Monochromatic subgraphs in graphs

Gary Chartrand, Emma Jent, Ping Zhang

Department of Mathematics, Western Michigan University, Kalamazoo, Michigan 49008-5248, USA

(Received: 10 June 2024. Accepted: 2 July 2024. Published online: 3 July 2024.)

© 2024 the authors. This is an open-access article under the CC BY (International 4.0) license (www.creativecommons.org/licenses/by/4.0/).

Abstract

For a positive integer \( t \) and a graph \( F \), the numbers \( ER_t(F) \) and \( VR_t(F) \) of \( F \) are the minimum positive integer \( n \) such that every red-blue coloring of the edges of the complete graph \( K_n \), results in \( t \) pairwise edge-disjoint and vertex-disjoint, respectively, monochromatic copies of \( F \) in \( K_n \). The number \( ER_t(F) \) is determined when \( F = K_3 \) for \( t \leq 4 \) and when \( F \) is the path \( P_3 \) of order 3 for every positive integer \( t \), while \( VR_t(F) \) is determined when \( F \in \{ K_3, P_3 \} \) for every positive integer \( t \).

Keywords: red-blue coloring; edge-disjoint and vertex-disjoint monochromatic graphs.

2020 Mathematics Subject Classification: 05C15, 05C35, 05C55.

1. Introduction

In a red-blue coloring of a graph \( G \), every edge of \( G \) is colored red or blue. For two graphs \( F \) and \( H \), the well-known Ramsey number \( R(F, H) \) is the minimum positive integer \( n \) such that for every red-blue coloring of the complete graph \( K_n \), of order \( n \), there is either a subgraph of \( K_n \) isomorphic to \( F \) all of whose edges are colored red (a red \( F \)) or a subgraph of \( K_n \) isomorphic to \( H \) all of whose edges are colored blue (a blue \( H \)). Therefore, for a single graph \( F \), the Ramsey number \( R(F, F) \), also denoted by \( R(F) \), is the minimum positive integer \( n \) such that for every red-blue coloring of \( K_n \), there is a subgraph of \( K_n \) isomorphic to \( F \) all of whose edges are colored the same (a monochromatic \( F \)). That these numbers exist for every graph \( F \) is due to Ramsey [3]. We refer to the book [1] for notation and terminology not defined here.

An introduction to Ramsey numbers in graph theory often begins with a question that is sometimes stated in the following manner.

\textbf{How many people must be present at a party to be guaranteed that there are three mutual acquaintances or three mutual strangers?}

It may already be clear that this question has a graph theory interpretation. For example, suppose that there are \( n \) people at the party. These \( n \) people are the \( n \) vertices of the complete graph \( K_n \). Two vertices are joined by a red edge if the two people are acquaintances and joined by a blue edge if they are strangers. The question then becomes the following.

\textbf{What is the smallest positive integer \( n \) such that if the \( \binom{n}{2} \) edges of \( K_n \) are colored red or blue in any manner whatsoever, we are guaranteed that a subgraph \( K_3 \) (a triangle) all of whose edges are colored red (a red triangle) or colored blue (a blue triangle) appears?}

In order to answer this question, it is useful to make the following observation.

\textbf{Observation 1.1.} Let there be given an arbitrary red-blue coloring of a complete graph \( K_n \) where \( n \geq 4 \). If a vertex of \( K_n \) is incident with three or more edges of the same color, then \( K_n \) contains a monochromatic triangle.

\textbf{Proof.} Suppose that a vertex \( v \) of \( K_n \) is incident with at least three edges of the same color, say \( vv_1, vv_2, vv_3 \) are red. If any two of \( v_1, v_2, v_3 \) are joined by a red edge, then there is a red triangle; otherwise, \((v_1, v_2, v_3, v_1)\) is a blue triangle. \qed

First, \( n = 5 \) does not work. By Observation 1.1, any red-blue coloring of \( K_5 \) in which some vertex is incident with at least three edges of the same color results in a monochromatic triangle. Therefore, the only possible red-blue coloring of \( K_5 \) without any monochromatic triangle is for every vertex to be incident with exactly two edges of each color. Consequently, we have the next observation.

\textbf{*Corresponding author (ping.zhang@wmich.edu).}
Observation 1.2. The only red-blue coloring of the complete graph $K_5$ for which there is no monochromatic triangle is one that produces a red cycle $C_5$ of order 5 and a blue $C_5$.

This red-blue coloring of $K_5$ is shown in Figure 1.1 where a bold edge represents a red edge and a thin edge represents a blue edge.

![Figure 1.1: A red-blue coloring of $K_5$.](image)

However, $n = 6$ does work. Every red-blue coloring of $K_6$ results in at least three edges incident with each vertex colored the same. By Observation 1.1, there is a red triangle or a blue triangle. Hence, the solution to the question above is the Ramsey number $R(K_3) = 6$, which appeared in [2]. This problem essentially appeared as Problem A2 in the 1953 Putnam Exam.

A2. The complete graph with 6 points and 15 edges has each edge colored red or blue. Show that we can find 3 points such that the 3 edges joining them are the same color.

Not only does every red-blue coloring of $K_6$ always produce a monochromatic triangle, it always produces at least two monochromatic triangles. To see this, let there be given a red-blue coloring of $G = K_6$ with red subgraph $G_r$ and blue subgraph $G_b$. First, suppose that there is a vertex $v$ that is incident with four edges of the same color, say $vv_1, vv_2, vv_3, vv_4$ are red. If there are two red edges in the subgraph $G[(v_1, v_2, v_3, v_4)]$ of $G$ induced by the set $\{v_1, v_2, v_3, v_4\}$, then there are two red triangles. If at most one edge, say $v_1v_2$, is colored red, then $\{v_1, v_2, v_3\}$ and $\{v_2, v_3, v_4\}$ are the vertices of two blue triangles. Next, suppose that $G_r$ is 3-regular. Then $G_r$ is either the Cartesian product $K_3 \square K_2$ of $K_3$ and $K_2$ or the complete bipartite graph $K_{3,3}$. If $G_r = K_3 \square K_2$, then $G_r$ has two triangles; while if $G_r = K_{3,3}$, then $G_b = 2K_3$ (the union of two vertex-disjoint copies of $K_3$) also contains two triangles.

Finally, suppose that neither $G_r$ nor $G_b$ contains a vertex of degree 4 or more or is 3-regular. Then each of these graphs must contain only vertices of degree 2 or 3, necessarily some of each. Since every graph contains an even number of odd vertices, one of these two graphs contains four vertices of degree 3 and two vertices of degree 2 while the other contains two vertices of degree 3 and four vertices of degree 2. Suppose that $G_b$ contains four vertices of degree 3 and two vertices $u$ and $v$ of degree 2. Thus, $G_b$ is one of graphs shown in Figures 1.2(a), (b), and (c).

![Figure 1.2: Three possible blue subgraphs $G_b$.](image)

If (a) occurs, then $u$ and $v$ are adjacent and $G_b$ has two triangles. If (b) occurs, then $(v, w, z, v)$ and $(u, x, y, u)$ are two red triangles. If (c) occurs, then $(x, y, z, x)$ is a blue triangle and $(u, x, y, u)$ is a red triangle. Therefore, we have the following.

In every red-blue coloring of the edges of $K_6$, there are always at least two monochromatic triangles.

2. Monochromatic triangles

The red-blue coloring of $K_6$ with red subgraph $K_{2,4}$ and blue subgraph $K_2 + K_4$ has four monochromatic triangles, all blue. However, every two blue triangles have an edge in common. When $n = 7$, however, every red-blue coloring of $K_7$ results in two edge-disjoint monochromatic triangles.
**Proposition 2.1.** Every red-blue coloring of $K_7$ results in two edge-disjoint monochromatic triangles.

**Proof.** Let there be given an arbitrary red-blue coloring of $G = K_7$. Since $R(K_3, K_3) = 6$, there is a monochromatic triangle $T$ in $G$. Let $V(T) = \{u_1, u_2, u_3\}$ and let $V(G) - V(T) = \{v_1, v_2, v_3, v_4\}$. Let $F_i = G[[u_1, v_1, v_2, v_3, v_4]] = K_5$ for $i = 1, 2, 3$. If any $F_i$ contains a monochromatic triangle, then $G$ contains two edge-disjoint (in fact, vertex-disjoint) monochromatic triangles. Thus, we may assume that no $F_i$ $(i = 1, 2, 3)$ contains a monochromatic triangle. It follows by Observation 1.2 that each $F_i$ consists of a red $C_5$ and a blue $C_5$ for $i = 1, 2, 3$. Assume, without loss of generality, that $(u_1, v_1, v_2, v_3, v_4)$ is a red $C_5$ in $F_1$ and $(u_1, v_2, v_4, v_1, v_3)$ is a blue $C_5$ in $F_1$. We now consider $F_i$ for $i = 2, 3$. Since

(a) $(v_1, v_2, v_3, v_4)$ is a red $P_4$ in $F_i$ and $(v_2, v_4, v_1, v_3)$ is a blue $P_4$ in $F_i$ and

(b) $F_i$ consists of a red $C_5$ and a blue $C_5$,

it follows that $u_1v_1, u_1v_4$ are red and $u_1v_2, u_1v_3$ are blue. If $T$ is a red triangle, then $(u_1, v_1, u_2, u_1)$ and $(u_2, v_4, u_3, u_2)$ are two edge-disjoint red triangles in $G$; while if $T$ is a blue triangle, then $(u_1, v_2, u_2, u_1)$ and $(u_2, v_3, u_3, u_2)$ are two edge-disjoint blue triangles in $G$. Therefore, $G$ contains two edge-disjoint monochromatic triangles.

Consequently, 6 is the smallest order $n$ of a complete graph $K_n$ for which every red-blue coloring of $K_n$ results in a monochromatic triangle (or two monochromatic triangles which may have an edge in common), while 7 is the smallest order $n$ of a complete graph $K_n$ for which every red-blue coloring of $K_n$ results in two edge-disjoint monochromatic triangles. These facts suggest the problem of determining the smallest positive integer $n$ such that every red-blue coloring of $K_n$ results in two vertex-disjoint monochromatic triangles.

The red-blue coloring of $K_7$ with red subgraph $K_5 + K_2$ and blue subgraph $K_{2,5}$ does not contain two vertex-disjoint monochromatic copies of $K_3$. When $n = 8$, however, every red-blue coloring of $K_8$ results in two vertex-disjoint monochromatic triangles.

**Proposition 2.2.** Every red-blue coloring of $K_8$ results in two vertex-disjoint monochromatic triangles.

**Proof.** Let there be given an arbitrary red-blue coloring of $G = K_8$ with vertex set $\{v_1, v_2, \ldots, v_8\}$. Since $R(K_3, K_3) = 6$, there is a monochromatic copy of $K_3$ in $G$. Let $T$ be a monochromatic triangle in $G$ where $V(T) = \{v_1, v_2, v_3\}$. Let $F = G[[v_4, v_5, v_6, v_7, v_8]] = K_5$. If $F$ contains a monochromatic triangle, then $G$ contains two vertex-disjoint monochromatic triangles. Suppose that $F$ does not contain a monochromatic triangle. Then $F$ consists of a red $C_5$ and a blue $C_5$. We may assume that the red $C_5$ is $(v_4, v_5, v_6, v_7, v_8)$ and the blue $C_5$ is $(v_4, v_5, v_6, v_7, v_8)$. Since $v_1$ is joined to at least three vertices of $V(F) = \{v_4, v_5, v_6, v_7, v_8\}$ by edges of the same color, we may assume that $v_1$ is adjacent to three vertices of $V(F)$ by red edges. Necessarily, $v_1$ is joined to two adjacent vertices of $V(F)$ that are joined by a red edge. Consequently, we may assume that $v_1v_4$ and $v_1v_5$ are red. See Figure 2.1.

Let $H = G[[v_2, v_3, v_6, v_7, v_8]] = K_5$. If $H$ contains a monochromatic triangle, then $G$ contains two vertex-disjoint monochromatic triangles. Suppose that $H$ does not contain a monochromatic triangle and so $H$ consists of a red subgraph $H_r = C_5$ and a blue subgraph $H_b = C_5$. First, suppose that $T$ is blue. Since $\deg_{H_r} v_7 = 2$, it follows that $v_2v_6, v_2v_8, v_3v_6, v_3v_8$ are red. However, then $\deg_{H_b} v_6 = \deg_{H_b} v_8 = 3$, a contradiction. We may assume that $T$ is red. We may further assume that $H_r = (v_2, v_3, v_6, v_7, v_6)$ and $H_b = (v_2, v_8, v_6, v_3, v_7, v_2)$. We consider two cases, according to whether $v_1v_6$ is red or $v_1v_6$ is blue.

**Case 1.** $v_1v_6$ is red. This implies that $v_2v_4$ is red for otherwise, there are two vertex-disjoint monochromatic triangles $(v_1, v_5, v_6, v_1)$ and $(v_2, v_4, v_7, v_2)$. Also, this implies that $v_3v_4$ is red for otherwise, there are two vertex-disjoint monochromatic triangles $(v_1, v_5, v_6, v_1)$ and $(v_3, v_4, v_8, v_3)$. However then, there are two vertex-disjoint monochromatic triangles $(v_1, v_5, v_6, v_1)$ and $(v_2, v_3, v_4, v_2)$.
Case 2. \(v_1v_6\) is blue. First, suppose that \(v_1v_6\) is red. So, \(v_2v_3\) is blue, for otherwise there are two vertex-disjoint monochromatic triangles \(\{v_1, v_4, v_8, v_1\}\) and \(\{v_2, v_5, v_6, v_2\}\). Then \(\{v_1, v_4, v_8, v_1\}\) and \(\{v_2, v_5, v_6, v_2\}\) are two vertex-disjoint monochromatic triangles. Next, suppose that \(v_1v_8\) is blue. If \(v_2v_3\) is blue, then there are two vertex-disjoint monochromatic triangles \(\{v_2, v_5, v_7, v_2\}\) and \(\{v_1, v_6, v_8, v_1\}\). Thus, we may assume that \(v_2v_5\) is red. If \(v_2v_4\) is blue, then there are two vertex-disjoint monochromatic triangles \(\{v_2, v_4, v_7, v_2\}\) and \(\{v_1, v_6, v_8, v_1\}\). Thus, we may assume that \(v_2v_4\) is red. Then there are two vertex-disjoint monochromatic triangles \(\{v_2, v_4, v_5, v_2\}\) and \(\{v_1, v_6, v_8, v_1\}\).

The results above concerning two edge-disjoint and vertex-disjoint monochromatic triangles in an edge-colored complete graph suggest more general concepts. Let \(t\) be a positive integer and let \(F\) be a graph without isolated vertices. The vertex-disjoint Ramsey number \(VR_t(F)\) is the minimum positive integer \(n\) such that every red-blue coloring of \(K_n\) results in at least \(t\) pairwise vertex-disjoint monochromatic copies of \(F\). Then \(VR_1(K_3) = R(K_3) = 6\) and \(VR_2(K_3) = 8\) by Proposition 2.2 and the red-blue coloring of \(K_7\) with red subgraph \(K_5 + K_2\) and blue subgraph \(K_2,5\). For a graph \(F\) without isolated vertices, \(VR_t(F) \geq t|V(F)|\) and the Ramsey number \(R(tF)\) exists where \(tF\) is a union of \(t\) vertex-disjoint copies of \(F\). Hence, we have the following observation.

**Observation 2.1.** For every graph \(F\) without isolated vertices and every positive integer \(t\), the number \(VR_t(F)\) exists and \(t|V(F)| \leq VR_t(F) \leq R(tF)\). Furthermore, \(VR_t(F) \leq VR_{t+1}(F)\).

**Theorem 2.1.** For an integer \(t \geq 2\), \(VR_t(K_3) = 3t + 2\).

**Proof.** We proceed by induction on \(t\). We saw that \(VR_2(K_3) = 8\) and so the result is true for \(t = 2\). Assume that \(VR_k(K_3) = 3k + 2\) for an integer \(k \geq 2\). We show that \(VR_{k+1}(K_3) = 3k + 5\). The red-blue coloring of \(K_{3k+4}\) with red subgraph \(K_{3k+2} + K_2\) and blue subgraph \(K_{2,3k+2}\) has \(k\) vertex-disjoint red triangles and no blue triangle. Thus, \(VR_{k+1}(K_3) \geq 3k + 5\). Next, let there be given a red-blue coloring of \(K_{3k+5}\). Since \(3k + 5 \geq 11 > 6\) and \(R(K_3) = 6\), there is a monochromatic triangle \(T\). Let \(H = K_{3k+5} - V(T) = K_{3k+2}\). Since \(VR_k(K_3) = 3k + 2\), it follows that \(H\) contains \(k\) vertex-disjoint monochromatic triangles. Hence, there are \(k + 1\) vertex-disjoint monochromatic triangles in \(K_{3k+5}\) and so \(VR_{k+1}(K_3) \leq 3k + 5\). Therefore, \(VR_{k+1}(K_3) = 3k + 5\).

Let \(t\) be a positive integer and let \(F\) be a graph without isolated vertices. The edge-disjoint Ramsey number \(ER_t(F)\) of \(F\) is the minimum positive integer \(n\) such that for every red-blue coloring of \(K_n\), there are at least \(t\) pairwise edge-disjoint monochromatic copies of \(F\). Hence, \(ER_1(F)\) is the Ramsey number \(R(F)\). Therefore, \(ER_1(K_3) = 6\) and \(ER_2(K_3) = 7\) by Proposition 2.1 and the red-blue coloring of \(K_6\) with red subgraph \(K_{2,4}\) and blue subgraph \(K_{2,4}\). There is an observation for edge-disjoint Ramsey numbers similar to Observation 2.1.

**Observation 2.2.** For every graph \(F\) without isolated vertices and every positive integer \(t\), the number \(ER_t(F)\) exists and \(ER_t(F) \leq VR_t(F)\). Furthermore, \(ER_t(F) \leq ER_{t+1}(F)\).

We now determine \(ER_t(K_3)\) for \(t = 3, 4\). First, we present a useful lemma.

**Lemma 2.1.** For each positive integer \(t\), \(ER_{t+1}(K_3) \leq ER_t(K_3) + 2\).

**Proof.** Let \(ER_t(K_3) = k\). Let there be given an arbitrary red-blue coloring of \(G = K_{k+2}\). Then there is a monochromatic copy \(F_0\) of \(K_3\) in \(G\). Let \(u, v \in V(F_0)\) and let \(H = G - \{u, v\}\). Then \(H = K_k\) contains no edge of \(F_0\). Since \(ER_t(K_3) = k\), there are \(t\) pairwise edge-disjoint monochromatic copies \(F_1, F_2, \ldots, F_t\) of \(K_3\) in \(H\) that are edge-disjoint from \(F_0\). Therefore, \(F_0, F_1, F_2, \ldots, F_t\) are \(t + 1\) pairwise edge-disjoint monochromatic copies of \(K_3\) in \(G\) and so \(ER_{t+1}(K_3) \leq k + 2 = ER_t(K_3) + 2\).

Both strict inequality and equality in Lemma 2.1 can occur, as we show next.

**Proposition 2.3.** \(ER_3(K_3) = 9\)

**Proof.** Since \(ER_3(K_3) \leq ER_2(K_3) + 2 = 9\) by Lemma 2.1 and Proposition 2.1, it remains to show that \(ER_3(K_3) \geq 9\). For the red-blue coloring of \(K_9\) with red subgraph \(K_{5,4}\) and blue graph \(2K_4\), there is no red triangle and only two edge-disjoint blue triangles. Since this red-blue coloring of \(K_9\) does not produce three pairwise edge-disjoint monochromatic triangles, it follows that \(ER_3(K_3) \geq 9\) and so \(ER_3(K_3) = 9\).

**Theorem 2.2.** \(ER_4(K_3) = 10\).

**Proof.** The red-blue coloring of \(K_9\) with red subgraph \(K_{5,4}\) and blue subgraph \(K_{5} + K_4\) contains no red triangle and three edge-disjoint blue triangles. Since this red-blue coloring of \(K_9\) does not produce four pairwise edge-disjoint monochromatic triangles, it follows that \(ER_4(K_3) \geq 10\).
It remains to show that $ER_4(K_3) \leq 10$. First, we verify the following claim.

**Claim:** For every red-blue coloring of $G = K_{10}$ containing two edge-disjoint monochromatic triangles $T$ and $T'$ having a vertex in common, there are four pairwise edge-disjoint monochromatic triangles in $G$.

To verify the claim, consider a red-blue coloring of $G$ with two edge-disjoint monochromatic triangles $T = (v_1, v_2, v_3, v_4)$ and $T' = (v_1, v_4, v_5, v_6)$. Let $H = G[V(G) - \{v_1, v_2, v_3\}] = K_7$. Since $ER_4(K_3) = 7$ by Proposition 2.1, there are two edge-disjoint monochromatic triangles $T_1$ and $T_2$ in $H$ edge-disjoint from $T$ and $T'$. Thus, $T, T', T_1, T_2$ are four pairwise edge-disjoint monochromatic triangles in $G$ and so the claim is true.

Next, we show that every red-blue coloring of $G = K_{10}$ produces four pairwise edge-disjoint monochromatic triangles in $G$. By the claim, we may assume that $T_1, T_2, T_3$ are pairwise vertex-disjoint. Hence, the subgraph $F$ of $G$ induced by $E(T_1) \cup E(T_2) \cup E(T_3)$ is $3K_3$. Let $T_1 = (v_1, v_2, v_3, v_4) \cup (v_4, v_5, v_6, v_7)$, $T_2 = (v_1, v_5, v_6, v_4)$, $T_3 = (v_7, v_8, v_9, v_7)$, and let $v$ be the vertex of $G$ not in $F$. Furthermore, let $H = G - E(F)$ be the spanning subgraph of $G$ whose edge set consists of all edges not belonging to any of $T_1, T_2, T_3$. Then $H \cong K_{1,3,3,3}$ and $\deg_H v = 9$. Let $r$ be the number of red edges incident with $v$ and $b$ the number of blue edges incident with $v$. Then $r + b = 9$. We may assume that $b \leq r$ and so $5 \leq r \leq 9$.

First, suppose that there is a red edge joining $v$ to at least one vertex in each of $T_1, T_2, T_3$. We may assume without loss of generality that $v_1, v_2, v_4, v_7$ are red. If there is a red edge joining two vertices in $\{v_1, v_4, v_7\}$, say $v_1v_4$ is red, then $(v_1, v_1, v_4, v_7)$ is a red triangle edge-disjoint from $T_1, T_2, T_3$; while if every two vertices in $\{v_1, v_4, v_7\}$ is joined by a blue edge, then $(v_1, v_4, v_7, v_1)$ is a blue triangle edge-disjoint from $T_1, T_2, T_3$. Thus, we may assume that $v$ is joined to exactly two of $T_1, T_2, T_3$ by red edges. Hence, $r = 5, 6$ and we may further assume that $v_1v_4$ is red for $i = 1, 2, 3, 4, 5$ and $v_1v_6$ is either red or blue.

Let $H' = K_{2,3}$ be the complete bipartite subgraph of $H$ with partite sets $\{v_1, v_2, v_3\}$ and $\{v_4, v_5\}$. If $H'$ contains a red edge, say $v_1v_4$ is red, then $(v, v_1, v_4, v)$ is a red triangle edge-disjoint from $T_1, T_2, T_3$. Thus, we may assume that $H'$ is a blue $K_{2,3}$. If $T_1$ is blue, then $(v_1, v_4, v_2, v_1)$ and $(v_1, v_3, v_5, v_1)$ are two edge-disjoint blue triangles having the vertex $v_1$ in common. It then follows by the claim that $G$ contains four pairwise edge-disjoint monochromatic triangles. Thus, we may assume that $T_1$ is red. If $T_2$ is red, then $(v_1, v, v_2, v_1)$ and $(v_4, v, v_5, v_4)$ are two edge-disjoint red triangles having the vertex $v$ in common; while if $T_2$ is blue, then $(v_1, v_2, v_4, v_1)$ and $(v_4, v_5, v_1)$ are two edge-disjoint monochromatic triangles having the vertex $v_1$ in common. Again, by the claim, $G$ contains four pairwise edge-disjoint monochromatic triangles. Therefore, $ER_4(K_3) \leq 10$ and so $ER_4(K_3) = 10$.

We close this section with the following conjecture.

**Conjecture 2.1.** For every integer $t \geq 4$, $ER_t(K_3) \leq ER_{t+1}(K_3) \leq ER_t(K_3) + 1$.

### 3. Monochromatic paths of order 3

We now turn our attention to the other connected graph of order 3, namely the path $P_3$ of order 3. Of course, $R_1(P_3) = VR_1(P_3) = ER_1(P_3) = 3$.

First, we determine $VR_t(P_3)$ for every positive integer $t$.

**Theorem 3.1.** For every positive integer $t$, $VR_t(P_3) = 3t$.

**Proof.** Since $VR_2(P_3) \geq 3t$ by Observation 2.1, it remains to show that $VR_t(P_3) \leq 3t$. We proceed by induction on $t$. Since $VR_1(P_3) = R(P_3) = 3$, the result is true for $t = 1$. Assume that $VR_k(P_3) \leq 3k$ for a positive integer $k$. We show that $VR_{k+1}(P_3) \leq 3k + 3$. Let there be given a red-blue coloring of $K_{k+3}$. Then there is a monochromatic copy $P$ of $P_3$. Let $H = K_{3k+3} - V(P) = K_{3k}$. Since $VR_k(P_3) = 3k$, it follows that $H$ contains $k$ vertex-disjoint monochromatic copies of $P_3$. Hence, there are $k + 1$ vertex-disjoint monochromatic copies of $P_3$ in $K_{3k}$ and so $VR_{k+1}(P_3) \leq 3k + 3$. Therefore, $VR_{k+1}(P_3) = 3k + 3$.

Next, we determine $ER_t(P_3)$ for every positive integer $t$, beginning with $t = 2, 3$.

**Proposition 3.1.** $ER_2(P_3) = 4$ and $ER_3(P_3) = 5$.

**Proof.** First, we show that $ER_2(P_3) = 4$. Since $K_2$ has size 3, it follows that $ER_2(P_3) \geq 4$. Let there be given a red-blue coloring of $K_4$ with $V(K_4) = \{v_1, v_2, v_3, v_4\}$. At least two edges incident with $v_1$ are colored the same, say $v_1v_2$ and $v_1v_3$, resulting in a monochromatic copy of $P_3$. The same is true for $v_4$. Thus, $ER_2(P_4) \leq 4$ and so $ER_2(P_3) = 4$. 

Next, we show that $ER_3(P_3) = 5$. The red-blue coloring of $K_4$ with red subgraph $2K_2$ and blue subgraph $C_4$ has only two edge-disjoint monochromatic copies of $P_3$ and so $ER_3(P_3) > 5$. Let there be given a red-blue coloring of $G = K_5$ with $V(K_5) = \{v_1, v_2, v_3, v_4, v_5\}$. Then $S = \{v_1, v_2, v_3, v_4\}$. Then $H = G[S] = K_4$. Since $ER_2(P_3) = 4$, it follows that $H$ contains two edge-disjoint monochromatic copies $F_1$ and $F_2$ of $P_3$. At least two edges incident with $v_5$ are colored the same, producing a monochromatic copy of $P_3$ edge-disjoint from $F_1$ and $F_2$. Thus, $ER_3(P_3) \leq 5$ and so $ER_3(P_3) = 5$.

We now determine $ER_t(P_3)$ for all positive integers $t$. For every positive integer $t$, there exists a unique nonnegative integer $k$ such that $k^2 - k < t \leq k^2 + k$.

**Theorem 3.2.** For a positive integer $k$, let $t$ be the unique integer with $k^2 - k < t \leq k^2 + k$.

(1) If $k^2 - k < t \leq k^2$, then $ER_t(P_3) = 2k + 1$.

(2) If $k^2 < t \leq k^2 + k$, then $ER_t(P_3) = 2k + 2$.

**Proof.** First, we verify (1). Since $ER_1(P_3) = 3$, we may assume that $t \geq 2$ is an integer such that $k^2 - k + 1 \leq t \leq k^2$ for a unique integer $k \geq 2$. We show that $ER_t(P_3) = 2k + 1$. By Observation 2.2, if $k^2 - k + 1 \leq t \leq k^2$, then $ER_{k^2-k+1+1}(P_3) \leq ER_t(P_3) \leq ER_{k^2}(P_3)$.

Hence, it suffices to show that $ER_{k^2-k+1}(P_3) \geq 2k + 1$ and $ER_{k^2}(P_3) \leq 2k + 1$.

First, we show that $ER_{k^2-k+1+1}(P_3) \geq 2k + 1$. Let $c$ be the red-blue coloring of $K_{2k}$ with red subgraph $kK_2$ and blue subgraph $K_{2k} - kK_2$. The red subgraph contains no $P_3$. Since the size of $K_{2k} - kK_2$ is $(\binom{k}{2} - k) = 2(k^2 - k)$, the blue subgraph contains at most $k^2 - k$ pairwise edge-disjoint copies of $P_3$. Since the coloring of $K_{2k}$ does not produce $k^2 - k + 1$ pairwise edge-disjoint monochromatic copies of $P_3$, it follows that $ER_{k^2-k+1+1}(P_3) \geq 2k + 1$.

To show that $ER_{k^2}(P_3) \leq 2k + 1$, we proceed by induction on $k \geq 1$. The statement is true for $k = 1, 2, 3$. Assume that $ER_{k^2}(P_3) \leq 2k + 1$ where $k \geq 3$. We show that $ER_{(k+1)^2}(P_3) \leq 2k + 3$. Let there be given a red-blue coloring of $G = K_{2k+3}$, where $G_r$ and $G_b$ are the red and blue subgraphs of $G$, respectively. Since $G_r$ has odd order, $G_r$ contains a vertex $v$ of even degree, say $\deg_{G_r} v = 2a + 2$ for a nonnegative integer $a$. Let $uv$ be a red edge of $G$ and let $H = G - \{u, v\} = K_{2k+1}$. By the induction hypothesis, $H$ contains $k^2$ pairwise edge-disjoint monochromatic copies of $P_3$ and so $ER_{(k+1)^2}(P_3) \leq 2k + 3$. Therefore, $ER_t(P_3) = 2k + 1$ for $k^2 - k < t \leq k^2$.

Next, we verify (2). Let $t \geq 2$ be an integer and let $k$ be the unique integer such that $k^2 + 1 \leq t \leq k^2 + k$. We show that $ER_{k^2+1}(P_3) \geq 2k + 2$ and $ER_{k^2+k}(P_3) \leq 2k + 2$.

First, we show that $ER_{k^2+1}(P_3) \geq 2k + 2$. Let $c$ be the red-blue coloring of $K_{2k+1}$ with red subgraph $kK_2$ and blue subgraph $K_{2k+1} - kK_2$. The red subgraph contains no $P_3$. Since the size of $K_{2k+1} - kK_2$ is $(\binom{k+1}{2} - k) = 2k^2$, the blue subgraph contains at most $k^2$ pairwise edge-disjoint copies of $P_3$. Since the coloring of $K_{2k+1}$ does not produce $k^2 + 1$ pairwise edge-disjoint monochromatic copies of $P_3$, it follows that $ER_{k^2+1}(P_3) \geq 2k + 2$.

To show that $ER_{k^2+k}(P_3) \leq 2k + 2$, we proceed by induction on $k \geq 1$. The statement is true for $k = 1, 2, 3$. Assume that $ER_{k^2+k}(P_3) \leq 2k + 2$ where $k \geq 3$. We show that $ER_{(k+1)^2+k}(P_3) \leq 2k + 4$. Let there be given a red-blue coloring of $G = K_{2k+4}$, where $G_r$ and $G_b$ are the red and blue subgraphs of $G$, respectively. Let $x$ be a vertex of $G$. Regardless of the colors of these $2k + 3$ edges incident with $x$, there are $k$ pairwise edge-disjoint monochromatic copies of $P_3$ centered at $x$. Hence, $G$ contains $2k + 1$ pairwise edge-disjoint monochromatic copies of $P_3$ centered at either $u$ or $v$ that are edge-disjoint from $F_1$, $F_2$, ..., $F_{k^2}$. Thus, $G$ contains $k^2 + 2k + 1 = (k + 1)^2$ pairwise edge-disjoint monochromatic copies of $P_3$ and so $ER_{(k+1)^2+k}(P_3) \leq 2k + 4$. Therefore, $ER_t(P_3) = 2k + 2$ for $k^2 - k < t \leq k^2 + k$. 

□
Corollary 3.1. For a positive integer \( t \), \( ER_t(P_3) = \left\lceil 2\sqrt{t} + 1 \right\rceil \).

Proof. Let \( t \) be a positive integer. Then there is a unique integer \( k \) such that \( k^2 - k + 1 \leq t \leq k^2 + k \). We consider two cases, according to whether \( k^2 - k + 1 \leq t \leq k^2 \) or \( k^2 + 1 \leq t \leq k^2 + k \).

Case 1. \( k^2 - k + 1 \leq t \leq k^2 \). By Theorem 3.2, \( ER_t(P_3) = 2k + 1 \). Since \( k^2 - k + 1 > \left( k - \frac{1}{2} \right)^2 \), it follows that

\[
\sqrt{k^2 - k + 1} > k - \frac{1}{2}.
\]

Thus, \( 2\sqrt{k^2 - k + 1} + 1 > 2 \left( k - \frac{1}{2} \right) + 1 = 2k \) and so \( \left\lceil 2\sqrt{k^2 - k + 1} + 1 \right\rceil \geq 2k + 1 \). Since \( 2\sqrt{k^2 + 1} = 2k + 1 \), it follows that

\[
2k + 1 \leq \left\lceil 2\sqrt{k^2 - k + 1} + 1 \right\rceil \leq \left\lceil 2\sqrt{t} + 1 \right\rceil \leq \left\lceil 2\sqrt{k^2 + 1} \right\rceil = 2k + 1.
\]

Therefore, \( \left\lceil 2\sqrt{t} + 1 \right\rceil = 2k + 1 = R_t(P_3) \).

Case 2. \( k^2 + 1 \leq t \leq k^2 + k \). By Theorem 3.2, \( ER_t(P_3) = 2k + 2 \). Since \( k^2 + k < \left( k + \frac{1}{2} \right)^2 \), it follows that \( \sqrt{k^2 + k} < k + \frac{1}{2} \).

Thus, \( 2\sqrt{k^2 + k} + 1 < 2 \left( k + \frac{1}{2} \right) + 1 = 2k + 2 \) and so \( \left\lceil 2\sqrt{k^2 + k} + 1 \right\rceil \leq 2k + 2 \). Since \( \sqrt{k^2 + 1} > k \), it follows that

\[
2\sqrt{k^2 + 1} + 1 > 2k + 1
\]

and so \( \left\lceil 2\sqrt{k^2 + 1} + 1 \right\rceil \geq 2k + 2 \). For each integer \( t \) with \( k^2 + 1 \leq t \leq k^2 + k \),

\[
2k + 2 \leq \left\lceil 2\sqrt{k^2 + 1} + 1 \right\rceil \leq \left\lceil 2\sqrt{t} + 1 \right\rceil \leq \left\lceil 2\sqrt{k^2 + k} + 1 \right\rceil \leq 2k + 2.
\]

Therefore, \( \left\lceil 2\sqrt{t} + 1 \right\rceil = 2k + 2 = ER_t(P_3) \). \qed

References