Contributions to Mathematics
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Contrib. Math. **9** (2024) 62–68
DOI: 10.47443/cm.2024.036

Research Article

Monochromatic subgraphs in graphs

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(Received: 10 June 2024. Accepted: 2 July 2024. Published online: 3 July 2024.)

Abstract

For a positive integer t and a graph F, the numbers $ER_t(F)$ and $VR_t(F)$ of F are the minimum positive integer n such that every red-blue coloring of the edges of the complete graph K_n results in t pairwise edge-disjoint and vertex-disjoint, respectively, monochromatic copies of F in K_n. The number $ER_t(F)$ is determined when $F = K_3$ for $t \leq 4$ and when F is the path P_3 of order 3 for every positive integer t, while $VR_t(F)$ is determined when $F \in \{K_3, P_3\}$ for every positive integer t.

Keywords: red-blue coloring; edge-disjoint and vertex-disjoint monochromatic graphs.

2020 Mathematics Subject Classification: 05C15, 05C35, 05C55.

1. Introduction

In a *red-blue coloring* of a graph G, every edge of G is colored red or blue. For two graphs F and H, the well-known *Ramsey number* $R(F, H)$ is the minimum positive integer n such that for every red-blue coloring of the complete graph K_n of order n, there is either a subgraph of K_n isomorphic to F all of whose edges are colored red (a *red* F) or a subgraph of K_n isomorphic to H all of whose edges are colored blue (a *blue* H). Therefore, for a single graph F, the *Ramsey number* $R(F, F)$, also denoted by $R(F)$, is the minimum positive integer n such that for every red-blue coloring of K_n, there is a subgraph of K_n isomorphic to F all of whose edges are colored the same (a *monochromatic* F). That these numbers exist for every graph F is due to Ramsey [3]. We refer to the book [1] for notation and terminology not defined here.

An introduction to Ramsey numbers in graph theory often begins with a question that is sometimes stated in the following manner.

> *How many people must be present at a party to be guaranteed that there are three mutual acquaintances or three mutual strangers?*

It may already be clear that this question has a graph theory interpretation. For example, suppose that there are n people at the party. These n people are the n vertices of the complete graph K_n. Two vertices are joined by a red edge if the two people are acquaintances and joined by a blue edge if they are strangers. The question then becomes the following.

> *What is the smallest positive integer n such that if the $\binom{n}{2}$ edges of K_n are colored red or blue in any manner whatsoever, we are guaranteed that a subgraph K_3 (a triangle) all of whose edges are colored red (a red triangle) or colored blue (a blue triangle) appears?*

In order to answer this question, it is useful to make the following observation.

Observation 1.1. *Let there be given an arbitrary red-blue coloring of a complete graph K_n where $n \geq 4$. If a vertex of K_n is incident with three or more edges of the same color, then K_n contains a monochromatic triangle.*

Proof. Suppose that a vertex v of K_n is incident with at least three edges of the same color, say vv_1, vv_2, vv_3 are red. If any two of v_1, v_2, v_3 are joined by a red edge, then there is a red triangle; otherwise, (v_1, v_2, v_3, v_1) is a blue triangle. □

First, $n = 5$ does not work. By Observation 1.1, any red-blue coloring of K_5 in which some vertex is incident with at least three edges of the same color results in a monochromatic triangle. Therefore, the only possible red-blue coloring of K_5 without any monochromatic triangle is for every vertex to be incident with exactly two edges of each color. Consequently, we have the next observation.

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Observation 1.2. *The only red-blue coloring of the complete graph K_5 for which there is no monochromatic triangle is one that produces a red cycle C_5 of order 5 and a blue C_5.*

This red-blue coloring of K_5 is shown in Figure 1.1 where a bold edge represents a red edge and a thin edge represents a blue edge.

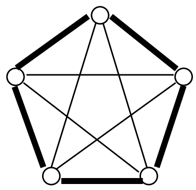

Figure 1.1: A red-blue coloring of K_5.

However, $n = 6$ does work. Every red-blue coloring of K_6 results in at least three edges incident with each vertex colored the same. By Observation 1.1, there is a red triangle or a blue triangle. Hence, the solution to the question above is the Ramsey number $R(K_3) = 6$, which appeared in [2]. This problem essentially appeared as Problem A2 in the 1953 Putnam Exam.

A2. *The complete graph with 6 points and 15 edges has each edge colored red or blue. Show that we can find 3 points such that the 3 edges joining them are the same color.*

Not only does every red-blue coloring of K_6 always produce a monochromatic triangle, it always produces at least two monochromatic triangles. To see this, let there be given a red-blue coloring of $G = K_6$ with red subgraph G_r and blue subgraph G_b. First, suppose that there is a vertex v that is incident with four edges of the same color, say vv_1, vv_2, vv_3, vv_4 are red. If there are two red edges in the subgraph $G[\{v_1, v_2, v_3, v_4\}]$ of G induced by the set $\{v_1, v_2, v_3, v_4\}$, then there are two red triangles. If at most one edge, say v_1v_2, is colored red, then $\{v_1, v_3, v_4\}$ and $\{v_2, v_3, v_4\}$ are the vertices of two blue triangles. Next, suppose that G_r is 3-regular. Then G_r is either the Cartesian product $K_3 \square K_2$ of K_3 and K_2 or the complete bipartite graph $K_{3,3}$. If $G_r = K_3 \square K_2$, then G_r has two triangles; while if $G_r = K_{3,3}$, then $G_b = 2K_3$ (the union of two vertex-disjoint copies of K_3) also contains two triangles.

Finally, suppose that neither G_r nor G_b contains a vertex of degree 4 or more or is 3-regular. Then each of these graphs must contain only vertices of degree 2 or 3, necessarily some of each. Since every graph contains an even number of odd vertices, one of these two graphs contains four vertices of degree 3 and two vertices of degree 2 while the other contains two vertices of degree 3 and four vertices of degree 2. Suppose that G_b contains four vertices of degree 3 and two vertices u and v of degree 2. Thus, G_b is one of graphs shown in Figures 1.2(a), (b), and (c).

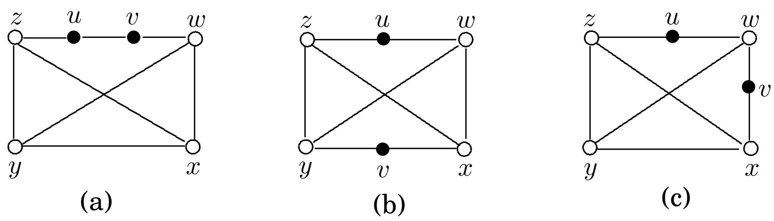

Figure 1.2: Three possible blue subgraphs G_b.

If (a) occurs, then u and v are adjacent and G_b has two triangles. If (b) occurs, then (v, w, z, v) and (u, x, y, u) are two red triangles. If (c) occurs, then (x, y, z, x) is a blue triangle and (u, x, y, u) is a red triangle. Therefore, we have the following.

In every red-blue coloring of the edges of K_6, there are always at least two monochromatic triangles.

2. Monochromatic triangles

The red-blue coloring of K_6 with red subgraph $K_{2,4}$ and blue subgraph $K_2 + K_4$ has four monochromatic triangles, all blue. However, every two blue triangles have an edge in common. When $n = 7$, however, every red-blue coloring of K_7 results in two edge-disjoint monochromatic triangles.

Proposition 2.1. *Every red-blue coloring of K_7 results in two edge-disjoint monochromatic triangles.*

Proof. Let there be given an arbitrary red-blue coloring of $G = K_7$. Since $R(K_3, K_3) = 6$, there is a monochromatic triangle T in G. Let $V(T) = \{u_1, u_2, u_3\}$ and let $V(G) - V(T) = \{v_1, v_2, v_3, v_4\}$. Let $F_i = G[\{u_i, v_1, v_2, v_3, v_4\}] = K_5$ for $i = 1, 2, 3$. If any F_i contains a monochromatic triangle, then G contains two edge-disjoint (in fact, vertex-disjoint) monochromatic triangles. Thus, we may assume that no F_i $(i = 1, 2, 3)$ contains a monochromatic triangle. It follows by Observation 1.2 that each F_i consists of a red C_5 and a blue C_5 for $i = 1, 2, 3$. Assume, without loss of generality, that $(u_1, v_1, v_2, v_3, v_4, u_1)$ is a red C_5 in F_1 and $(u_1, v_2, v_4, v_1, v_3, u_1)$ is a blue C_5 in F_1. We now consider F_i for $i = 2, 3$. Since

(a) (v_1, v_2, v_3, v_4) is a red P_4 in F_i and (v_2, v_4, v_1, v_3) is a blue P_4 in F_i and

(b) F_i consists of a red C_5 and a blue C_5,

it follows that u_iv_1, u_iv_4 are red and u_iv_2, u_iv_3 are blue . If T is a red triangle, then (u_1, v_1, u_2, u_1) and (u_2, v_4, u_3, u_2) are two edge-disjoint red triangles in G; while if T is a blue triangle, then (u_1, v_2, u_2, u_1) and (u_2, v_3, u_3, u_2) are two edge-disjoint blue triangles in G. Therefore, G contains two edge-disjoint monochromatic triangles. □

Consequently, 6 is the smallest order n of a complete graph K_n for which every red-blue coloring of K_n results in a monochromatic triangle (or two monochromatic triangles which may have an edge in common), while 7 is the smallest order n of a complete graph K_n for which every red-blue coloring of K_n results in two edge-disjoint monochromatic triangles. These facts suggest the problem of determining the smallest positive integer n such that every red-blue coloring of K_n results in two vertex-disjoint monochromatic triangles.

The red-blue coloring of K_7 with red subgraph $K_5 + K_2$ and blue subgraph $K_{2,5}$ does not contain two vertex-disjoint monochromatic copies of K_3. When $n = 8$, however, every red-blue coloring of K_8 results in two vertex-disjoint monochromatic triangles.

Proposition 2.2. *Every red-blue coloring of K_8 results in two vertex-disjoint monochromatic triangles.*

Proof. Let there be given an arbitrary red-blue coloring of $G = K_8$ with vertex set $\{v_1, v_2, \ldots, v_8\}$. Since $R(K_3, K_3) = 6$, there is a monochromatic copy of K_3 in G. Let T be a monochromatic triangle in G where $V(T) = \{v_1, v_2, v_3\}$. Let $F = G[\{v_4, v_5, v_6, v_7 v_8\} = K_5$. If F contains a monochromatic triangle, then G contains two vertex-disjoint monochromatic triangles. Suppose that F does not contain a monochromatic triangle. Then F consists of a red C_5 and a blue C_5. We may assume that the red C_5 is $(v_4, v_5, v_6, v_7, v_8, v_4)$ and the blue C_5 is $(v_4, v_6, v_8, v_5, v_7, v_4)$. Since v_1 is jointed to at least three vertices of $V(F) = \{v_4, v_5, v_6, v_7, v_8\}$ by edges of the same color, we may assume that v_1 is adjacent to three vertices of $V(F)$ by red edges. Necessarily, v_1 is joined to two adjacent vertices of $V(F)$ that are joined by a red edge. Consequently, we may assume that v_1v_4 and v_1v_5 are red. See Figure 2.1.

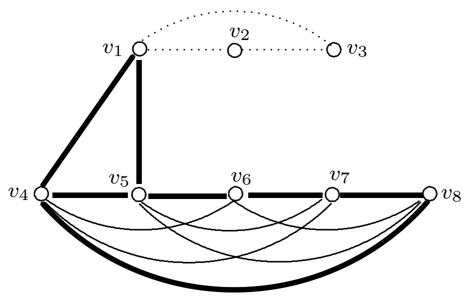

Figure 2.1: A step in the proof of Proposition 2.2.

Let $H = G[\{v_2, v_3, v_6, v_7, v_8\}] = K_5$. If H contains a monochromatic triangle, then G contains two vertex-disjoint monochromatic triangles. Suppose that H does not contain a monochromatic triangle and so H consists of a red subgraph $H_r = C_5$ and a blue subgraph $H_b = C_5$. First, suppose that T is blue. Since $\deg_{H_r} v_7 = 2$, it follows that $v_2v_6, v_2v_8, v_3v_6, v_3v_8$ are red. However then, $\deg_{H_r} v_6 = \deg_{H_r} v_8 = 3$, a contradiction. We may assume that T is red. We may further assume that $H_r = (v_2, v_3, v_8, v_7, v_6, v_2)$ and $H_b = (v_2, v_8, v_6, v_3, v_7, v_2)$. We consider two cases, according to whether v_1v_6 is red or v_1v_6 is blue.

Case 1. *v_1v_6 is red.* This implies that v_2v_4 is red for otherwise, there are two vertex-disjoint monochromatic triangles (v_1, v_5, v_6, v_1) and (v_2, v_4, v_7, v_2). Also, this implies that v_3v_4 is red for otherwise, there are two vertex-disjoint monochromatic triangles (v_1, v_5, v_6, v_1) and (v_3, v_4, v_8, v_3). However then, there are two vertex-disjoint monochromatic triangles (v_1, v_5, v_6, v_1) and (v_2, v_3, v_4, v_2).

Case 2. *v_1v_6 is blue.* First, suppose that v_1v_8 is red. So, v_2v_5 is blue, for otherwise there are two vertex-disjoint monochromatic triangles (v_1, v_4, v_8, v_1) and (v_2, v_5, v_6, v_2). Then (v_1, v_4, v_8, v_1) and (v_2, v_5, v_7, v_2) are two vertex-disjoint monochromatic triangles. Next, suppose that v_1v_8 is blue. If v_2v_5 is blue, then there are two vertex-disjoint monochromatic triangles (v_2, v_5, v_7, v_2) and (v_1, v_6, v_8, v_1). Thus, we may assume that v_2v_5 is red. If v_2v_4 is blue, then there are two vertex-disjoint monochromatic triangles (v_2, v_4, v_7, v_2) and (v_1, v_6, v_8, v_1). Thus, we may assume that v_2v_4 is red. Then there are two vertex-disjoint monochromatic triangles (v_2, v_4, v_5, v_2) and (v_1, v_6, v_8, v_1). □

The results above concerning two edge-disjoint and vertex-disjoint monochromatic triangles in an edge-colored complete graph suggest more general concepts. Let t be a positive integer and let F be a graph without isolated vertices. The *vertex-disjoint Ramsey number* $VR_t(F)$ is the minimum positive integer n such that every red-blue coloring of K_n results in at least t pairwise vertex-disjoint monochromatic copies of F. Then $VR_1(K_3) = R(K_3) = 6$ and $VR_2(K_3) = 8$ by Proposition 2.2 and the red-blue coloring of K_7 with red subgraph $K_5 + K_2$ and blue subgraph $K_{2,5}$. For a graph F without isolated vertices, $VR_t(F) \geq t|V(F)|$ and the Ramsey number $R(tF)$ exists where tF is a union of t vertex-disjoint copies of F. Hence, we have the following observation.

Observation 2.1. *For every graph F without isolated vertices and every positive integer t, the number $VR_t(F)$ exists and $t|V(F)| \leq VR_t(F) \leq R(tF)$. Furthermore, $VR_t(F) \leq VR_{t+1}(F)$.*

Theorem 2.1. *For an integer $t \geq 2$, $VR_t(K_3) = 3t + 2$.*

Proof. We proceed by induction on t. We saw that $VR_2(K_3) = 8$ and so the result is true for $t = 2$. Assume that $VR_k(K_3) = 3k+2$ for an integer $k \geq 2$. We show that $VR_{k+1}(K_3) = 3k+5$. The red-blue coloring of K_{3k+4} with red subgraph $K_{3k+2}+K_2$ and blue subgraph $K_{2,3k+2}$ has k vertex-disjoint red triangles and no blue triangle. Thus, $VR_{k+1}(K_3) \geq 3k + 5$. Next, let there be given a red-blue coloring of K_{3k+5}. Since $3k + 5 \geq 11 > 6$ and $R(K_3) = 6$, there is a monochromatic triangle T. Let $H = K_{3k+5} - V(T) = K_{3k+2}$. Since $VR_k(K_3) = 3k + 2$, it follows that H contains k vertex-disjoint monochromatic triangles. Hence, there are $k + 1$ vertex-disjoint monochromatic triangles in K_{3k+5} and so $VR_{k+1}(K_3) \leq 3k + 5$. Therefore, $VR_{k+1}(K_3) = 3k + 5$. □

Let t be a positive integer and let F be a graph without isolated vertices. The *edge-disjoint Ramsey number* $ER_t(F)$ of F is the minimum positive integer n such that for every red-blue coloring of K_n, there are at least t pairwise edge-disjoint monochromatic copies of F. Hence, $ER_1(F)$ is the Ramsey number $R(F)$. Therefore, $ER_1(K_3) = 6$ and $ER_2(K_3) = 7$ by Proposition 2.1 and the red-blue coloring of K_6 with red subgraph $K_{2,4}$ and blue subgraph K_2+K_4. There is an observation for edge-disjoint Ramsey numbers similar to Observation 2.1.

Observation 2.2. *For every graph F without isolated vertices and every positive integer t, the number $ER_t(F)$ exists and $ER_t(F) \leq VR_t(F)$. Furthermore, $ER_t(F) \leq ER_{t+1}(F)$.*

We now determine $ER_t(K_3)$ for $t = 3, 4$. First, we present a useful lemma.

Lemma 2.1. *For each positive integer t, $ER_{t+1}(K_3) \leq ER_t(K_3) + 2$.*

Proof. Let $ER_t(K_3) = k$. Let there be given an arbitrary red-blue coloring of $G = K_{k+2}$. Then there is a monochromatic copy F_0 of K_3 in G. Let $u, v \in V(F_0)$ and let $H = G - \{u, v\}$. Then $H = K_k$ contains no edge of F_0. Since $ER_t(K_3) = k$, there are t pairwise edge-disjoint monochromatic copies F_1, F_2, …, F_t of K_3 in H that are edge-disjoint from F_0. Therefore, F_0, F_1, F_2, …, F_t are $t+1$ pairwise edge-disjoint monochromatic copies of K_3 in G and so $ER_{t+1}(K_3) \leq k+2 = ER_t(K_3)+2$. □

Both strict inequality and equality in Lemma 2.1 can occur, as we show next.

Proposition 2.3. $ER_3(K_3) = 9$

Proof. Since $ER_3(K_3) \leq ER_2(K_3) + 2 = 9$ by Lemma 2.1 and Proposition 2.1, it remains to show that $ER_3(K_3) \geq 9$. For the red-blue coloring of K_8 with red subgraph $K_{4,4}$ and blue graph $2K_4$, there is no red triangle and only two edge-disjoint blue triangles. Since this red-blue coloring of K_8 does not produce three pairwise edge-disjoint monochromatic triangles, it follows that $ER_3(K_3) \geq 9$ and so $ER_3(K_3) = 9$. □

Theorem 2.2. $ER_4(K_3) = 10$.

Proof. The red-blue coloring of K_9 with red subgraph $K_{5,4}$ and blue subgraph $K_5 + K_4$ contains no red triangle and three edge-disjoint blue triangles. Since this red-blue coloring of K_9 does not produce four pairwise edge-disjoint monochromatic triangles, it follows that $ER_4(K_3) \geq 10$.

It remains to show that $ER_4(K_3) \leq 10$. First, we verify the following claim.

Claim: For every red-blue coloring of $G = K_{10}$ containing two edge-disjoint monochromatic triangles T and T' having a vertex in common, there are four pairwise edge-disjoint monochromatic triangles in G.

To verify the claim, consider a red-blue coloring of G with two edge-disjoint monochromatic triangles $T = (v_1, v_2, v_3, v_1)$ and $T' = (v_1, v_4, v_5, v_1)$. Let $H = G[V(G) - \{v_1, v_2, v_4\}] = K_7$. Since $ER_2(K_3) = 7$ by Proposition 2.1, there are two edge-disjoint monochromatic triangles T_1 and T_2 in H edge-disjoint from T and T'. Thus, T, T', T_1, T_2 are four pairwise edge-disjoint monochromatic triangles in G and so the claim is true.

Next, we show that every red-blue coloring of $G = K_{10}$ produces four pairwise edge-disjoint monochromatic triangles in G. Let there be given a red-blue coloring of G. Since $ER_3(K_3) = 9$, it follows that G contains three pairwise edge-disjoint monochromatic triangles T_1, T_2, T_3. We show that G contains a fourth monochromatic triangle edge-disjoint from T_1, T_2, T_3. By the claim, we may assume that T_1, T_2, T_3 are pairwise vertex-disjoint. Hence, the subgraph F of G induced by $E(T_1) \cup E(T_2) \cup E(T_3)$ is $3K_3$. Let $T_1 = (v_1, v_2, v_3, v_1)$, $T_2 = (v_4, v_5, v_6, v_4)$, $T_3 = (v_7, v_8, v_9, v_7)$, and let v be the vertex of G not in F. Furthermore, let $H = G - E(F)$ be the spanning subgraph of G whose edge set consists of all edges not belonging to any of T_1, T_2, and T_3. Then $H \cong K_{1,3,3,3}$ and $\deg_H v = 9$. Let r be the number of red edges incident with v and b the number of blue edges incident with v. Then $r + b = 9$. We may assume that $b \leq r$ and so $5 \leq r \leq 9$.

First, suppose that there is a red edge joining v to at least one vertex in each of T_1, T_2, T_3. We may assume without loss of generality that vv_1, vv_4, vv_7 are red. If there is a red edge joining two vertices in $\{v_1, v_4, v_7\}$, say v_1v_4 is red, then (v, v_1, v_4, v) is a red triangle edge-disjoint from T_1, T_2, T_3; while if every two vertices in $\{v_1, v_4, v_7\}$ is joined by a blue edge, then (v_1, v_4, v_7, v_1) is a blue triangle edge-disjoint from T_1, T_2, T_3. Thus, we may assume that v is joined to exactly two of T_1, T_2, T_3 by red edges. Hence, $r = 5, 6$ and we may further assume that vv_i is red for $i = 1, 2, 3, 4, 5$ and vv_6 is either red or blue.

Let $H' = K_{2,3}$ be the complete bipartite subgraph of H with partite sets $\{v_1, v_2, v_3\}$ and $\{v_4, v_5\}$. If H' contains a red edge, say v_1v_4 is red, then (v, v_1, v_4, v) is a red triangle edge-disjoint from T_1, T_2, T_3. Thus, we may assume that H' is a blue $K_{2,3}$. If T_1 is blue, then (v_1, v_4, v_2, v_1) and (v_1, v_5, v_3, v_1) are two edge-disjoint blue triangles having the vertex v_1 in common. It then follows by the claim that G contains four pairwise edge-disjoint monochromatic triangles. Thus, we may assume that T_1 is red. If T_2 is red, then (v_1, v, v_2, v_1) and (v_4, v, v_5, v_4) are two edge-disjoint red triangles having the vertex v in common; while if T_2 is blue, then (v_1, v_2, v_3, v_1) and (v_1, v_4, v_5, v_1) are two edge-disjoint monochromatic triangles having the vertex v_1 in common. Again, by the claim, G contains four pairwise edge-disjoint monochromatic triangles. Therefore, $ER_4(K_3) \leq 10$ and so $ER_4(K_3) = 10$. □

We close this section with the following conjecture.

Conjecture 2.1. *For every integer $t \geq 4$, $ER_t(K_3) \leq ER_{t+1}(K_3) \leq ER_t(K_3) + 1$.*

3. Monochromatic paths of order 3

We now turn our attention to the other connected graph of order 3, namely the path P_3 of order 3. Of course,

$$R_1(P_3) = VR_1(P_3) = ER_1(P_3) = 3.$$

First, we determine $VR_t(P_3)$ for every positive integer t.

Theorem 3.1. *For every positive integer t, $VR_t(P_3) = 3t$.*

Proof. Since $VR_t(P_3) \geq 3t$ by Observation 2.1, it remains to show that $VR_t(P_3) \leq 3t$. We proceed by induction on t. Since $VR_1(P_3) = R(P_3) = 3$, the result is true for $t = 1$. Assume that $VR_k(P_3) \leq 3k$ for a positive integer k. We show that $VR_{k+1}(P_3) \leq 3k + 3$. Let there be given a red-blue coloring of K_{k+3}. Then there is a monochromatic copy P of P_3. Let $H = K_{3k+3} - V(P) = K_{3k}$. Since $VR_k(P_3) = 3k$, it follows that H contains k vertex-disjoint monochromatic copies of P_3. Hence, there are $k + 1$ vertex-disjoint monochromatic copies of P_3 in K_{3k} and so $VR_{k+1}(P_3) \leq 3k + 3$. Therefore, $VR_{k+1}(P_3) = 3k + 3$. □

Next, we determine $ER_t(P_3)$ for every positive integer t, beginning with $t = 2, 3$.

Proposition 3.1. *$ER_2(P_3) = 4$ and $ER_3(P_3) = 5$.*

Proof. First, we show that $ER_2(P_3) = 4$. Since K_3 has size 3, it follows that $ER_2(P_3) \geq 4$. Let there be given a red-blue coloring of K_4 with $V(K_4) = \{v_1, v_2, v_3, v_4\}$. At least two edges incident with v_1 are colored the same, say v_1v_2 and v_1v_3, resulting in a monochromatic copy of P_3. The same is true for v_4. Thus, $ER_2(P_4) \leq 4$ and so $ER_2(P_3) = 4$.

Next, we show that $ER_3(P_3) = 5$. The red-blue coloring of K_4 with red subgraph $2K_2$ and blue subgraph C_4 has only two edge-disjoint monochromatic copies of P_3 and so $ER_3(P_3) \geq 5$. Let there be given a red-blue coloring of $G = K_5$ with $V(K_5) = \{v_1, v_2, v_3, v_4, v_5\}$. Let $S = \{v_1, v_2, v_3, v_4\}$. Then $H = G[S] = K_4$. Since $ER_2(P_3) = 4$, it follows that H contains two edge-disjoint monochromatic copies F_1 and F_2 of P_3. At least two edges incident with v_5 are colored the same, producing a monochromatic copy of P_3 edge-disjoint from F_1 and F_2. Thus, $ER_3(P_3) \leq 5$ and so $ER_3(P_3) = 5$. □

We now determine $ER_t(P_3)$ for all positive integers t. For every positive integer t, there exists a unique nonnegative integer k such that $k^2 - k < t \leq k^2 + k$.

Theorem 3.2. *For a positive integer t, let k be the unique integer with $k^2 - k < t \leq k^2 + k$.*

(1) *If $k^2 - k < t \leq k^2$, then $ER_t(P_3) = 2k + 1$.*

(2) *If $k^2 < t \leq k^2 + k$, then $ER_t(P_3) = 2k + 2$.*

Proof. First, we verify (1). Since $ER_1(P_3) = 3$, we may assume that $t \geq 2$ is an integer such that $k^2 - k + 1 \leq t \leq k^2$ for a unique integer $k \geq 2$. We show that $ER_t(P_3) = 2k + 1$. By Observation 2.2, if $k^2 - k + 1 \leq t \leq k^2$, then

$$ER_{k^2-k+1}(P_3) \leq ER_t(P_3) \leq ER_{k^2}(P_3).$$

Hence, it suffices to show that $ER_{k^2-k+1}(P_3) \geq 2k + 1$ and $ER_{k^2}(P_3) \leq 2k + 1$.

First, we show that $ER_{k^2-k+1}(P_3) \geq 2k + 1$. Let c be the red-blue coloring of K_{2k} with red subgraph kK_2 and blue subgraph $K_{2k} - kK_2$. The red subgraph contains no P_3. Since the size of $K_{2k} - kK_2$ is $\binom{2k}{2} - k = 2(k^2 - k)$, the blue subgraph contains at most $k^2 - k$ pairwise edge-disjoint copies of P_3. Since the coloring c of K_{2k} does not produce $k^2 - k + 1$ pairwise edge-disjoint monochromatic copies of P_3, it follows that $ER_{k^2-k+1}(P_3) \geq 2k + 1$.

To show that $ER_{k^2}(P_3) \leq 2k + 1$, we proceed by induction on $k \geq 1$. The statement is true for $k = 1, 2, 3$. Assume that $ER_{k^2}(P_3) \leq 2k + 1$ where $k \geq 3$. We show that $ER_{(k+1)^2}(P_3) \leq 2k + 3$. Let there be given a red-blue coloring of $G = K_{2k+3}$, where G_r and G_b are the red and blue subgraphs of G, respectively. Since G_r has odd order, G_r contains a vertex u of even degree, say $\deg_{G_r} u = 2a + 2$ for a nonnegative integer a. Let uv be a red edge of G and let $H = G - \{u, v\} = K_{2k+1}$. By the induction hypothesis, H contains k^2 pairwise edge-disjoint monochromatic copies $F_1, F_2, \ldots, F_{k^2}$ of P_3. The vertex u is incident with $2a + 1$ red edges that join u to vertices in H and $(2k - 2a)$ blue edges that join u to vertices in H. Together with the red edge uv, there are $a + 1$ pairwise edge-disjoint red copies of P_3 centered at u and $k - a$ pairwise edge-disjoint blue copies of P_3 centered at u. Hence, there are $(a + 1) + (k - a) = k + 1$ pairwise edge-disjoint monochromatic copies of P_3 centered at u. Regardless of the colors of these $2k + 1$ edges incident with v, there are k pairwise edge-disjoint monochromatic copies of P_3 centered at v. Hence, G contains $2k + 1$ pairwise edge-disjoint monochromatic copies of P_3 centered at either u or v that are edge-disjoint from $F_1, F_2, \ldots, F_{k^2}$. Thus, G contains $k^2 + 2k + 1 = (k + 1)^2$ pairwise edge-disjoint monochromatic copies of P_3 and so $ER_{(k+1)^2}(P_3) \leq 2k + 3$. Therefore, $ER_t(P_3) = 2k + 1$ for $k^2 - k < t \leq k^2$.

Next, we verify (2). Let $t \geq 2$ be an integer and let k be the unique integer such that $k^2 + 1 \leq t \leq k^2 + k$. We show that $ER_t(P_3) = 2k + 2$. By Observation 2.2, if $k^2 + 1 \leq t \leq k^2 + k$, then $ER_{k^2+1}(P_3) \leq ER_t(P_3) \leq ER_{k^2+k}(P_3)$. Hence, it suffices to show that $ER_{k^2+1}(P_3) \geq 2k + 2$ and $ER_{k^2+k}(P_3) \leq 2k + 2$.

First, we show that $ER_{k^2+1}(P_3) \geq 2k + 2$. Let c be the red-blue coloring of K_{2k+1} with red subgraph kK_2 and blue subgraph $K_{2k+1} - kK_2$. The red subgraph contains no P_3. Since the size of $K_{2k+1} - kK_2$ is $\binom{2k+1}{2} - k = 2k^2$, the blue subgraph contains at most k^2 pairwise edge-disjoint copies of P_3. Since the coloring c of K_{2k+1} does not produce $k^2 + 1$ pairwise edge-disjoint monochromatic copies of P_3, it follows that $ER_{k^2+1}(P_3) \geq 2k + 2$.

To show that $ER_{k^2+k}(P_3) \leq 2k + 2$, we proceed by induction on $k \geq 1$. The statement is true for $k = 1, 2, 3$. Assume that $ER_{k^2+k}(P_3) \leq 2k + 2$ where $k \geq 3$. We show that $ER_{(k+1)^2+(k+1)}(P_3) \leq 2k + 4$. Let there be given a red-blue coloring of $G = K_{2k+4}$, where G_r and G_b are the red and blue subgraphs of G, respectively. Let x be a vertex of G. Regardless of the colors of these $2k + 3$ edges incident with x, there are $k + 1$ pairwise edge-disjoint monochromatic copies of P_3 centered at x in G. Let $G' = G - x = K_{2k+3}$ where G'_r and G'_b are the red and blue subgraphs of G', respectively. Since G' has odd order, there is a vertex y in G' such that the degree $\deg_{G'_r} y$ of y in G'_r is even, say $\deg_{G'_r} y = 2a$ for some nonnegative integer a. Let $H = G - \{x, y\} = K_{2k+2}$. By the induction hypothesis, H contains $k^2 + k$ pairwise edge-disjoint monochromatic copies $F_1, F_2, \ldots, F_{k^2+k}$ of P_3. The vertex y is incident with $2a$ red edges that join y to vertices in H and incident with $(2k + 2 - 2a) = 2(k + 1 - a)$ blue edges that join y to vertices in H. Thus, G' contains a pairwise edge-disjoint red copies of P_3 centered at y and $k + 1 - a$ pairwise edge-disjoint blue copies of P_3 centered at y. Hence, there are $k + 1$ pairwise edge-disjoint monochromatic copies of P_3 centered at y in $G' = G - x$. Thus, G contains $2k + 2$ pairwise edge-disjoint monochromatic copies of P_3 centered at either x or y that are edge-disjoint from $F_1, F_2, \ldots, F_{k^2+k}$. Thus, G contains $k^2 + k + 2k + 2 = (k + 1)^2 + (k + 1)$ pairwise edge-disjoint monochromatic copies of P_3 and so $ER_{k^2+k}(P_3) \leq 2k + 2$. Therefore, $ER_t(P_3) = 2k + 2$ for $k^2 < t \leq k^2 + k$. □

Corollary 3.1. *For a positive integer t,* $ER_t(P_3) = \lceil 2\sqrt{t}+1 \rceil$.

Proof. Let t be a positive integer. Then there is a unique integer k such that $k^2-k+1 \le t \le k^2+k$. We consider two cases, according to whether $k^2-k+1 \le t \le k^2$ or $k^2+1 \le t \le k^2+k$.

Case 1. $k^2-k+1 \le t \le k^2$. By Theorem 3.2, $ER_t(P_3) = 2k+1$. Since $k^2-k+1 > \left(k-\frac{1}{2}\right)^2$, it follows that

$$\sqrt{k^2-k+1} > k - \frac{1}{2}.$$

Thus, $2\sqrt{k^2-k+1}+1 > 2\left(k-\frac{1}{2}\right)+1 = 2k$ and so $\lceil 2\sqrt{k^2-k+1}+1 \rceil \ge 2k+1$. Since $2\sqrt{k^2}+1 = 2k+1$, it follows that $\left\lceil 2\sqrt{k^2}+1 \right\rceil = 2k+1$. For each integer t with $k^2-k+1 \le t \le k^2$,

$$2k+1 \le \left\lceil 2\sqrt{k^2-k+1}+1 \right\rceil \le \left\lceil 2\sqrt{t}+1 \right\rceil \le \left\lceil 2\sqrt{k^2}+1 \right\rceil = 2k+1.$$

Therefore, $\lceil 2\sqrt{t}+1 \rceil = 2k+1 = R_t(P_3)$.

Case 2. $k^2+1 \le t \le k^2+k$. By Theorem 3.2, $ER_t(P_3) = 2k+2$. Since $k^2+k < \left(k+\frac{1}{2}\right)^2$, it follows that $\sqrt{k^2+k} < k+\frac{1}{2}$. Thus, $2\sqrt{k^2+k}+1 < 2\left(k+\frac{1}{2}\right)+1 = 2k+2$ and so $\lceil 2\sqrt{k^2+k}+1 \rceil \le 2k+2$. Since $\sqrt{k^2+1} > k$, it follows that

$$2\sqrt{k^2+1}+1 > 2k+1$$

and so $\lceil 2\sqrt{k^2+1}+1 \rceil \ge 2k+2$. For each integer t with $k^2+1 \le t \le k^2+k$,

$$2k+2 \le \left\lceil 2\sqrt{k^2+1}+1 \right\rceil \le \left\lceil 2\sqrt{t}+1 \right\rceil \le \left\lceil 2\sqrt{k^2+k}+1 \right\rceil \le 2k+2.$$

Therefore, $\lceil 2\sqrt{t}+1 \rceil = 2k+2 = ER_t(P_3)$. □

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