$$(h,\varphi)(xaz) \nleqslant_T (h,\varphi)(yaz),$$

i.e. $h(x)\varphi(a)h(z) \nleqslant_T h(y)\varphi(a)h(z)$. Thus $h(x) \nleqslant_T h(y)$. So, $x\rho \not\preceq y\rho$ by definition of $\not\preceq$.

Let $x, y \in S$ be arbitrary elements such that $(\pi,i)(x) = x\rho \not\preceq y\rho = (\pi,i)(y)$. Then $x \not\prec y$. This means that

$$(h,\varphi)(x) \nleqslant_T (h,\varphi)(y).$$

Thus, $x \nleqslant_S y$ because (h,φ) is a reverse isotone mapping. We have proved that (π,i) is a reverse isotone mapping. □

Lemma 3.3 ([37]). *If $(h,\varphi) : S \longrightarrow T$ is a reverse isotone homomorphism then $q = Coker(h,\varphi) = \not\prec \cup \not\prec^{-1}$.*

Proof. The proof of this Lemma is analogous to the proof of Lemma 1 of [29], so we omit it. □

Although the statement in the following lemma is taken from [37], its proof is included here due to the consistency of the presented material.

Lemma 3.4. *The structure $(S/(\rho,q), =_1, \neq_1, \not\preceq)$ is an ordered Γ-semigroup with apartness under the co-order relation $\not\preceq$, defined by the flowing way*

$$(\forall\, x, y \in S)\ (x\rho \not\preceq y\rho \iff x \not\prec y),$$

and there exists a unique reverse isotone se-epimorphism $(\pi, i) : S \ni x \longmapsto x\rho \in S/(\rho,q)$ such that the following holds

$$w_{S/(\rho,q)} \circ (\pi, i, \pi) = (\pi, i) \circ w_S$$

where $i : \Gamma \longrightarrow \Gamma$ is the identical mapping.

Proof. It is clear that $w_{S/(\rho,q)}$ is a well-defined total function. Let us prove that $w_{S/(\rho,q)}$ is strongly extensional. Let $x\rho, y\rho, u\rho, v\rho \in S/(\rho,q)$ and $a \in \Gamma$ be arbitrary elements such that

$$w_{S/(\rho,q)}(x\rho, a, y\rho) \neq_1 w_{S/(\rho,q)}(u\rho, b, v\rho).$$

Then $(xay)\rho \neq_1 (ubv)\rho$. The latter means that $(xay, ubv) \in q$. Thus $xay \neq ubv$ by consistency of q. Hence, $w_{S/(\rho,q)}$ is a strongly extensional total function.

Let us now show that $w_{S/(\rho,q)}$ satisfies the associativity condition of Definition 2.3. For $x\rho, y\rho, z\rho \in S/(\rho,q)$ and $a, b \in \Gamma$, we have

$$\begin{aligned} w_{S/(\rho,q)}(x\rho a y\rho, b, z\rho) &=_1 w_{S/(\rho,q)}((xay)\rho, b, z\rho) \\ &=_1 ((xay)bz)\rho =_1 (xa(ybz))\rho =_1 w_{S/(\rho,q)}(x\rho, a, (ybz)\rho). \end{aligned}$$

If we define the function $(\pi, i) : S \longrightarrow S/(\rho,q)$ in the following way $(\forall\, x \in S)\ ((\pi,i)(x) := x\rho \in S/(\rho,q)$ then it can be checked without major difficulties that it is correctly defined a unique strongly extensional total surjective homomorphism.

Finally, let $x, y \in S$ and $a \in \Gamma$ be arbitrary elements. We have

$$\begin{aligned} &((\pi,i)\circ w_S)(x,a,y) =_1 (\pi,i)(w(x,a,y)) =_1 (\pi,i)(xay) =_1 (xay)\rho \\ &=_1 w_{S/(\rho,q)}(x\rho, a, y\rho) =_1 w_{S/(\rho,q)}(\pi(x), i(a), \pi(y)) =_1 w_{S/(\rho,q)}((\pi,i,\pi)(x,a,y)) \\ &=_1 (w_{S/\rho,q)} \circ (\pi,i,\pi))(x,a,y). \end{aligned}$$

Let $x, y \in S$ be arbitrary elements such that $x\rho \not\preceq y\rho$. Then $x \not\prec y$. This means that $(h,\varphi)(x) \nleqslant_T (h,\varphi)(y)$. Thus $x \nleqslant_S y$ because (h,φ) is a reverse isotone mapping. Thus, $\not\preceq$ is a consistent relation on $S/(\rho,q)$.

Let $x, y, z \in S$ be arbitrary elements such that $x\rho \not\preceq z\rho$. Then $x \not\prec z$. Thus $x \not\prec y \vee y \not\prec z$ by co-transitivity of $\not\prec$. Hence, $x\rho \not\preceq y\rho \vee y\rho \not\preceq z\rho$. So, the relation $\not\preceq$ is a co-transitive relation on $S/(\rho,q)$.

Let $x, y \in S$ be arbitrary elements such that $x\rho \neq_1 y\rho$. Then $(x,y) \in q$. This means that $(h,\varphi)(x) \neq_T (h,\varphi)(y)$. From here, we have

$$(h,\varphi)(x) \nleqslant_T (h,\varphi)(y) \ \vee\ (h,\varphi)(y) \nleqslant_T (h,\varphi)(x)$$

since $\nleqslant_T$ is a co-order relation. Therefore, $x\rho \not\preceq y\rho \vee y\rho \not\preceq x\rho$ which shows that $\not\preceq$ satisfies the condition of linearity.

Let $x, y, z \in S$ and $a \in \Gamma$ be arbitrary elements such that $(xaz)\rho \not\preceq (ybz)\rho$. Then $(xaz) \not\prec (yaz)$. This means that $(h,\varphi)(xaz) \nleqslant_T (h,\varphi)(yaz)$, i.e. $h(x)\varphi(a)h(z) \nleqslant_T h(y)\varphi(a)h(z)$. Thus $h(x) \nleqslant_T h(y)$. So, $x\rho \not\preceq y\rho$ by definition of $\not\preceq$.

Let $x, y \in S$ be arbitrary elements such that $(\pi,i)(x) = x\rho \not\preceq y\rho = (\pi,i)(y)$. Then $x \not\prec y$. This means that

$$(h,\varphi)(x) \nleqslant_T (h,\varphi)(y).$$

Thus, $x \nleqslant_S y$ because (h,φ) is a reverse isotone mapping. We have proved that (π,i) is a reverse isotone mapping. □

Propositions 3.4 and Lemma 3.4 allow us to prove the next theorem.

Theorem 3.8. *Let $(h,\varphi) : S \longrightarrow T$ be a reverse isotone se-homomorphism from a co-ordered Γ-semigroup with apartness $(S, =_S, \neq_S, w_S, \nleqslant_S)$ to a co-ordered Λ-semigroup with apartness $(T, =_T, \neq_T, w_T, \nleqslant_T)$. Then, for each left (right, both sided) co-ordered Γ-coideal B in a co-ordered Γ-semigroup with apartness $S/(\rho,q)$ there is a co-ordered left (right, both sided) Γ-coideal A in S such that $(\pi,i)(A) = B$.*

Proof. Lemma 3.4 provides the existence of co-ordered Γ-semigroup with apartness $S/(\rho,q)$ and provides the existence of a reverse isotonic se-epimorphism $(\pi,i) : S \longrightarrow S/(\rho,q)$. We now can apply Proposition 3.4 to the reverse isotone se-mapping (π,i). □

We can define the set $[S:q] := \{xq : x \in S\}$ where the equality '$=_2$' and the apartness '$\neq_2$' are determined in the following way

$$(\forall\, xq, yq \in [S:q])\ (xq =_2 yq \iff (x,y) \lhd q \ \text{ and } \ xq \neq_2 yq \iff (x,y) \in q).$$

The operation $w_{[S:q]}$ on $[S:q] \times \Gamma \times [S:q]$ is defined as

$$(\forall\, xq, yq, \in [S:q])\ (\forall\, a \in \Gamma)\ (w_{[S:q]}(xq, a, yq) := (xay)q).$$

Similar to the previous lemma, although the statement of the following lemma is taken from [37], its proof is included here due to the consistency and clarity of the presentation of the presented material.

Lemma 3.5. *The structure* $([S:q], =_2, \neq_2, \npreceq)$ *is an ordered* Γ*-semigroup with apartness under a co-order* $\npreceq$*, defined by the flowing way*

$$\npreceq := \{(xq, yq) \in [S:q] \times [S;q] : x \nprec y\},$$

and there is a unique reverse isotone se-epimorphism $(\vartheta, i) : S \longrightarrow [S:q]$ *such that the following holds*

$$w_{[S:q]} \circ (\vartheta, i, \vartheta) = (\vartheta, i) \circ w_S.$$

Proof. For the arbitrary elements $xq, yq, uq, vq \in [S:q]$ and $a, b \in \Gamma$ satisfying $(xq, a, yq) =_{[S:q]\times\Gamma\times[S:q]} (uq, b, vq)$ we have $xq =_2 uq$, $a = b$ and $yq =_2 vq$. This means that $(x, u) \lhd q$ and $(y, v) \lhd q$. Take $s, t \in S$ such that $(s, t) \in q$. Then $(s, xay) \in q$ or $(xay, ubv) \in q$ or $(ubv, t) \in q$. since the second possibility gives $(x, u) \in q \vee a = b \vee (y, v) \in q$ which contradicts the adopted assumption, we conclude that the following holds $sq \neq_2 (xay)q$ or $(ubv)q \neq_2 tq$. Since $s, t \in S$ were arbitrary, we conclude that $(xay, ubv) \lhd q$ is valid. Hence

$$w_{[S:q]}(xq, a, yq) =_2 (xay)q =_2 (ubv)q =_2 w_{[S:q]}(uq, b, vq).$$

This shows that $w_{[S:q]}$ is a well-defined function.

Take $xq, yq, uq, vq \in [S:q]$ and $a, b \in \Gamma$ such that

$$w_{[S:q]}(xq, a, yq) =_2 (xay)q \neq_2 (ubv)q =_2 w_{[S:q]}(uq, b, vq).$$

It follows that

$$(x, u) \in q \vee a \neq b \vee (y, v) \in q.$$

So, $(xq, a, yq) \neq_{[S:q]\times\Gamma\times[S:q]} (uq, b, vq)$, which shows that $w_{[S:q]}$ is a se-mapping.

For $xq, yq, zq \in [S:q]$ and $a, b \in \Gamma$ we have

$$w_{[S:q]}(xq, a, (yz)a) =_2 (xa(ybz))q =_2 ((xay)bz)q =_2 w_{[S:q]}(xay, b, z).$$

Therefore, $([S:q], =_2, \neq_2)$ is a Γ-semigroup.

Finally, if $x, y \in S$ and $a \in \Gamma$ are arbitrary elements, then we have

$$\begin{aligned}(w_{[S:q]} \circ (\vartheta, i, \vartheta))(x, a, y) &=_2 w_{[S:q]}(xq, a, yq) =_2 (xay)q =_2 (\vartheta, i)(xay)\\ &=_2 ((\vartheta, i) \circ w_S)(x, a, t).\end{aligned}$$

We will show that $\npreceq$ is a co-order relation on $[S:q]$, that is, we will show that $([S,q], \doteq_2, \neq_2, \npreceq)$ is co-ordered Γ-semigroup.

Take $x, y \in S$ such that $xq \npreceq yq$. Then $x \nprec y$, i.e. $(h, \varphi)(x) \nleqslant_T (h, \varphi)(y)$. Thus $(h, \varphi)(x) \neq_T (h, \varphi)(y)$ by consistency of $\nleqslant_T$. This means that $(x, y) \in q$ and hence $xq \neq_2 yq$.

Take $x, y, z \in S$ such that $xq \npreceq zq$. Then $x \nprec z$. This means that $(h, \varphi)(x) \nleqslant_T (h, \varphi)(z)$. Thus

$$(h, \varphi)(x) \nleqslant_T (h, \varphi)(y) \vee (h, \varphi)(y) \nleqslant_T (h, \varphi)(z)$$

by co-transitivity of $\nleqslant_T$. So, we have

$$xq \npreceq yq \vee yq \npreceq zq.$$

Take $x, y \in S$ such that $xq \neq_2 yq$. Then $(h, \varphi)(x) \neq_T (h, \varphi)(y)$. Thus

$$(h, \varphi)(x) \nleqslant_T (h, \varphi)(y) \ \vee (h, \varphi)(y) \nleqslant_T (h, \varphi)(x)$$

by linearity of $\nleqslant_T$.

If $x, y, z \in S$ and $a \in \Gamma$ such that $(xaz)q \npreceq (yaz)q$, then we have

$$(h, \varphi)(xaz) \nleqslant_T (h, \varphi)(yaz),$$

i.e. we have

$$h(x)\varphi(a)h(z) \nleqslant_T h(y)\varphi(a)h(z).$$

Thus $h(x) \nleqslant_T h(y)$. This means that $xq \npreceq yq$.

It remains to show that (ϑ, i) is a reverse isotone se-epimorphism. Take $xq, yq \in [S:q]$ such that $(\vartheta, i)(x) \npreceq (\vartheta, i)(y)$. Then $xq \npreceq yq$. Thus $x \nprec y$ by definition of $\npreceq$. Hence $x \nleqslant_T y$. □

Propositions 3.4 and Lemma 3.5 allow us to prove the following theorem.

Theorem 3.9. *Let $(h,\varphi) : S \longrightarrow T$ be a reverse isotone se-homomorphism from a co-ordered Γ-semigroup with apartness $(S, =_S, \neq_S, w_S, \nleqslant_S)$ to a co-ordered Λ-semigroup with apartness $(T, =_T, \neq_T, w_T, \nleqslant_T)$. Then, for each left (right, both sided) co-ordered Γ-coideal B in a co-ordered Γ-semigroup with apartness $[S : q]$ there is a co-ordered left (right, both sided) Γ-coideal A in S such that $(\vartheta, i)(A) = B$.*

Proof. Lemma 3.5 provides the existence of co-ordered Γ-semigroup with apartness $[S : q]$ and provides the existence of a reverse isotonic se-epimorphism $(\vartheta, i) : S \longrightarrow [S : q]$. We now can apply Proposition 3.4 to the reverse isotone se-mapping (ϑ, i). □

This structure is one of the specificities of the chosen principle-logical orientation and it cannot be found in the classical theory of Γ-semigroups.

Proposition 3.5. *Let $(h,\varphi) : S \longrightarrow T$ be an isotone se-homomorphism from a co-ordered Γ-semigroup with apartness $(S, =_S, \neq_S, w_S, \nleqslant_S)$ onto a co-ordered Λ-semigroup with apartness $(T, =_T, \neq_T, w_T, \nleqslant_T)$ and let A be a left (right, two sided) Γ-coideal of S. Then $(h,\varphi)(A)$ is a left (right, both sided) Λ-coideal in T.*

Proof. Take $u, v \in T$ and $b \in \Lambda$ such that $ubv \in (h,\varphi)(A)$. Then there are elements $x, y \in S$ and $a \in \Gamma$ such that $h(x) =_T u$, $\varphi(a) =_\Lambda b$ and $h(y) =_T v$ such that $(h,\varphi)(xay) =_T h(x)\varphi(a)h(y) \in (h,\varphi)(A)$. This means that $xay \in A$. Thus $y \in A$ because A is a co-ordered left Γ-coideal in S. Hence $v =_T h(y(\in (h,\varphi)(A)$. Also, for $h(y) =_T v \in (h,\varphi)(A)$, we have $y \in A$ and $y \in A \implies y \nleqslant_S x \vee x \in A$ since A is a co-ordered left Γ-coideal in S. Then $v =_T h(y) \nleqslant_T h(x) =_T u$ or $u \in (h,\varphi)(A)$ since (h,φ) is an isotone se-mapping. This shows that $(h,\varphi)(A)$ is a co-ordered left Λ-coideal in T.

Similarly, it can be shown that if A is a co-ordered right Γ-coideal, then $(h,\varphi)(A)$ is a co-ordered right Λ-coideal in T. □

3.3. Notes

Note 3.1. *For the purpose of the following analysis, we need to recall some previously defined notions.*

The concept of semi-lattice co-congruences on an ordered Γ-semigroup with apartness S under a co-order '$\nleqslant_S$' is given in [33]. A co-congruence q on a co-ordered Γ-semigroup with apartness S is a semi-lattice co-congruence on S if the following hold

$(\forall\, x \in S)\ (\forall\, a \in \Gamma)((x, xax) \lhd q)$ *and*

$(\forall\, x, y \in S)\ (\forall\, a \in \Gamma)((xay, yax) \lhd q)$.

A semi-lattice co-congruence q on S is called a co-ordered semi-lattice co-congruence if the following holds

$(\forall\, x, y \in S)\ (\forall\, a \in \Gamma)((x, xay) \in q \implies x \nleqslant_S y)$.

In [33], it was shown that for a class xq of this relation generated by the element $x \in S$ the following holds:

- *The class xq is a strongly extensional subset in S;*
- $(\forall\, u, v \in S)\ (\forall\, a \in \Gamma)(uav \in xq \implies (u \in xq \wedge v \in xq))$ *and*
- $(\forall\, u, v \in S)\ (\forall\, a \in \Gamma)(u \in xq \implies (uav \in xq \wedge vau \in xq))$.

So, xq is a Γ-ideal and a Γ-filter in S. (The concept of Γ-cofilter in a co-ordered Γ-semigroup with apartness was introduced in [35].) The constraint

$$(\forall\, u, v \in S)\ (\forall\, a \in \Gamma)(u \in P \wedge v \in P \implies uav \in P),$$

imposed on a co-ordered (left, right) Γ-coideal P in S could be a dual of the classical term "s-prime Γ-ideal in Γ-semigroup" [42] since in [35], it was shown that $P^{\lhd}$ is an ordered Γ-filter in S.

Note 3.2. *For the purpose of the following analysis, we need a new notion. An ordered Γ-semigroup with apartness $(S, =_S, \neq_S, w_S)$ under a co-order $\nleqslant_S$ is called regular if for each $x \in S$ and for every pair $a, b \in \Gamma$ there exists an element $x' \in S$ such that $x \nleqslant_S^{\lhd} xax'bx$ according to a definition given in the paper [1]. In this case, a co-ordered left L and a right R co-ideals in a co-ordered regular Γ-semigroup with apartness S have some specific features.*

For elements $u, v \in S$ such that $u \in L$, the following $u \nleqslant_S v \vee v \in L$ is valid by (2). On the other hand, since S is a co-irdered regular Γ-semigroup with apartness, for the element $u \in L$ and arbitrary elements $a, b \in \Gamma$ there exists an element $u' \in S$ such that $u \nleqslant_S^{\lhd} uau'bu$. Thus, for the elements $u \in L$ and $uau'bu \in S$, it holds that $uau'bu \in L \vee u \nleqslant_S uau'bu$.

Hence $uau'bu \in L$ because the second option is impossible according to the accepted hypothesis of this analysis. Now, from $ua(u'bu) \in L$, it follows that $u'bu \in L$ since L is a left ideal in S. We conclude that

$$(\forall\, u \in S)\ (\forall\, b \in \Gamma)(\exists\, u' \in S)(u \in L \implies u'bu \in L).$$

By analogy with the previous demonstration procedure, for a co-ordered right Γ-coideal R in a co-ordered regular Γ-semigroup with apartness S, the following implication is valid:

$$(\forall\, v \in S)\ (\forall\, a \in \Gamma)(\exists\, v' \in S)(v \in R \implies vav' \in R).$$

4. Final comments

In the principle-logical environment of Bishop's constructive algebra, which implies the use of intuitionist logic instead of the classics logic, the author develops the concept of ordered Γ-semigroups with apartness ordered under a co-order relation. In addition to designing the concept of co-ordered (left, right) Γ-coideals in a co-ordered Γ-semigroup with apartness as a dual to the notion of ordered (left, right) Γ-ideals (Definition 3.3), the author exposes his knowledge of the connection between these duals (Theorem 3.1). At the end of Section 3, a constructive dual of the classical notion of quasi-Γ-ideal in such algebraic structures is also given (Theorem 3.7).

One open question that still remains unanswered is how to design the dual of construction $A\Gamma B$ for the subsets A and B of a Γ-semigroup with apartness. It is expected that the answer to this question could be the basis for designing many other classes of duals of ideals in such an algebraic structure.

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