

Research Article

A note on Chebyshev inequality via k -generalized fractional integrals

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Abstract

In this paper, by using the k -generalized fractional integrals, we establish certain integral inequalities for the Chebyshev functional in case of synchronous function. The obtained inequalities generalize several known integral inequalities.

Keywords: generalized k -proportional fractional integral operators with general kernel; Chebyshev inequality; synchronous functions.

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1. Introduction

One of the most developed mathematical areas in recent years is that of integral inequalities, particularly involving various fractional and generalized integral operators; for example, see [1, 5–8, 10, 15, 17, 18, 21, 25, 31, 43, 44]. Recently, the generalized k -proportional fractional integral operators with general kernel were defined, which contain many of the known fractional operators. In order to give detail of this work, we need to first present some preliminary results.

Throughout this paper, we use the functions Γ (see [33, 36, 46, 47]) and Γ_k (see [13]) defined as follows:

$$\Gamma(z) = \int_0^\infty \tau^{z-1} e^{-\tau} d\tau, \quad \Re(z) > 0$$

and

$$\Gamma_k(z) = \int_0^\infty \tau^{z-1} e^{-\tau^k/k} d\tau, \quad k > 0.$$

It is clear that if $k \rightarrow 1$ we have $\Gamma_k(z) \rightarrow \Gamma(z)$, $\Gamma_k(z) = (k)^{\frac{z}{k}-1} \Gamma\left(\frac{z}{k}\right)$ and $\Gamma_k(z+k) = z\Gamma_k(z)$. Also, we define the k -beta function as follows

$$B_k(u, v) = \frac{1}{k} \int_0^1 \tau^{\frac{u}{k}-1} (1-\tau)^{\frac{v}{k}-1} d\tau.$$

Notice that

$$B_k(u, v) = \frac{1}{k} B\left(\frac{u}{k}, \frac{v}{k}\right)$$

and

$$B_k(u, v) = \frac{\Gamma_k(u)\Gamma_k(v)}{\Gamma_k(u+v)}.$$

The fractional integral operator of Riemann-Liouville are being extended and generalized in various ways. One of such ways are presented in this paper. From the point of view of differential operators and by manipulating simple algebraic identities, one can follow the idea of fractional differential operators of the Riemann-Liouville or Caputo type. From the simple facts $\alpha = 1 + \alpha - 1$ and $\alpha = \alpha - 1 + 1$ we have

$${}^{RL}D^\alpha f(t) = \frac{d}{dt} \{J_{a_1}^{1-\alpha}(f)(t)\},$$

and

$${}^CD^\alpha f(t) = J_{a_1}^{1-\alpha} \left(\frac{df}{dt} \right) (t),$$

respectively.

Next, we present several definitions of fractional integrals, some very recent (with $0 \leq a_1 < \tau < a_2 \leq \infty$). One of the first operators that can be called fractional is the Riemann-Liouville fractional derivative of order $\alpha \in \mathbb{C}$, with $\Re(\alpha) > 0$, defined as follows (see [16]).

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Definition 1.1. Let $a_1 < a_2$ and $f \in L^1((a_1, a_2); \mathbb{R})$. The right and left side Riemann-Liouville fractional integrals of order α , with $\text{Re}(\alpha) > 0$, are defined, respectively, by

$${}^{RL}J_{a_1^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^t (t - s)^{\alpha-1} f(s) ds,$$

and

$${}^{RL}J_{a_2^-}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^{a_2} (s - t)^{\alpha-1} f(s) ds,$$

with $t \in (a_1, a_2)$.

Other definitions of fractional integral operators are the following ones.

Definition 1.2. [19] Let $a_1 < a_2$ and $f \in L^1((a_1, a_2); \mathbb{R})$. The right and left side Hadamard fractional integrals of order α , with $\text{Re}(\alpha) > 0$, are defined, respectively, by

$$H_{a_1^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds,$$

and

$$H_{a_2^-}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^{a_2} \left(\log \frac{s}{t}\right)^{\alpha-1} \frac{f(s)}{s} ds,$$

with $t \in (a_1, a_2)$.

In [22], the author introduced new fractional integral operators, called the Katugampola fractional integrals, in the following way.

Definition 1.3. Let $0 < a_1 < a_2$, $f : [a_1, a_2] \rightarrow \mathbb{R}$ be an integrable function, and $\alpha \in (0, 1), \rho > 0$ be two fixed real numbers. The right and left side Katugampola fractional integrals of order α are defined, respectively, by

$$K_{a_1^+}^{\alpha, \rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{a_1}^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} f(s) ds,$$

and

$$K_{a_2^-}^{\alpha, \rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^{a_2} \frac{t^{\rho-1}}{(s^\rho - t^\rho)^{1-\alpha}} f(s) ds,$$

with $t \in (a_1, a_2)$.

The left-sided and right-sided Riemann-Liouville k -fractional integrals are given in [28].

Definition 1.4. Let $f \in L_1[a_1, a_2]$. Then the Riemann-Liouville k -fractional integrals of order $\alpha \in \mathbb{C}, \Re(\alpha) > 0$ and $k > 0$ are given by the expressions:

$${}^\alpha I_{a_1^+}^k f(u) = \frac{1}{k\Gamma_k(\alpha)} \int_{a_1}^u (u - \tau)^{\frac{\alpha}{k}-1} f(\tau) d\tau, \quad u > a_1,$$

$${}^\alpha I_{a_2^-}^k f(u) = \frac{1}{k\Gamma_k(\alpha)} \int_u^{a_2} (\tau - u)^{\frac{\alpha}{k}-1} f(\tau) d\tau, \quad u < a_2.$$

A more general definition of the Riemann-Liouville fractional integrals is given in [24].

Definition 1.5. Let $f : [a_1, a_2] \rightarrow \mathbb{R}$ be an integrable function. Also, let g be an increasing and positive function on $(a_1, a_2]$ with a continuous derivative g' on (a_1, a_2) . The left and right sided fractional integrals of a function f with respect to another function g on $[a_1, a_2]$ of order $\alpha \in \mathbb{C}, \Re(\alpha) > 0$, are expressed by:

$${}^\alpha I_{a_1^+} f(u) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^u (g(u) - g(\tau))^{\alpha-1} g'(\tau) f(\tau) d\tau, \quad u > a_1,$$

$${}^\alpha I_{a_2^-} f(u) = \frac{1}{\Gamma(\alpha)} \int_u^{a_2} (g(\tau) - g(u))^{\alpha-1} g'(\tau) f(\tau) d\tau, \quad u < a_2.$$

A k -fractional analogue of Definition 1.5 is given in the following (see [4, 26, 37]).

Definition 1.6. Let $f : [a_1, a_2] \rightarrow \mathbb{R}$ be an integrable function. Also, let g be an increasing and positive function on $(a_1, a_2]$ with a continuous derivative g' on (a_1, a_2) . The left and right sided k -fractional integrals of a function f with respect to another function g on $[a_1, a_2]$ of order $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ and $k > 0$ are expressed by:

$${}^\alpha I_{a_1^+}^k f(u) = \frac{1}{k\Gamma_k(\alpha)} \int_{a_1}^u (g(u) - g(\tau))^{\frac{\alpha}{k}-1} g'(\tau) f(\tau) d\tau, \quad u > a_1,$$

$${}^\alpha I_{a_2^-}^k f(u) = \frac{1}{k\Gamma_k(\alpha)} \int_u^{a_2} (g(\tau) - g(u))^{\frac{\alpha}{k}-1} g'(\tau) f(\tau) d\tau, \quad u < a_2.$$

Next, we have the definition of the generalized proportional fractional (GPF) integral operator (see [38]).

Definition 1.7. Let $U \in X_\Psi^q(0, +\infty)$, $0 < a_1 < a_2$, and there be an increasing and positive monotone function Ψ defined on $[0, +\infty)$ having continuous derivative Ψ' on $[0, +\infty)$ with $\Psi(0) = 0$. Then the left-sided and right-sided GPF-integral operator of a function U in the sense of another function Ψ of order η are stated as:

$${}^\Psi T_{v_1^+}^{\eta, \varsigma} U(\varsigma) = \frac{1}{\varsigma^\eta \Gamma(\eta)} \int_{v_1}^\varsigma \frac{\exp\left[\frac{\varsigma-1}{\varsigma}(\Psi(\varsigma) - \Psi(\xi))\right]}{(\Psi(\varsigma) - \Psi(\xi))^{1-\eta}} U(\xi) \Psi'(\xi) d\xi, \quad v_1 < \varsigma,$$

and

$${}^\Psi T_{v_2^-}^{\eta, \varsigma} U(\varsigma) = \frac{1}{\varsigma^\eta \Gamma(\eta)} \int_{v_1}^\varsigma \frac{\exp\left[\frac{\varsigma-1}{\varsigma}(\Psi(\xi) - \Psi(\varsigma))\right]}{(\Psi(\xi) - \Psi(\varsigma))^{1-\eta}} U(\xi) \Psi'(\xi) d\xi, \quad \varsigma < v_2,$$

where the proportionality index $\varsigma \in (0, 1]$, $\eta \in \mathbb{C}$, $\Re(\eta) > 0$, and Γ is the gamma function.

The functional space on which we develop our work is the following.

Definition 1.8. Let $h \in L_1[0, +\infty)$ and F be a continuous and positive function on $[0, +\infty)$ with $F(0) = 0$. The space $X_F^q(0, +\infty)$ ($1 \leq q < +\infty$) consists of those real-valued Lebesgue measurable functions h on $[0, +\infty)$ for which

$$\|h\|_{X_F^q} = \left(\int_{a_1}^{a_2} |h(s)|^q F(s) ds \right)^{\frac{1}{q}} < +\infty, \quad 1 \leq q < +\infty,$$

and for the case $q = +\infty$

$$\|h\|_{X_F^\infty} = \operatorname{ess\,sup}_{0 \leq s < \infty} [F(s)h(s)].$$

We are now in a position to define the generalized integral operators that we use in our work (see [30]).

Definition 1.9. Let $h \in X_F^q(0, +\infty)$ and F be a continuous and positive function on $[0, +\infty)$ with $F(0) = 0$. The right and left side generalized k -proportional fractional integral operators with general kernel of order γ of h are defined, respectively, by

$$J_{F, a_1^+}^{\gamma, \lambda} h(\chi) = \frac{1}{\lambda^\gamma k \Gamma_k(\gamma)} \int_{a_1}^\chi \frac{G(\mathbb{F}_+(\chi, s), \lambda) F(s) h(s)}{(\mathbb{F}_+(\chi, s))^{1-\frac{\gamma}{k}}} ds, \tag{1}$$

and

$$J_{F, a_2^-}^{\gamma, \lambda} h(\chi) = \frac{1}{\lambda^\gamma k \Gamma_k(\gamma)} \int_\chi^{a_2} \frac{G(\mathbb{F}_-(s, \chi), \lambda) F(s) h(s)}{(\mathbb{F}_-(s, \chi))^{1-\frac{\gamma}{k}}} ds, \tag{2}$$

where the proportionality index $\lambda \in (0, 1)$, $\gamma \in \mathbb{C}$, $\Re(\gamma) > 0$, $\chi \in (a_1, a_2)$,

$$\mathbb{F}_+(\chi, s) = \int_s^\chi F(r) dr,$$

$$\mathbb{F}_-(s, \chi) = \int_\chi^s F(r) dr,$$

and

$$G(\mathbb{F}_+(\chi, s), 1) = G(\mathbb{F}_-(\chi, s), 1) = 1.$$

Of course there are other integral fractional operators and their variations can be considered, however we do not discuss them here.

Remark 1.1. Next, we will show how many integral operators are particular cases of (1) and (2).

1. If in Definition 1.9 we take $k = 1$, $F = 1$, and $\lambda = 1$, we obtain the Riemann-Liouville operators given in Definition 1.1.
2. Under the above conditions, if $k \neq 1$ then, from Definition 1.9, the k -fractional operators defined in [28] are obtained.

3. If $F(s) = \frac{1}{s}$, $\lambda = 1$ and $k = 1$, then the Hadamard fractional operator is reproduced, see Definition 1.2, and [19, 23, 39].
4. If $F(s) = \frac{1}{s^\rho}$, $\lambda = 1$, and $k = 1$, then we obtain the Katugampola’s fractional operator given in Definition 1.3, see [22].
5. The Choices $\lambda = 1$, $F(s) = g'(s)$, and $k = 1$, lead us to the integral operator defined in [24].
6. If we take $F(s) = \frac{1}{s}$, $k \neq 1$ and

$$G(\mathbb{F}_+(\chi, s), \lambda) = \exp \left[\frac{\lambda - 1}{\lambda} \left(\ln \frac{\chi}{s} \right) \right],$$

then we obtain the integral operator defined in [35].

7. We can obtain an integral operator with a non-singular nucleus, of the Riemann-Liouville type, by putting $\gamma = k = 1$, $F(t) = 1$, and

$$G(\mathbb{F}_+(x, s), \alpha) = \exp \left[-\frac{1 - \alpha}{\alpha} (x - s) \right],$$

that is a slight modification of the operator defined in [3].

8. If we Choose $\lambda \neq 1$, $F(s) = g'(s)$, $k = 1$ and

$$G(\mathbb{F}_+(\chi, s), \lambda) = \exp \left[\frac{\lambda - 1}{\lambda} (g(\chi) - g(s)) \right],$$

then we obtain the integral operator defined in [38], known as GFP and is given in Definition 1.7.

2. Results

One of the best known integral inequalities is Chebyshev’s inequality (see [11]), which establishes relationships between the integral of the product of two functions and the product of their integrals. This inequality is stated in the framework of the classical Riemann integral:

$$\frac{1}{b - a} \int_a^b f(x)g(x)dx \geq \left(\frac{1}{b - a} \int_a^b f(x)dx \right) \left(\frac{1}{b - a} \int_a^b g(x)dx \right), \tag{3}$$

where f and g are two integrable and synchronous functions on $[a, b]$, $a < b$, $a, b \in \mathbb{R}$. Inequality (3) has many applications in diverse research subjects such as numerical quadrature, transform theory, probability, existence of solutions of differential equations and statistical problems. Many authors have investigated generalizations of the Chebyshev’s inequality (3), which are called Chebyshev type inequalities (for example, see [2, 9, 12, 14, 20, 27, 29, 32, 34, 41–43, 45]). This inequality is generalized also in the present paper by using the generalized operator of Definition 1.9, which contain many inequalities reported in the literature as particular cases.

Definition 2.1. Two functions f and g are said to synchronous (asynchronous) on $[0, \infty)$, if

$$((f(u) - f(v))(g(u) - g(v))) \geq (\leq) 0,$$

for all $u, v \in [0, +\infty)$.

In Chebyshev’s work cited above, the following functional is presented, which has been the subject of researches:

$$T(f, g) = \frac{1}{b - a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b - a} \int_a^b f(x)dx \right) \left(\frac{1}{b - a} \int_a^b g(x)dx \right), \tag{4}$$

where f and g are two integrable functions which are synchronous on $[a, b]$.

In our work, we consider the following generalization of the functional $T(f, g)$ (see (4)):

$$C_\lambda^{\frac{\gamma}{k}}(f, g) = J_{F, a_1+}^{\frac{\gamma}{k}, \lambda} (fg)(\chi) - \left(\frac{J_{F, a_1+}^{\frac{\gamma}{k}, \lambda} f(\chi) J_{F, a_1+}^{\frac{\gamma}{k}, \lambda} g(\chi)}{J_{F, a_1+}^{\frac{\gamma}{k}, \lambda} (1)(\chi)} \right),$$

with

$$J_{F, a_1+}^{\frac{\gamma}{k}, \lambda} (1)(\chi) = \frac{1}{\lambda^\gamma k \Gamma_k(\gamma)} \int_{a_1}^\chi \frac{G(\mathbb{F}_+(\chi, u), \lambda) F(u)(1) du}{(\mathbb{F}_+(\chi, u))^{1 - \frac{\gamma}{k}}}.$$

Our first result is given in the form of following theorem.

Theorem 2.1. *Let f and g be two synchronous on $[0, \infty)$. Then for all $\chi > a_1 \geq 0, \alpha > 0, \lambda \in (0, 1)$, and $\gamma \in \mathbb{C}$ with $Re(\gamma) > 0$, it holds that*

$$C_{\lambda}^{\frac{\gamma}{k}}(f, g) \geq 0.$$

Proof. As f and g are synchronous functions on $[0, \infty)$, it follows that

$$(f(u) - f(v))(g(u) - g(v)) \geq 0,$$

for all $u, v \geq 0$. Therefore,

$$f(u)g(u) + f(v)g(v) \geq f(u)g(v) + f(v)g(u). \tag{5}$$

Multiplying both sides of (5) by

$$\frac{1}{\lambda^{\gamma} k \Gamma_k(\gamma)} \frac{G(\mathbb{F}_+(\chi, u), \lambda) F(u)}{(\mathbb{F}_+(\chi, u))^{1-\frac{\gamma}{k}}},$$

and integrating the result with respect to u over (a_1, χ) , we get

$$\begin{aligned} & \frac{1}{\lambda^{\gamma} k \Gamma_k(\gamma)} \int_{a_1}^{\chi} \frac{G(\mathbb{F}_+(\chi, u), \lambda) F(u) f(u) g(u) du}{(\mathbb{F}_+(\chi, u))^{1-\frac{\gamma}{k}}} + \frac{f(v)g(v)}{\lambda^{\gamma} k \Gamma_k(\gamma)} \int_{a_1}^{\chi} \frac{G(\mathbb{F}_+(\chi, u), \lambda) F(u) du}{(\mathbb{F}_+(\chi, u))^{1-\frac{\gamma}{k}}} \\ & \geq \frac{g(v)}{\lambda^{\gamma} k \Gamma_k(\gamma)} \int_{a_1}^{\chi} \frac{G(\mathbb{F}_+(\chi, u), \lambda) f(u) F(u) du}{(\mathbb{F}_+(\chi, u))^{1-\frac{\gamma}{k}}} + \frac{f(v)}{\lambda^{\gamma} k \Gamma_k(\gamma)} \int_{a_1}^{\chi} \frac{G(\mathbb{F}_+(\chi, u), \lambda) g(u) F(u) du}{(\mathbb{F}_+(\chi, u))^{1-\frac{\gamma}{k}}}. \end{aligned}$$

We can write

$$J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(fg)(\chi) + f(v)g(v)J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(1)(\chi) \geq g(v)J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(f)(\chi) + f(v)J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(g)(\chi). \tag{6}$$

Multiplying both sides of (6) by

$$\frac{1}{\lambda^{\gamma} k \Gamma_k(\gamma)} \frac{G(\mathbb{F}_+(\chi, v), \lambda) F(v)}{(\mathbb{F}_+(\chi, v))^{1-\frac{\gamma}{k}}},$$

then integrating the resulting inequality with respect to v over (a_1, χ) , we obtain

$$\begin{aligned} & J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(1)(\chi)J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(fg)(\chi) + J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(fg)(\chi)J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(1)(\chi) \\ & \geq J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(g)(\chi)J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(f)(\chi) + J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(f)(\chi)J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(g)(\chi) \end{aligned} \tag{7}$$

that is,

$$J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(1)(\chi)J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(fg)(\chi) \geq J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(g)(\chi)J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(f)(\chi).$$

Reordering it and taking into account the definition of $C_{\lambda}^{\frac{\gamma}{k}}(f, g)$, the desired inequality is obtained. This completes the proof. \square

Remark 2.1. *If we take $k = 1, a_1 = 1$ and $F(s) = \frac{1}{s}$ in Theorem 2.1 then we get the Hadamard Integral and the above result cover Theorem 3.1 of [12].*

Theorem 2.2. *Let f and g be two synchronous functions on $[0, \infty)$. Then for all $\chi > a_1 \geq 0, \lambda \in (0, 1)$ and $\gamma, \delta \in \mathbb{C}$ with $Re(\gamma) > 0, Re(\delta) > 0$ it holds that*

$$C_{\lambda}^{\frac{\gamma}{k}, \frac{\delta}{k}}(f, g) \geq 0,$$

where

$$C_{\lambda}^{\frac{\gamma}{k}, \frac{\delta}{k}}(f, g) = \frac{J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(fg)(\chi)}{J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(1)(\chi)} + \frac{J_{F, a_1+}^{\frac{\delta}{k}, \lambda}(fg)(\chi)}{J_{F, a_1+}^{\frac{\delta}{k}, \lambda}(1)(\chi)} - \frac{J_{F, a_1+}^{\frac{\delta}{k}, \lambda}(g)(\chi)J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(f)(\chi) + J_{F, a_1+}^{\frac{\delta}{k}, \lambda}(f)(\chi)J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(g)(\chi)}{J_{F, a_1+}^{\frac{\delta}{k}, \lambda}(1)(\chi)J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(1)(\chi)}$$

is the generalization of the Functional $T(f, g)$ for γ and δ .

Proof. Multiplying both sides of (6) by

$$\frac{1}{\lambda^{\delta} k \Gamma_k(\delta)} \frac{G(\mathbb{F}_+(\chi, v), \lambda) F(v)}{(\mathbb{F}_+(\chi, v))^{1-\frac{\delta}{k}}},$$

then integrating the resulting inequality with respect to v over (a_1, χ) , we obtain

$$\begin{aligned} & J_{F, a_1+}^{\frac{\delta}{k}, \lambda}(1)(\chi)J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(fg)(\chi) + J_{F, a_1+}^{\frac{\delta}{k}, \lambda}(fg)(\chi)J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(1)(\chi) \\ & \geq J_{F, a_1+}^{\frac{\delta}{k}, \lambda}(g)(\chi)J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(f)(\chi) + J_{F, a_1+}^{\frac{\delta}{k}, \lambda}(f)(\chi)J_{F, a_1+}^{\frac{\gamma}{k}, \lambda}(g)(\chi). \end{aligned} \tag{8}$$

Dividing both sides of (8) by

$$J_{F,a_1+}^{\frac{\delta}{k},\lambda}(1)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi),$$

we get

$$\frac{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi)}{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi)} + \frac{J_{F,a_1+}^{\frac{\delta}{k},\lambda}(fg)(\chi)}{J_{F,a_1+}^{\frac{\delta}{k},\lambda}(1)(\chi)} \geq \frac{J_{F,a_1+}^{\frac{\delta}{k},\lambda}(g)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f)(\chi) + J_{F,a_1+}^{\frac{\delta}{k},\lambda}(f)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(g)(\chi)}{J_{F,a_1+}^{\frac{\delta}{k},\lambda}(1)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi)}.$$

Reordering it and taking into account the definition of $C_{\lambda}^{\frac{\gamma}{k},\frac{\delta}{k}}(f, g)$, the desired inequality is obtained. This completes the proof. □

Remark 2.2. Note that

1. The setting $\gamma = \beta$ in Theorem 2.2 yields Theorem 2.1.
2. If we take $k = 1$ and $\gamma = \delta = 1$ in Theorem 2.2, then

$$C_{\lambda}^{\frac{\gamma}{k},\frac{\delta}{k}}(f, g) = T(f, g).$$

Theorem 2.3. Let $(f_i)_{i=1,2,\dots,n}$ be positive increasing function on $[0, \infty)$. Then for all $\chi > a_1 \geq 0, \lambda \in (0, 1)$ and $\gamma \in \mathbb{C}$ with $Re(\gamma) > 0$, the following inequality holds

$$J_{F,a_1+}^{\frac{\gamma}{k},\lambda} \left(\prod_{i=1}^n f_i \right) (\chi) \geq \left[J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi) \right]^{1-n} \prod_{i=1}^n J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f_i)(\chi).$$

Proof. We prove the result by induction on n . Clearly, for $n = 1$, it holds that

$$J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f_1)(\chi) \geq J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f_1)(\chi)$$

for all $\chi > a_1 \geq 0, \alpha > 0, \lambda \in (0, 1)$ and $\gamma \in \mathbb{C}$ with $Re(\gamma) > 0$.

For $n = 2$, applying Equation (7), we obtain

$$J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f_1 f_2)(\chi) \geq \left[J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi) \right]^{-1} J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f_1)(\chi) J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f_2)(\chi).$$

By induction hypothesis, we have

$$J_{F,a_1+}^{\frac{\gamma}{k},\lambda} \left(\prod_{i=1}^{n-1} f_i \right) (\chi) \geq \left[J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi) \right]^{2-n} \prod_{i=1}^{n-1} J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f_i)(\chi)$$

for all $\chi > a_1 \geq 0, \alpha > 0, \lambda \in (0, 1)$ and $\gamma \in \mathbb{C}$ with $Re(\gamma) > 0$.

Now, since $(f_i)_{i=1,2,\dots,n}$ are positive increasing functions on $[0, \infty)$, $\left(\prod_{i=1}^{n-1} f_i \right) (\chi)$ is also a positive increasing function on $[0, \infty)$. Therefore, by applying Theorem 2.1 to the functions $g = \prod_{i=1}^{n-1} f_i$ and $f = f_n$, we obtain

$$\begin{aligned} J_{F,a_1+}^{\frac{\gamma}{k},\lambda} \left(\prod_{i=1}^n f_i \right) (\chi) &\geq J_{F,a_1+}^{\frac{\gamma}{k},\lambda} \left(\prod_{i=1}^{n-1} f_i f_n \right) (\chi) J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(gf)(\chi) \\ &\geq \left[J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi) \right]^{-1} J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(g)(\chi) J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f)(\chi) \\ &\geq \left[J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi) \right]^{-1} J_{F,a_1+}^{\frac{\gamma}{k},\lambda} \left(\prod_{i=1}^{n-1} f_i \right) (\chi) J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f_n)(\chi) \\ &\geq \left[J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi) \right]^{-1} \left[J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi) \right]^{2-n} \prod_{i=1}^{n-1} J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f_i)(\chi) J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f_n)(\chi) \\ &\geq \left[J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi) \right]^{1-n} \prod_{i=1}^n J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f_i)(\chi). \end{aligned}$$

This completes the proof. □

The previous results can be extended if we consider a certain positive “weight” function h .

Theorem 2.4. *Let f and g be two synchronous functions on $[0, \infty)$ and $h \geq 0$. Then for all $\chi > a_1 \geq 0, \lambda > 0$ and $\gamma \in \mathbb{C}$ with $\text{Re}(\gamma) > 0$, the following inequality holds:*

$$J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fgh)(\chi) \geq J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(g)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fh)(\chi) + J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(gh)(\chi) - J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(h)(\chi).$$

Proof. Since $h \geq 0$ and the functions f and g are synchronous on $[0, \infty)$, we obtain

$$(f(u) - f(v))(g(u) - g(v))(h(u) + h(v)) \geq 0,$$

for all $u, v \geq 0$. Then,

$$f(u)g(u)h(u) + f(v)g(v)h(v) \geq f(u)g(v)h(u) + f(v)g(u)h(u) - f(v)g(v)h(u) - f(u)g(u)h(v) + f(u)g(v)h(v) + f(v)g(u)h(v). \tag{9}$$

Multiplying both sides of (9) by

$$\frac{1}{\lambda^\gamma k \Gamma_k(\gamma)} \frac{G(\mathbb{F}_+(\chi, u), \lambda)F(u)}{(\mathbb{F}_+(\chi, u))^{1-\frac{\gamma}{k}}},$$

then integrating the resulting inequality with respect to u over (a_1, χ) , we obtain

$$\begin{aligned} & \frac{1}{\lambda^\gamma k \Gamma_k(\gamma)} \int_{a_1}^\chi \frac{G(\mathbb{F}_+(\chi, u), \lambda)F(u)f(u)g(u)h(u)du}{(\mathbb{F}_+(\chi, u))^{1-\frac{\gamma}{k}}} + \frac{f(v)g(v)h(v)}{\lambda^\gamma k \Gamma_k(\gamma)} \int_{a_1}^\chi \frac{G(\mathbb{F}_+(\chi, u), \lambda)F(u)du}{(\mathbb{F}_+(\chi, u))^{1-\frac{\gamma}{k}}} \\ \geq & \frac{g(v)}{\lambda^\gamma k \Gamma_k(\gamma)} \int_{a_1}^\chi \frac{G(\mathbb{F}_+(\chi, u), \lambda)F(u)f(u)h(u)du}{(\mathbb{F}_+(\chi, u))^{1-\frac{\gamma}{k}}} + \frac{f(v)}{\lambda^\gamma k \Gamma_k(\gamma)} \int_{a_1}^\chi \frac{G(\mathbb{F}_+(\chi, u), \lambda)F(u)g(u)h(u)du}{(\mathbb{F}_+(\chi, u))^{1-\frac{\gamma}{k}}} \\ & - \frac{f(v)g(v)}{\lambda^\gamma k \Gamma_k(\gamma)} \int_{a_1}^\chi \frac{G(\mathbb{F}_+(\chi, u), \lambda)F(u)h(u)du}{(\mathbb{F}_+(\chi, u))^{1-\frac{\gamma}{k}}} - \frac{h(v)}{\lambda^\gamma k \Gamma_k(\gamma)} \int_{a_1}^\chi \frac{G(\mathbb{F}_+(\chi, u), \lambda)F(u)f(u)g(u)du}{(\mathbb{F}_+(\chi, u))^{1-\frac{\gamma}{k}}} \\ & + \frac{g(v)h(v)}{\lambda^\gamma k \Gamma_k(\gamma)} \int_{a_1}^\chi \frac{G(\mathbb{F}_+(\chi, u), \lambda)F(u)f(u)du}{(\mathbb{F}_+(\chi, u))^{1-\frac{\gamma}{k}}} + \frac{f(v)h(v)}{\lambda^\gamma k \Gamma_k(\gamma)} \int_{a_1}^\chi \frac{G(\mathbb{F}_+(\chi, u), \lambda)F(u)g(u)du}{(\mathbb{F}_+(\chi, u))^{1-\frac{\gamma}{k}}}. \end{aligned}$$

We can write as

$$\begin{aligned} J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fgh)(\chi) + f(v)g(v)h(v)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi) & \geq g(v)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fh)(\chi) + f(v)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(gh)(\chi) \\ & - f(v)g(v)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(h)(\chi) - h(v)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi) \\ & + g(v)h(v)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f)(\chi) \\ & + f(v)h(v)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(g)(\chi). \end{aligned} \tag{10}$$

Multiplying both sides of (10) by

$$\frac{1}{\lambda^\gamma k \Gamma_k(\gamma)} \frac{G(\mathbb{F}_+(\chi, v), \lambda)F(v)}{(\mathbb{F}_+(\chi, v))^{1-\frac{\gamma}{k}}},$$

then integrating the resulting inequality with respect to v over (a_1, χ) , we obtain

$$\begin{aligned} J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fgh)(\chi) + J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fgh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi) & \geq J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(g)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fh)(\chi) \\ & + J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(gh)(\chi) - J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(h)(\chi) \\ & - J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(h)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi) + J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(gh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f)(\chi) \\ & + J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(g)(\chi). \end{aligned}$$

That is,

$$J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fgh)(\chi) \geq J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(g)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fh)(\chi) + J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(gh)(\chi) - J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(h)(\chi).$$

□

Remark 2.3. If we take $h = 1$ in Theorem 2.4, we obtain Theorem 2.1.

Theorem 2.5. Let f and g be two synchronous functions on $[0, \infty)$, $h \geq 0$. Then for all $\chi > a_1 \geq 0$, $\lambda \in (0, 1)$, $\gamma \in \mathbb{C}$ with $Re(\gamma) > 0$ and $\delta \in \mathbb{C}$ with $Re(\delta) > 0$, the following inequality holds:

$$\begin{aligned} & J_{F,a_1+}^{\frac{\delta}{k},\lambda}(h)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fgh)(\chi) + J_{F,a_1+}^{\frac{\delta}{k},\lambda}(fgh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(h)(\chi) \\ & \geq J_{F,a_1+}^{\frac{\delta}{k},\lambda}(gh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fh)(\chi) + J_{F,a_1+}^{\frac{\delta}{k},\lambda}(fh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(gh)(\chi). \end{aligned}$$

Proof. Multiplying both sides of (10) by

$$\frac{1}{\lambda^\delta k \Gamma_k(\delta)} \frac{G(\mathbb{F}_+(\chi, v), \lambda)F(v)}{(\mathbb{F}_+(\chi, v))^{1-\frac{\delta}{k}}},$$

then integrating the resulting inequality with respect to v over (a_1, χ) , we obtain

$$\begin{aligned} & J_{F,a_1+}^{\frac{\delta}{k},\lambda}(1)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fgh)(\chi) + J_{F,a_1+}^{\frac{\delta}{k},\lambda}(fgh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi) \geq J_{F,a_1+}^{\frac{\delta}{k},\lambda}(g)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fh)(\chi) \\ & + J_{F,a_1+}^{\frac{\delta}{k},\lambda}(f)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(gh)(\chi) - J_{F,a_1+}^{\frac{\delta}{k},\lambda}(fg)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(h)(\chi) \\ & - J_{F,a_1+}^{\frac{\delta}{k},\lambda}(h)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi) + J_{F,a_1+}^{\frac{\delta}{k},\lambda}(gh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f)(\chi) \\ & + J_{F,a_1+}^{\frac{\delta}{k},\lambda}(fh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(g)(\chi). \end{aligned}$$

□

Remark 2.4. For $h = 1$, Theorem 2.5 gives Theorem 2.2.

More precise results can be obtained, if in the previous Theorem we impose additional conditions on the function h .

Theorem 2.6. Let f, g and h be three monotonic functions defined on $[0, \infty)$ satisfying the following inequality

$$(f(u) - f(v))(g(u) - g(v))(h(u) - h(v)) \geq 0$$

for all $u, v \in [a_1, \chi]$. Then, for all $\chi > a_1 \geq 0$, $\lambda \in (0, 1)$, $\gamma \in \mathbb{C}$ with $Re(\gamma) > 0$ and $\delta \in \mathbb{C}$ with $Re(\delta) > 0$, it holds that

$$\begin{aligned} & J_{F,a_1+}^{\frac{\delta}{k},\lambda}(1)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fgh)(\chi) + J_{F,a_1+}^{\frac{\delta}{k},\lambda}(fgh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi) \geq J_{F,a_1+}^{\frac{\delta}{k},\lambda}(g)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fh)(\chi) \\ & + J_{F,a_1+}^{\frac{\delta}{k},\lambda}(f)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(gh)(\chi) - J_{F,a_1+}^{\frac{\delta}{k},\lambda}(fg)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(h)(\chi) \\ & - J_{F,a_1+}^{\frac{\delta}{k},\lambda}(h)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi) + J_{F,a_1+}^{\frac{\delta}{k},\lambda}(gh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f)(\chi) \\ & + J_{F,a_1+}^{\frac{\delta}{k},\lambda}(fh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(g)(\chi). \end{aligned}$$

Proof. As in the proof of Theorem 2.5, if we multiply both sides of (10) by

$$\frac{1}{\lambda^\delta k \Gamma_k(\delta)} \frac{G(\mathbb{F}_+(\chi, v), \lambda)F(v)}{(\mathbb{F}_+(\chi, v))^{1-\frac{\delta}{k}}},$$

then integrate the resulting inequality with respect to v over (a_1, χ) , we obtain the desired inequality. □

An inequality involving the square of the functions f and g can be stated as follows.

Theorem 2.7. Let f and g be defined on $[0, \infty)$, then for all $\chi > a_1 \geq 0$, $\lambda \in (0, 1)$ and $\gamma \in \mathbb{C}$ with $Re(\gamma) > 0$, the following inequalities hold

$$J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)[J_{F,a_1+}^{\frac{\gamma}{k},\lambda}f^2(\chi) + J_{F,a_1+}^{\frac{\gamma}{k},\lambda}g^2(\chi)] \geq 2J_{F,a_1+}^{\frac{\gamma}{k},\lambda}f(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}g(\chi) \tag{11}$$

and

$$J_{F,a_1+}^{\frac{\gamma}{k},\lambda}f^2(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}g^2(\chi) \geq [J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi)]^2. \tag{12}$$

Proof. Since

$$(f(u) - g(v))^2 \geq 0,$$

we have

$$f^2(u) + g^2(v) \geq 2f(u)g(v). \tag{13}$$

Multiplying both sides of (13) by

$$\frac{1}{\lambda^\gamma k \Gamma_k(\gamma)} \frac{G(\mathbb{F}_+(\chi, u), \lambda) F(u)}{(\mathbb{F}_+(\chi, u))^{1-\frac{\gamma}{k}}},$$

then integrating the resulting inequality with respect to u and v over (a_1, χ) , we obtain (11).

On the other hand, since

$$(f(u)g(v) - f(v)g(u))^2 \geq 0,$$

then by using the same arguments as before, we have (12). □

Theorem 2.8. *Let f and g be defined on $[0, \infty)$. Then for all $\chi > a_1 \geq 0, \lambda \in (0, 1), \gamma \in \mathbb{C}$ with $Re(\gamma) > 0$ and $\delta \in \mathbb{C}$ with $Re(\delta) > 0$, the following inequalities hold*

$$J_{F, a_1^+}^{\frac{\gamma}{k}, \lambda} f^2(\chi) J_{F, a_1^+}^{\frac{\delta}{k}, \lambda} (1) + J_{F, a_1^+}^{\frac{\gamma}{k}, \lambda} (1) J_{F, a_1^+}^{\frac{\delta}{k}, \lambda} g^2(\chi) \geq 2 J_{F, a_1^+}^{\frac{\gamma}{k}, \lambda} f(\chi) J_{F, a_1^+}^{\frac{\delta}{k}, \lambda} g(\chi) \tag{14}$$

and

$$J_{F, a_1^+}^{\frac{\gamma}{k}, \lambda} f^2(\chi) J_{F, a_1^+}^{\frac{\delta}{k}, \lambda} g^2(\chi) + J_{F, a_1^+}^{\frac{\delta}{k}, \lambda} f^2(\chi) J_{F, a_1^+}^{\frac{\gamma}{k}, \lambda} g^2(\chi) \geq 2 J_{F, a_1^+}^{\frac{\gamma}{k}, \lambda} (fg)(\chi) J_{F, a_1^+}^{\frac{\delta}{k}, \lambda} (fg)(\chi). \tag{15}$$

Proof. Like before, since $(f(u) - g(v))^2 \geq 0$, we have

$$f^2(u) + g^2(v) \geq 2f(u)g(v). \tag{16}$$

Multiplying both sides of (16) by

$$\frac{1}{\lambda^\gamma k \Gamma_k(\gamma)} \frac{G(\mathbb{F}_+(\chi, u), \lambda) F(u)}{(\mathbb{F}_+(\chi, u))^{1-\frac{\gamma}{k}}}$$

and

$$\frac{1}{\lambda^\delta k \Gamma_k(\delta)} \frac{G(\mathbb{F}_+(\chi, v), \lambda) F(v)}{(\mathbb{F}_+(\chi, v))^{1-\frac{\delta}{k}}},$$

then integrating the resulting inequality with respect to u and v over (a_1, χ) respectively, we obtain (14).

On the other hand, since

$$(f(u)g(v) - f(v)g(u))^2 \geq 0,$$

then by using the same arguments as before, we have (15). □

Remark 2.5. *If we take $\gamma = \delta$ we obtain Theorem 2.7.*

Theorem 2.9. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ with*

$$\bar{f}(u) = \int_{a_1}^u F(z, s) f(z) dz,$$

$u > a_1 \geq 0, \lambda \in (0, 1)$ and $\gamma \in \mathbb{C}$ with $Re(\gamma) > 0$. Then, for $\gamma \geq k > 0$ it holds that

$$J_{F, a_1^+}^{\frac{\gamma+k}{k}, \lambda} f(\chi) = \frac{1}{k} J_{F, a_1^+}^{\frac{\gamma}{k}, \lambda} \bar{f}(\chi).$$

Proof. Here

$$\begin{aligned} J_{F, a_1^+}^{\frac{\gamma}{k}, \lambda} \bar{f}(\chi) &= \frac{1}{\lambda^\gamma k \Gamma_k(\gamma)} \int_{a_1}^\chi \frac{G(\mathbb{F}_+(\chi, u), \lambda) F(u) \bar{f}(u)}{(\mathbb{F}_+(\chi, u))^{1-\frac{\gamma}{k}}} du \\ &= \frac{1}{\lambda^\gamma k \Gamma_k(\gamma)} \int_{a_1}^\chi \frac{G(\mathbb{F}_+(\chi, u), \lambda) F(u)}{(\mathbb{F}_+(\chi, u))^{1-\frac{\gamma}{k}}} \int_{a_1}^u F(z, s) f(z) dz du. \end{aligned}$$

Then, by Dirichlet’s formula, the last expression becomes

$$\begin{aligned} &\frac{1}{\lambda^\gamma k \Gamma_k(\gamma)} \int_{a_1}^\chi F(z, s) f(z) \int_z^\chi G(\mathbb{F}_+(\chi, u), \lambda) F(u) (\mathbb{F}_+(\chi, u))^{\frac{\gamma}{k}-1} du dz \\ &= \frac{1}{\lambda^\gamma \Gamma_k(\gamma + k)} \int_{a_1}^\chi \frac{G(\mathbb{F}_+(\chi, u), \lambda) F(u) f(u)}{(\mathbb{F}_+(\chi, u))^{1-\frac{\gamma+k}{k}}} \\ &= k J_{F, a_1^+}^{\frac{\gamma+k}{k}, \lambda} f(u). \end{aligned}$$

This complete the proof. □

3. Conclusion

In this paper, we present a generalized formulation of the Riemann-Liouville fractional integral, which contains as particular cases many of the integral operators reported in the literature. We present several integral inequalities that generalize several known inequalities.

We highlight the strength of Definition 1.9 by pointing out the following fact. If we consider the kernel

$$F(\chi, s) = \chi^{1-s}$$

and $G \equiv 1$, we obtain a variant of the (k, s) -Riemann-Liouville fractional integral defined in [40]:

$${}_s I_{a_1}^\alpha f(u) = \frac{(2-s)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_{a_1}^u (u^{2-s} - \chi^{2-s})^{\frac{\alpha}{k}-1} \chi^{1-s} f(\chi) d\chi.$$

This opens up wide possibilities of obtaining new integral inequalities.

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