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Research Article

Hyper-Leonardo p-numbers and associated norms

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Abstract

In this paper, we introduce hyper-Leonardo p-numbers, which generalize "hyper-Leonardo numbers". We establish their various combinatorial properties, including recurrence relations, summation formulas, and the generating function. We also compute Euclidean norms and obtain bounds for spectral norms of different forms of k-circulant matrices associated with hyper-Leonardo p-numbers. Additionally, we derive some bounds for spectral norms of Kronecker and Hadamard products involving these matrices.

Keywords: Leonardo numbers; hyper-Leonardo p-numbers; generating function; k-circulant matrix; spectral norm.

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1. Introduction

The Leonardo numbers, denoted as $\mathcal{L}_n$ and introduced in [1], are defined as

$$\mathcal{L}_{n+1} = \mathcal{L}_n + \mathcal{L}_{n-1} + 1, \ \ n \geq 1,$$

with initial conditions $\mathcal{L}_0 = \mathcal{L}_1 = 1$. An alternative recurrence relation for Leonardo numbers, when $n \geq 2$, is given (see [1]) as follows: $\mathcal{L}_{n+1} = 2\mathcal{L}_n - \mathcal{L}_{n-2}$. Mersin and Bahşi [8] introduced hyper-Leonardo numbers, which are denoted as $\mathcal{L}_n^{(r)}$ and are defined via the following formula:

$$\mathcal{L}_n^{(r)} = \sum_{k=0}^{n} \mathcal{L}_k^{(r-1)}, \ \ \text{with} \ \ \mathcal{L}_n^{(0)} = \mathcal{L}_n, \ \mathcal{L}_0^{(r)} = \mathcal{L}_0, \ \ \text{and} \ \mathcal{L}_1^{(r)} = r+1,$$

where r is a positive integer. For $n \geq 1$ and $r \geq 1$, the following recurrence relation [8] holds:

$$\mathcal{L}_n^{(r)} = \mathcal{L}_{n-1}^{(r)} + \mathcal{L}_n^{(r-1)}. \tag{1}$$

For any given integer $p > 0$, the Fibonacci p-numbers [11, 12] are defined through the following recurrence relation:

$$F_{p,n+1} = F_{p,n} + F_{p,n-p}, \ \ n \geq p,$$

with initial values $F_{p,0} = 0$ and $F_{p,k} = 1$ for $k = 1, 2, \ldots, p$; here, we remark that the choice $p = 1$ yields Fibonacci numbers, which are denoted as F_n. The Leonardo p-numbers extend the concept of Leonardo numbers and are defined by the following non-homogeneous recurrence relation (see [13]):

$$\mathcal{L}_{p,n+1} = \mathcal{L}_{p,n} + \mathcal{L}_{p,n-p} + p, \ \ n \geq p, \tag{2}$$

with $\mathcal{L}_{p,k} = 1$ for $k = 0, 1, 2, \ldots, p$. When $p = 1$, Leonardo p-numbers coincide with Leonardo numbers. The exists a direct relation between Leonardo p-numbers and Fibonacci p-numbers for $n \geq 0$: $\mathcal{L}_{p,n} = (p+1)F_{p,n+1} - p$. The authors of [13] also defined incomplete Leonardo p-numbers, which are defined for $0 \leq k \leq \left\lfloor \frac{n}{p+1} \right\rfloor$ as follows:

$$\mathcal{L}_{p,n}(k) = (p+1) \sum_{i=0}^{k} \binom{n-pi}{i} - p. \tag{3}$$

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For $p = 1$, incomplete Leonardo p-numbers are the same as incomplete Leonardo numbers, which were defined by Catarino and Borges [2] for $0 \le k \le \lfloor n/2 \rfloor$ as

$$\mathcal{L}_n(k) = 2\sum_{i=0}^{k} \binom{n-i}{i} - 1.$$

In this article, we introduce and study hyper-Leonardo p-numbers. We explore their recurrence relations, explicit form, generating function, and various identities and summation formulas. We also establish a connection between incomplete Leonardo p-numbers and hyper-Leonardo p-numbers. Furthermore, we study different forms of k-circulant matrices whose entries are hyper-Leonardo p-numbers. We compute Euclidean norms and establish bounds for spectral norms of these particular circulant matrices. Moreover, we derive some bounds for spectral norms of Kronecker and Hadamard products involving these matrices.

2. Hyper-Leonardo p-numbers

In this section, we introduce hyper-Leonardo p-numbers and explore various properties associated with them.

Definition 2.1. *The n-th hyper-Leonardo p-number $\mathcal{L}_{p,n}^{(r)}$ is defined as*

$$\mathcal{L}_{p,n}^{(r)} = \sum_{k=0}^{n} \mathcal{L}_{p,k}^{(r-1)}, \quad \text{with} \ \ \mathcal{L}_{p,n}^{(0)} = \mathcal{L}_{p,n}, \ \ \mathcal{L}_{p,0}^{(r)} = \mathcal{L}_{p,0}, \ \text{and} \ \mathcal{L}_{p,1}^{(r)} = r+1,$$

where r is a positive integer.

The first few hyper-Leonardo p-numbers for $n \ge 0$ are given below.

- For $p = 1$, hyper-Leonardo p-numbers are the same as hyper-Leonardo numbers.
- For $p = 2$, we get hyper-Leonardo 2-numbers:
  - $\mathcal{L}_{2,n}^{(1)} = 1, 2, 3, 7, 14, 24, 40, 65, 102, 157, 239, 360, 538, 800, 1185, \cdots;$
  - $\mathcal{L}_{2,n}^{(2)} = 1, 3, 6, 13, 27, 51, 91, 156, 258, 415, 654, 1014, 1552, 2352, 3537, \cdots;$
  - $\mathcal{L}_{2,n}^{(3)} = 1, 4, 10, 23, 50, 101, 192, 348, 606, 1021, 1675, 2689, 4241, 6593, 10130, \cdots.$
- For $p = 3$, we get hyper-Leonardo 3-numbers:
  - $\mathcal{L}_{3,n}^{(1)} = 1, 2, 3, 4, 9, 18, 31, 48, 73, 110, 163, 236, 337, 478, 675, \cdots;$
  - $\mathcal{L}_{3,n}^{(2)} = 1, 3, 6, 10, 19, 37, 68, 116, 189, 299, 462, 698, 1035, 1513, 2188, \cdots;$
  - $\mathcal{L}_{3,n}^{(3)} = 1, 4, 10, 20, 39, 76, 144, 260, 449, 748, 1210, 1908, 2943, 4456, 6644, \cdots.$

By Definition 2.1, hyper-Leonardo p-numbers satisfy the following non-homogeneous relation for $n \ge 1$ and $r \ge 1$:

$$\mathcal{L}_{p,n}^{(r)} = \mathcal{L}_{p,n-1}^{(r)} + \mathcal{L}_{p,n}^{(r-1)}. \tag{4}$$

Note that for $p = 1$, Equation (4) yields Equation (1). Next, we give an explicit form of hyper-Leonardo p-numbers.

Proposition 2.1. *For $n, r \ge 0$, it holds that*

$$\mathcal{L}_{p,n}^{(r)} = (p+1)\sum_{k=0}^{\lfloor \frac{n}{p+1} \rfloor} \binom{n+r-pk}{r+k} - p\binom{n+r}{r}. \tag{5}$$

Proof. We prove (5) by induction on $m = n + r$. The result is true when $n = 0$ or $r = 0$. Suppose that the result holds for all $i \le m$. We show that (5) holds for $m + 1$, where $m + 1 = (n + 1) + r$ and $n, r \ge 1$. According to (4), we have

$$\begin{aligned}
\mathcal{L}_{p,n+1}^{(r)} &= \mathcal{L}_{p,n}^{(r)} + \mathcal{L}_{p,n+1}^{(r-1)} \\
&= (p+1)\sum_{k=0}^{\lfloor \frac{n}{p+1} \rfloor} \binom{n+r-pk}{r+k} - p\binom{n+r}{r} + (p+1)\sum_{k=0}^{\lfloor \frac{n+1}{p+1} \rfloor} \binom{n+r-pk}{r-1+k} - p\binom{n+r}{r-1} \\
&= (p+1)\sum_{k=0}^{\lfloor \frac{n+1}{p+1} \rfloor} \binom{n+1+r-pk}{r+k} - p\binom{n+1+r}{r}.
\end{aligned}$$

which gives the desired result. □

The next result provides a non-homogeneous recurrence relation for hyper-Leonardo p-numbers.

Theorem 2.1. *For $n \geq p+1$ and $r \geq 1$, the following recurrence relation holds:*

$$\mathcal{L}_{p,n}^{(r)} = \mathcal{L}_{p,n-1}^{(r)} + \mathcal{L}_{p,n-p-1}^{(r)} + p\binom{n+r-p-1}{r} + \binom{n+r-1}{r-1}. \tag{6}$$

Proof. By Proposition 2.1, we have

$$\begin{aligned}\mathcal{L}_{p,n}^{(r)} &= (p+1)\sum_{k=0}^{\lfloor \frac{n}{p+1} \rfloor}\binom{n+r-pk}{r+k} - p\binom{n+r}{r} = (p+1)\sum_{k=0}^{\lfloor \frac{n}{p+1} \rfloor}\left(\binom{n+r-pk-1}{r+k} + \binom{n+r-pk-1}{r+k-1}\right) - p\binom{n+r}{r}\\ &= (p+1)\sum_{k=0}^{\lfloor \frac{n-1}{p+1} \rfloor}\binom{n+r-pk-1}{r+k} - p\binom{n+r-1}{r} - p\binom{n+r-1}{r-1} + (p+1)\left(\sum_{k=0}^{\lfloor \frac{n-p-1}{p+1} \rfloor}\binom{n+r-p-1-pk}{r+k} + \binom{n+r-1}{r-1}\right)\\ &= \mathcal{L}_{p,n-1}^{(r)} + \mathcal{L}_{p,n-p-1}^{(r)} + p\binom{n+r-p-1}{r} + \binom{n+r-1}{r-1}.\end{aligned}$$

□

The relation (6) can be transformed into the subsequent recurrence relation given in the next result.

Proposition 2.2. *For $n \geq 2p+1$ and $r \geq 1$, the following relation holds:*

$$\mathcal{L}_{p,n}^{(r)} = \mathcal{L}_{p,n-1}^{(r)} + \mathcal{L}_{p,n-p}^{(r)} - \mathcal{L}_{p,n-2p-1}^{(r)} + p\binom{n+r-p-1}{r} - p\binom{n+r-2p-1}{r} + \binom{n+r-1}{r-1} - \binom{n+r-p-1}{r-1}.$$

If we take $p = 1$ in Proposition 2.2, we obtain the following result (see [8]):

$$\mathcal{L}_n^{(r)} = 2\mathcal{L}_{n-1}^{(r)} - \mathcal{L}_{n-3}^{(r)} + \binom{n+r-3}{r-1} - \binom{n+r-2}{r-1} + \binom{n+r-1}{r-1}.$$

Next, we provide a relation involving Leonardo p-numbers, incomplete Leonardo p-numbers, and hyper-Leonardo p-numbers.

Theorem 2.2. *For all $n \geq 0$ and $r \geq 1$, the following relation holds:*

$$\mathcal{L}_{p,n+(p+1)r} = \mathcal{L}_{p,n}^{(r)} + \mathcal{L}_{p,n+(p+1)r}(r-1) + p\binom{n+r}{r}.$$

Proof. The result follows from Equations (3) and (5). □

Remark 2.1. *The substitution $p = 1$ in Theorem 2.2 yields the following relation:*

$$\mathcal{L}_{n+2r} = \mathcal{L}_n^{(r)} + \mathcal{L}_{n+2r}(r-1) + \binom{n+r}{r}, \quad \textit{for } \ n \geq 0 \textit{ and } r \geq 1.$$

The next result establishes a relationship between hyper-Leonardo p-numbers and Leonardo p-numbers.

Theorem 2.3. *For $n \geq 1$ and $r \geq 1$, the following equation holds:*

$$\mathcal{L}_{p,n}^{(r)} = \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}\mathcal{L}_{p,s}.$$

Proof. Let (a_n) and $(a^{(n)})$ be two real initial sequences. The symmetric infinite matrix associated with these sequences is defined recursively [3] by

$$\begin{cases} a_n^{(0)} = a_n,\ a_0^{(n)} = a^{(n)} & (n \geq 0),\\ a_n^{(r)} = a_{n-1}^{(r)} + a_n^{(r-1)}, & (n \geq 1, r \geq 1).\end{cases}$$

The entries of the symmetric infinite matrix are given by

$$a_n^{(r)} = \sum_{i=1}^{r}\binom{n+r-i-1}{n-1}a_0^{(i)} + \sum_{s=1}^{n}\binom{n+r-s-1}{r-1}a_s^{(0)}. \tag{7}$$

Let $a_n^{(r)} = \mathcal{L}_{p,n}^{(r)}$, then Equation (7) gives

$$\mathcal{L}_{p,n}^{(r)} = \sum_{i=1}^{r}\binom{n+r-i-1}{n-1}\mathcal{L}_{p,0}^{(i)} + \sum_{s=1}^{n}\binom{n+r-s-1}{r-1}\mathcal{L}_{p,s}^{(0)} = \sum_{i=1}^{r}\binom{n+r-i-1}{n-1} + \sum_{s=1}^{n}\binom{n+r-s-1}{r-1}\mathcal{L}_{p,s}$$
$$= \sum_{i=0}^{r-1}\binom{n+r-i-2}{n-1} + \sum_{s=0}^{n-1}\binom{n+r-s-2}{r-1}\mathcal{L}_{p,s+1} = \sum_{i=0}^{r-1}\binom{n+i-1}{n-1} + \sum_{s=0}^{n-1}\binom{r+s-1}{r-1}\mathcal{L}_{p,n-s}.$$

Using the binomial identity

$$\sum_{i=a}^{c}\binom{i}{a} = \binom{c+1}{a+1}, \tag{8}$$

we obtain

$$\mathcal{L}_{p,n}^{(r)} = \binom{n+r-1}{r-1} + \sum_{s=0}^{n-1}\binom{r+s-1}{r-1}\mathcal{L}_{p,n-s} = \sum_{s=0}^{n}\binom{r+s-1}{r-1}\mathcal{L}_{p,n-s} = \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}\mathcal{L}_{p,s}.$$

□

When $p=1$, Theorem 2.3 gives the following identity (see [8]):

$$\mathcal{L}_{n}^{(r)} = \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}\mathcal{L}_{s}, \quad \text{for } n,r\geq 1.$$

Next, we establish a recurrence formula for hyper-Leonardo p-numbers in terms of the Fibonacci p-numbers.

Theorem 2.4. *For $n \geq p-1$ and $r\geq 1$, it holds that*

$$\sum_{s=0}^{r}\mathcal{L}_{p,n}^{(s)} = \mathcal{L}_{p,n+1}^{(r)} - (p+1)F_{p,n-p+1}. \tag{9}$$

Proof. By using Theorem 2.3 and Equation (8), we have

$$\sum_{s=1}^{r}\mathcal{L}_{p,n}^{(s)} = \sum_{s=1}^{r}\sum_{i=0}^{n}\binom{n+s-i-1}{s-1}\mathcal{L}_{p,i} = \sum_{i=0}^{n}\mathcal{L}_{p,i}\sum_{s=1}^{r}\binom{n+s-i-1}{s-1} = \sum_{i=0}^{n}\binom{n+r-i}{n-i+1}\mathcal{L}_{p,i}$$
$$= \sum_{i=0}^{n+1}\binom{n+r-i}{r-1}\mathcal{L}_{p,i} - \mathcal{L}_{p,n+1} = \mathcal{L}_{p,n+1}^{(r)} - \mathcal{L}_{p,n+1}.$$

We recall that $\mathcal{L}_{p,n} = (p+1)F_{p,n+1} - p$. Hence, by using (2), we obtain

$$\sum_{s=0}^{r}\mathcal{L}_{p,n}^{(s)} = \mathcal{L}_{p,n+1}^{(r)} - \mathcal{L}_{p,n+1} + \mathcal{L}_{p,n} = \mathcal{L}_{p,n+1}^{(r)} - \mathcal{L}_{p,n-p} - p = \mathcal{L}_{p,n+1}^{(r)} - (p+1)F_{p,n-p+1}.$$

□

When $p=1$, Theorem 2.4 yields the following identity (see [8]): $\sum_{s=0}^{r}\mathcal{L}_{n}^{(s)} = \mathcal{L}_{n+1}^{(r)} - 2F_n$. The next result provides the alternating convolution identity.

Theorem 2.5. *For $r,m\geq 0$, the following identity holds:*

$$\sum_{k=0}^{m}(-1)^k\binom{r}{k}\mathcal{L}_{p,m-k}^{(r)} = \mathcal{L}_{p,m}.$$

Proof. We prove the identity by induction on m. Certainly, the identity is true for $m=0$. Suppose that the result holds for all $i\leq m$. Then, we can prove it for $m+1$. By (2) and (6), we have

$$\mathcal{L}_{p,m+1} = \mathcal{L}_{p,m} + \mathcal{L}_{p,m-p} + p = \sum_{k=0}^{m}(-1)^k\binom{r}{k}\mathcal{L}_{p,m-k}^{(r)} + \sum_{k=0}^{m-p}(-1)^k\binom{r}{k}\mathcal{L}_{p,m-p-k}^{(r)} + p$$
$$= \sum_{k=0}^{m+1}(-1)^k\binom{r}{k}\mathcal{L}_{p,m+1-k}^{(r)} - p\sum_{k=0}^{r}(-1)^k\binom{r}{k}\binom{m-k+r-p}{r} - \sum_{k=0}^{r}(-1)^k\binom{r}{k}\binom{m-k+r}{r-1} + p.$$

Since

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x-k}{m}=\binom{x-n}{m-n},$$

we have

$$\mathcal{L}_{p,m+1}=\sum_{k=0}^{m+1}(-1)^k\binom{r}{k}\mathcal{L}^{(r)}_{p,m+1-k}-p\binom{m-p}{r-r}-\binom{m}{r-1-r}+p=\sum_{k=0}^{m+1}(-1)^k\binom{r}{k}\mathcal{L}^{(r)}_{p,m+1-k}.$$

□

Remark 2.2. *The choice $p=1$ in Theorem 2.5 yields the following identity:*

$$\sum_{k=0}^{m}(-1)^k\binom{r}{k}\mathcal{L}^{(r)}_{m-k}=\mathcal{L}_m.$$

3. Generating function for hyper-Leonardo p-numbers

In this section, we provide the generating function for hyper-Leonardo p-numbers.

Lemma 3.1 (see [9]). *Let $\{s_n\}$ be a complex sequence satisfying the recurrence relation $s_n=s_{n-1}+s_{n-p-1}+r_n$ $(n>p)$, where r_n is a given complex sequence. The generating function $\varphi(t)$ of s_n is*

$$\varphi(t)=\sum_{n\geq 0}s_nt^n=\frac{H(t)+s_0-r_0+\sum_{i=1}^{p}t^i\left(s_i-s_{i-1}-r_i\right)}{1-t-t^{p+1}},$$

where $H(t)$ is the generating function of the sequence r_n.

Theorem 3.1. *The generating function of hyper-Leonardo p-numbers $\mathcal{L}^{(r)}_{p,n}$ is given by*

$$\Phi(t)=\sum_{n\geq 0}\mathcal{L}^{(r)}_{p,n}t^n=\frac{1-t+pt^{p+1}}{\left(1-t-t^{p+1}\right)(1-t)^{r+1}}.$$

Proof. Let $s_0=\mathcal{L}^{(r)}_{p,0}=\mathcal{L}_{p,0}$ and $s_n=\mathcal{L}^{(r)}_{p,n}$. From (6), we have

$$s_n=\mathcal{L}^{(r)}_{p,n-1}+\mathcal{L}^{(r)}_{p,n-p-1}+p\binom{n+r-p-1}{r}+\binom{n+r-1}{r-1}.$$

Using Lemma 3.1, we have $r_0=1$ and $r_n=p\binom{n+r-p-1}{r}+\binom{n+r-1}{r-1}$. It is clear that for $0\leq n\leq p$, $r_n=\binom{n+r-1}{r-1}$. The generating function of (r_n) is

$$G(t)=\frac{1-t+pt^{p+1}}{(1-t)^{r+1}}.$$

Thus, by Lemma 3.1, the generating function of the sequence (s_n) satisfies the following equation

$$\Phi(t)=\frac{1-1+\sum_{i=1}^{p}t^i\left(\mathcal{L}^{(r)}_{p,i}-\mathcal{L}^{(r)}_{p,i-1}-\binom{i+r-1}{r-1}\right)}{1-t-t^{p+1}}+\frac{1-t+pt^{p+1}}{\left(1-t-t^{p+1}\right)(1-t)^{r+1}}=\frac{1-t+pt^{p+1}}{\left(1-t-t^{p+1}\right)(1-t)^{r+1}}.$$

□

When we take $p=1$ in Theorem 3.1, we obtain the generating function of hyper-Leonardo numbers (see [8])

$$g(t)=\sum_{n\geq 0}\mathcal{L}^{(r)}_{n}t^n=\frac{1-t+t^2}{\left(1-t-t^2\right)(1-t)^{r+1}}=\frac{1-t+t^2}{\left(1-2t+t^3\right)(1-t)^{r}}.$$

4. Spectral norms of k-circulant matrices whose entries are hyper-Leonardo p-numbers

Let $n\geq 2$ be an integer and k be a nonzero complex number. The $n\times n$ k-circulant matrix C_k, with the first row $(c_0,c_1,\ldots,c_{n-1})$, is defined as follows:

$$C_k=\begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1}\\ kc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2}\\ kc_{n-2} & kc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ kc_2 & kc_3 & kc_4 & \cdots & c_0 & c_1\\ kc_1 & kc_2 & kc_3 & \cdots & kc_{n-1} & c_0 \end{pmatrix}.$$

For brevity, we write $C_k = Circ_{n,k}(c_0, c_1, \dots, c_{n-1})$. When $k = 1$, the k-circulant matrix reduce to the circulant matrix $C = Circ_n(c_0, c_1, \dots, c_{n-1})$. The circulant matrices are normal matrices [4], i.e., $AA^H = A^H A$, where A^H is the conjugate transpose matrix of A. The eigenvalues of C are computed as follows:

$$\lambda_s = \sum_{k=0}^{n-1} c_k \mu_s^k, \quad s = 0, 1, \dots, n-1, \tag{10}$$

where $\mu_s = \exp(-\frac{2\pi i}{n} s)$ and $i^2 = -1$ (see [4,6]). Let $A = (a_{ij})$ be any $m \times n$ complex matrix. The Euclidean norm and the spectral norm of the matrix A are respectively given by

$$\|A\|_E = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} \tag{11}$$

and

$$\|A\|_2 = \sqrt{\max_i \lambda_i(A^H A)}, \tag{12}$$

where $\lambda_i(A^H A)$'s are eigenvalues of $A^H A$. The following inequalities hold (see [5]):

$$\frac{1}{\sqrt{n}} \|A\|_E \le \|A\|_2 \le \|A\|_E \,, \tag{13}$$

$$\|A\|_2 \le \|A\|_E \le \sqrt{n} \|A\|_2 \,. \tag{14}$$

Lemma 4.1 (see [5]). *Let A be a normal matrix with eigenvalues $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$. The spectral norm of A is*

$$\|A\|_2 = \max_{0 \le i \le n-1} \lambda_i. \tag{15}$$

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ matrices. The Hadamard product of A and B is defined (see [7,10,14]) by

$$A \circ B = (a_{ij} b_{ij}).$$

Lemma 4.2 (see [5]). *If $A = (a_{ij})$ and $B = (b_{ij})$ are any $m \times n$-matrices, then*

$$\|A \circ B\|_2 \le \|A\|_2 \|B\|_2. \tag{16}$$

Lemma 4.3 (see [7]). *If $A = (a_{ij})$ and $B = (b_{ij})$ are any $m \times n$-matrices, then*

$$\|A \circ B\|_2 \le r_1(A) c_1(B), \tag{17}$$

where

$$r_1(A) = \max_{1 \le i \le m} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} \quad \textit{and} \quad c_1(B) = \max_{1 \le j \le n} \sqrt{\sum_{i=1}^{m} |b_{ij}|^2}.$$

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ and $p \times q$ matrices, respectively. The Kronecker product of A and B is defined by

$$A \otimes B = (a_{ij} B).$$

Lemma 4.4 ([5,7]). *Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices. Then*

$$\|A \otimes B\|_2 = \|A\|_2 \|B\|_2. \tag{18}$$

Let $\mathcal{N}_{p,k}$ and $\mathcal{T}_{p,k}$ be the k-circulant matrices, associated with hyper-Leonardo p-numbers, defined as

$$\mathcal{N}_{p,k} = Circ_{n,k}\left(\mathcal{L}_{p,0}^{(r)}, \mathcal{L}_{p,1}^{(r)}, \dots, \mathcal{L}_{p,n-1}^{(r)}\right) \quad \text{and} \quad \mathcal{T}_{p,k} = Circ_{n,k}\left(\mathcal{L}_{p,r}^{(0)}, \mathcal{L}_{p,r}^{(1)}, \dots, \mathcal{L}_{p,r}^{(n-1)}\right).$$

The next result provides the spectral norm of the circulant matrix $\mathcal{N}_{p,1}$.

Theorem 4.1. *For $n \ge 1$ and $r \ge 0$, the spectral norm of the matrix $\mathcal{N}_{p,1}$ is $\|\mathcal{N}_{p,1}\|_2 = \mathcal{L}_{p,n-1}^{(r+1)}$.*

Proof. By (10), the eigenvalues of $\mathcal{N}_{p,1}$ are of the form

$$\lambda_s = \sum_{j=0}^{n-1} \mathcal{L}_{p,j}^{(r)} \mu_s^j, \quad \text{for all } \ 0 \leq s \leq n-1.$$

For $s=0$, we have $\lambda_0 = \sum_{j=0}^{n-1} \mathcal{L}_{p,j}^{(r)}$. By using Definition 2.1, we get $\lambda_0 = \mathcal{L}_{p,n-1}^{(r+1)}$. Next, for $1 \leq s \leq n-1$, we have

$$|\lambda_s| = \left| \sum_{j=0}^{n-1} \mathcal{L}_{p,j}^{(r)} \mu_s^j \right| \leq \sum_{j=0}^{n-1} \left| \mathcal{L}_{p,j}^{(r)} \right| \left| \mu_s^j \right| \leq \sum_{j=0}^{n-1} \mathcal{L}_{p,j}^{(r)} = \lambda_0.$$

Since $\mathcal{N}_{p,1}$ is a normal matrix, we have $\|\mathcal{N}_{p,1}\|_2 = \mathcal{L}_{p,n-1}^{(r+1)}$. □

For example, note that $\|\mathcal{N}_{3,1}\|_2 = \mathcal{L}_{3,4}^{(r+1)} = 9, 19, 39, 74, 130, \ldots$ for $r = 0, 1, 2, 3, 4, \ldots$ and $n = 5$.

Remark 4.1. *For $p = 1$, the spectral norm of $\mathcal{N}_{p,1}$ associated with hyper-Leonardo numbers is given by $\|\mathcal{N}_{p,1}\|_2 = \mathcal{L}_{n-1}^{(r+1)}$.*

For example, note that $\|\mathcal{N}_{1,1}\|_2 = \mathcal{L}_{6}^{(r+1)} = 59, 130, 268, 520, 956, \ldots$ for $r = 0, 1, 2, 3, 4, \ldots$ and $n = 7$.

Corollary 4.1. *The Euclidean norm of the matrix $\mathcal{N}_{p,1}$ satisfies*

$$\mathcal{L}_{p,n-1}^{(r+1)} \leq \|\mathcal{N}_{p,1}\|_E \leq \sqrt{n} \mathcal{L}_{p,n-1}^{(r+1)}. \tag{19}$$

Proof. The result follows from Theorem 4.1 and the connection between spectral and Euclidean norms in (14). □

Corollary 4.2. *The sum of squares of hyper-Leonardo p-numbers satisfies*

$$\frac{1}{\sqrt{n}} \mathcal{L}_{p,n-1}^{(r+1)} \leq \sqrt{\sum_{j=0}^{n-1} \left(\mathcal{L}_{p,j}^{(r)} \right)^2} \leq \mathcal{L}_{p,n-1}^{(r+1)}. \tag{20}$$

Proof. The proof follows from the definition of the Euclidean norm (11) and Corollary 4.1. □

The next result gives the spectral norm of the circulant matrix $\mathcal{T}_{p,1}$.

Theorem 4.2. *For $n \geq 1$ and $r \geq p-1$, the spectral norm of the matrix $\mathcal{T}_{p,1}$ is $\|\mathcal{T}_{p,1}\|_2 = \mathcal{L}_{p,r+1}^{(n-1)} - (p+1)F_{p,r-p+1}$.*

Proof. By (10), the eigenvalues of $\mathcal{T}_{p,1}$ are of the form $\lambda_s = \sum_{j=0}^{n-1} \mathcal{L}_{p,r}^{(j)} \mu_s^j$. For $s = 0$, we have $\lambda_0 = \sum_{j=0}^{n-1} \mathcal{L}_{p,r}^{(j)}$. From (9), it follows that $\lambda_0 = \mathcal{L}_{p,r+1}^{(n-1)} - (p+1)F_{p,r-p+1}$. Now, for $1 \leq s \leq n-1$, we have

$$|\lambda_s| = \left| \sum_{j=0}^{n-1} \mathcal{L}_{p,r}^{(j)} \mu_s^j \right| \leq \sum_{j=0}^{n-1} \left| \mathcal{L}_{p,r}^{(j)} \right| \left| \mu_s^j \right| \leq \sum_{j=0}^{n-1} \mathcal{L}_{p,r}^{(j)} = \lambda_0.$$

Since $\mathcal{T}_{p,1}$ is a normal matrix, we have $\|\mathcal{T}_{p,1}\|_2 = \mathcal{L}_{p,r+1}^{(n-1)} - (p+1)F_{p,r-p+1}$. □

For example, note that $\|\mathcal{T}_{3,1}\|_2 = \mathcal{L}_{3,r+1}^{(5)} - 4F_{3,r-2} = 56, 126, 276, 570, 1124, \ldots$ for $r = 2, 3, 4, 5, 6, \ldots$ and $n = 6$.

Remark 4.2. *For $p = 1$, the spectral norm of $\mathcal{T}_{p,1}$ associated with hyper-Leonardo numbers is $\|\mathcal{T}_{p,1}\|_2 = \mathcal{L}_{r+1}^{(n-1)} - 2F_r$.*

For example, note that $\|\mathcal{T}_{1,1}\|_2 = \mathcal{L}_{r+1}^{(8)} - 2F_r = 9, 45, 183, 603, 1743, \ldots$ for $r = 0, 1, 2, 3, 4, \ldots$ and $n = 9$.

Corollary 4.3. *The Euclidean norm of the matrix $\mathcal{T}_{p,1}$ satisfies*

$$\mathcal{L}_{p,r+1}^{(n-1)} - (p+1)F_{p,r-p+1} \leq \|\mathcal{T}_{p,1}\|_E \leq \sqrt{n} \left(\mathcal{L}_{p,r+1}^{(n-1)} - (p+1)F_{p,r-p+1} \right). \tag{21}$$

Proof. The result follows from Theorem 4.2 and the connection between spectral and Euclidean norms in (14). □

Corollary 4.4. *The sum of squares of hyper-Leonardo p-numbers satisfies*

$$\frac{1}{\sqrt{n}} \left(\mathcal{L}_{p,r+1}^{(n-1)} - (p+1)F_{p,r-p+1} \right) \leq \sqrt{\sum_{j=0}^{n-1} \left(\mathcal{L}_{p,r}^{(j)} \right)^2} \leq \mathcal{L}_{p,r+1}^{(n-1)} - (p+1)F_{p,r-p+1}. \tag{22}$$

Proof. The result follows from the definition of the Euclidean norm (11) and Corollary 4.3. □

Corollary 4.5. *The spectral norm of the Hadamard product of $\mathcal{N}_{p,1}$ and $\mathcal{T}_{p,1}$ satisfies*

$$\|\mathcal{N}_{p,1} \circ \mathcal{T}_{p,1}\|_2 \leq \mathcal{L}_{p,n-1}^{(r+1)} \left(\mathcal{L}_{p,r+1}^{(n-1)} - (p+1)F_{p,r-p+1}\right).$$

The spectral norm of the Kronecker product of $\mathcal{N}_{p,1}$ and $\mathcal{T}_{p,1}$ satisfies $\|\mathcal{N}_{p,1} \otimes \mathcal{T}_{p,1}\|_2 = \mathcal{L}_{p,n-1}^{(r+1)} \left(\mathcal{L}_{p,r+1}^{(n-1)} - (p+1)F_{p,r-p+1}\right)$.

In the next result, we establish upper and lower bounds for the spectral norm of the k-circulant matrix $\mathcal{N}_{p,k}$.

Theorem 4.3. *Let $k \in \mathbb{C}$ and $\mathcal{N}_{p,k}$ be an $n \times n$ k-circulant matrix. Then*

(i). *For $|k| \geq 1$, it holds that $\frac{1}{\sqrt{n}}\mathcal{L}_{p,n-1}^{(r+1)} \leq \|\mathcal{N}_{p,k}\|_2 \leq \sqrt{(n-1)|k|^2+1}\mathcal{L}_{p,n-1}^{(r+1)}$.*

(ii). *For $|k| < 1$, it holds that $\frac{|k|}{\sqrt{n}}\mathcal{L}_{p,n-1}^{(r+1)} \leq \|\mathcal{N}_{p,k}\|_2 \leq \sqrt{n}\mathcal{L}_{p,n-1}^{(r+1)}$.*

Proof. Let

$$\mathcal{N}_{p,k} := \begin{pmatrix} \mathcal{L}_{p,0}^{(r)} & \mathcal{L}_{p,1}^{(r)} & \cdots & \mathcal{L}_{p,n-2}^{(r)} & \mathcal{L}_{p,n-1}^{(r)} \\ k\mathcal{L}_{p,n-1}^{(r)} & \mathcal{L}_{p,0}^{(r)} & \cdots & \mathcal{L}_{p,n-3}^{(r)} & \mathcal{L}_{p,n-2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k\mathcal{L}_{p,2}^{(r)} & k\mathcal{L}_{p,3}^{(r)} & \cdots & \mathcal{L}_{p,0}^{(r)} & \mathcal{L}_{p,1}^{(r)} \\ k\mathcal{L}_{p,1}^{(r)} & k\mathcal{L}_{p,2}^{(r)} & \cdots & k\mathcal{L}_{p,n-1}^{(r)} & \mathcal{L}_{p,0}^{(r)} \end{pmatrix}.$$

By the definition of the Euclidean norm, we have

$$\|\mathcal{N}_{p,k}\|_E = \sqrt{\sum_{j=0}^{n-1}(n-j)|c_j|^2 + \sum_{j=0}^{n-1} j|k|^2|c_j|^2} = \sqrt{\sum_{j=0}^{n-1}\left((n-j)+j|k|^2\right)\left(\mathcal{L}_{p,j}^{(r)}\right)^2}.$$

(i). For $|k| \geq 1$, using (20), we have

$$\|\mathcal{N}_{p,k}\|_E \geq \sqrt{\sum_{j=0}^{n-1}\left((n-j)+j\right)\left(\mathcal{L}_{p,j}^{(r)}\right)^2} = \sqrt{\sum_{j=0}^{n-1} n\left(\mathcal{L}_{p,j}^{(r)}\right)^2} \geq \mathcal{L}_{p,n-1}^{(r+1)}.$$

From (13), it follows that

$$\|\mathcal{N}_{p,k}\|_2 \geq \frac{1}{\sqrt{n}}\mathcal{L}_{p,n-1}^{(r+1)}.$$

Next, let $\mathcal{N}_{p,k} = A \circ B$, where

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ k & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k & k & \cdots & 1 & 1 \\ k & k & \cdots & k & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \mathcal{L}_{p,0}^{(r)} & \mathcal{L}_{p,1}^{(r)} & \cdots & \mathcal{L}_{p,n-2}^{(r)} & \mathcal{L}_{p,n-1}^{(r)} \\ \mathcal{L}_{p,n-1}^{(r)} & \mathcal{L}_{p,0}^{(r)} & \cdots & \mathcal{L}_{p,n-3}^{(r)} & \mathcal{L}_{p,n-2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{L}_{p,2}^{(r)} & \mathcal{L}_{p,3}^{(r)} & \cdots & \mathcal{L}_{p,0}^{(r)} & \mathcal{L}_{p,1}^{(r)} \\ \mathcal{L}_{p,1}^{(r)} & \mathcal{L}_{p,2}^{(r)} & \cdots & \mathcal{L}_{p,n-1}^{(r)} & \mathcal{L}_{p,0}^{(r)} \end{pmatrix}.$$

Then

$$r_1(A) = \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|a_{ij}|^2} = \sqrt{\sum_{j=1}^{n}|a_{nj}|^2} = \sqrt{(n-1)|k|^2+1}$$

and

$$c_1(B) = \max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}|b_{ij}|^2} = \sqrt{\sum_{i=1}^{n}|b_{in}|^2} = \sqrt{\sum_{i=0}^{n-1}\left(\mathcal{L}_{p,i}^{(r)}\right)^2}.$$

Using (17) and (20), we obtain $\|\mathcal{N}_{p,k}\|_2 \leq r_1(A)c_1(B) \leq \sqrt{(n-1)|k|^2+1}\mathcal{L}_{p,n-1}^{(r+1)}$.

(ii). For $|k| < 1$, using (20), we have

$$\|\mathcal{N}_{p,k}\|_E \geq \sqrt{\sum_{j=0}^{n-1}\left((n-j)|k|^2+j|k|^2\right)\left(\mathcal{L}_{p,j}^{(r)}\right)^2} = |k|\sqrt{\sum_{j=0}^{n-1} n\left(\mathcal{L}_{p,j}^{(r)}\right)^2} \geq |k|\mathcal{L}_{p,n-1}^{(r+1)}.$$

From (13), we obtain

$$\|\mathcal{N}_{p,k}\|_2 \geq \frac{|k|}{\sqrt{n}}\mathcal{L}_{p,n-1}^{(r+1)}.$$

Now, let $\mathcal{N}_{p,k} = A \circ B$, where

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ k & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k & k & \cdots & 1 & 1 \\ k & k & \cdots & k & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} \mathcal{L}_{p,0}^{(r)} & \mathcal{L}_{p,1}^{(r)} & \cdots & \mathcal{L}_{p,n-2}^{(r)} & \mathcal{L}_{p,n-1}^{(r)} \\ \mathcal{L}_{p,n-1}^{(r)} & \mathcal{L}_{p,0}^{(r)} & \cdots & \mathcal{L}_{p,n-3}^{(r)} & \mathcal{L}_{p,n-2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{L}_{p,2}^{(r)} & \mathcal{L}_{p,3}^{(r)} & \cdots & \mathcal{L}_{p,0}^{(r)} & \mathcal{L}_{p,1}^{(r)} \\ \mathcal{L}_{p,1}^{(r)} & \mathcal{L}_{p,2}^{(r)} & \cdots & \mathcal{L}_{p,n-1}^{(r)} & \mathcal{L}_{p,0}^{(r)} \end{pmatrix}.$$

Then

$$r_1(A) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |a_{1j}|^2} = \sqrt{n}$$

and

$$c_1(B) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{i=1}^{n} |b_{in}|^2} = \sqrt{\sum_{i=0}^{n-1} \left(\mathcal{L}_{p,i}^{(r)}\right)^2}.$$

Using (17) and (20), we obtain $\|\mathcal{N}_{p,k}\|_2 \leq r_1(A)c_1(B) \leq \sqrt{n}\mathcal{L}_{p,n-1}^{(r+1)}$.

□

Remark 4.3. *If we take $p = 1$ in Theorem 4.3, then the following statements hold:*

(i). *For $|k| \geq 1$, we have $\frac{1}{\sqrt{n}}\mathcal{L}_{n-1}^{(r+1)} \leq \|\mathcal{N}_{1,k}\|_2 \leq \sqrt{(n-1)|k|^2+1}\mathcal{L}_{n-1}^{(r+1)}$.*

(ii). *For $|k| < 1$, we have $\frac{|k|}{\sqrt{n}}\mathcal{L}_{n-1}^{(r+1)} \leq \|\mathcal{N}_{1,k}\|_2 \leq \sqrt{n}\mathcal{L}_{n-1}^{(r+1)}$.*

The next theorem provides upper and lower bounds for the spectral norm of the k-circulant matrix $\mathcal{T}_{p,k}$.

Theorem 4.4. *Let $k \in \mathbb{C}$ and $\mathcal{T}_{p,k}$ be an $n \times n$ k-circulant matrix.*

(i). *For $|k| \geq 1$, we have $\frac{1}{\sqrt{n}}\left(\mathcal{L}_{p,r+1}^{(n-1)} - (p+1)F_{p,r-p+1}\right) \leq \|\mathcal{T}_{p,k}\|_2 \leq \sqrt{(n-1)|k|^2+1}\left(\mathcal{L}_{p,r+1}^{(n-1)} - (p+1)F_{p,r-p+1}\right)$.*

(ii). *For $|k| < 1$, we have $\frac{|k|}{\sqrt{n}}\left(\mathcal{L}_{p,r+1}^{(n-1)} - (p+1)F_{p,r-p+1}\right) \leq \|\mathcal{T}_{p,k}\|_2 \leq \sqrt{n}\left(\mathcal{L}_{p,r+1}^{(n-1)} - (p+1)F_{p,r-p+1}\right)$.*

Proof. The proof is similar to that of Theorem 4.3. □

Remark 4.4. *The substitution $p = 1$ in Theorem 4.4 yields the following statements:*

(i). *For $|k| \geq 1$, we have $\frac{1}{\sqrt{n}}\left(\mathcal{L}_{r+1}^{(n-1)} - 2F_r\right) \leq \|\mathcal{T}_{1,k}\|_2 \leq \sqrt{(n-1)|k|^2+1}\left(\mathcal{L}_{r+1}^{(n-1)} - 2F_r\right)$.*

(ii). *For $|k| < 1$, we have $\frac{|k|}{\sqrt{n}}\left(\mathcal{L}_{r+1}^{(n-1)} - 2F_r\right) \leq \|\mathcal{T}_{1,k}\|_2 \leq \sqrt{n}\left(\mathcal{L}_{r+1}^{(n-1)} - 2F_r\right)$.*

From (16) and (18), we obtain the next results.

Corollary 4.6. *For the spectral norm of the Hadamard product of $\mathcal{N}_{p,k}$ and $\mathcal{T}_{p,k}$, the following statements hold:*

(i). *For $|k| \geq 1$, the inequality $\|\mathcal{N}_{p,k} \circ \mathcal{T}_{p,k}\|_2 \leq \left((n-1)|k|^2+1\right)\mathcal{L}_{p,n-1}^{(r+1)}\left(\mathcal{L}_{p,r+1}^{(n-1)} - (p+1)F_{p,r-p+1}\right)$ holds.*

(ii). *For $|k| < 1$, the inequality $\|\mathcal{N}_{p,k} \circ \mathcal{T}_{p,k}\|_2 \leq n\mathcal{L}_{p,n-1}^{(r+1)}\left(\mathcal{L}_{p,r+1}^{(n-1)} - (p+1)F_{p,r-p+1}\right)$ holds.*

Remark 4.5. *If we take $p = 1$ in Corollary 4.6, we obtain the following statements:*

(i). *For $|k| \geq 1$, the inequality $\|\mathcal{N}_{1,k} \circ \mathcal{T}_{1,k}\|_2 \leq \left((n-1)|k|^2+1\right)\mathcal{L}_{n-1}^{(r+1)}\left(\mathcal{L}_{r+1}^{(n-1)} - 2F_r\right)$ holds.*

(ii). *For $|k| < 1$, the inequality $\|\mathcal{N}_{1,k} \circ \mathcal{T}_{1,k}\|_2 \leq n\mathcal{L}_{n-1}^{(r+1)}\left(\mathcal{L}_{r+1}^{(n-1)} - 2F_r\right)$ holds.*

Corollary 4.7. *For the spectral norms of the Kronecker product of $\mathcal{N}_{p,k}$ and $\mathcal{T}_{p,k}$, the following statements hold:*

(i). *For $|k| \geq 1$, we have*

$$\frac{1}{n}\mathcal{L}_{p,n-1}^{(r+1)}\left(\mathcal{L}_{p,r+1}^{(n-1)} - (p+1)F_{p,r-p+1}\right) \leq \|\mathcal{N}_{p,k} \otimes \mathcal{T}_{p,k}\|_2 \leq \left((n-1)|k|^2+1\right)\mathcal{L}_{p,n-1}^{(r+1)}\left(\mathcal{L}_{p,r+1}^{(n-1)} - (p+1)F_{p,r-p+1}\right).$$

(ii). *For $|k| < 1$, we have*

$$\frac{|k|^2}{n}\mathcal{L}_{p,n-1}^{(r+1)}\left(\mathcal{L}_{p,r+1}^{(n-1)} - (p+1)F_{p,r-p+1}\right) \leq \|\mathcal{N}_{p,k} \otimes \mathcal{T}_{p,k}\|_2 \leq n\mathcal{L}_{p,n-1}^{(r+1)}\left(\mathcal{L}_{p,r+1}^{(n-1)} - (p+1)F_{p,r-p+1}\right).$$

Remark 4.6. *If we take $p = 1$ in Corollary 4.7, then we obtain the following statements:*

(i). *For $|k| \geq 1$, we have* $\frac{1}{n}\mathcal{L}_{n-1}^{(r+1)}\left(\mathcal{L}_{r+1}^{(n-1)} - 2F_r\right) \leq \|\mathcal{N}_{1,k} \otimes \mathcal{T}_{1,k}\|_2 \leq \left((n-1)|k|^2+1\right)\mathcal{L}_{n-1}^{(r+1)}\left(\mathcal{L}_{r+1}^{(n-1)} - 2F_r\right)$.

(ii). *For $|k| < 1$, we have* $\frac{|k|^2}{n}\mathcal{L}_{n-1}^{(r+1)}\left(\mathcal{L}_{r+1}^{(n-1)} - 2F_r\right) \leq \|\mathcal{N}_{1,k} \otimes \mathcal{T}_{1,k}\|_2 \leq n\mathcal{L}_{n-1}^{(r+1)}\left(\mathcal{L}_{r+1}^{(n-1)} - 2F_r\right)$.

5. Conclusion

In this work, we have introduced a new family of generalized Leonardo numbers, namely hyper-Leonardo p-numbers. We have studied several properties of hyper-Leonardo p-numbers, including recurrence relations, explicit form, summation formulas, and the generating function. In addition, we have established some bounds for spectral norms of a specific form of k-circulant matrices associated with hyper-Leonardo p-numbers. In the particular case when $p = 1$, we have obtained bounds for spectral norms of k-circulant matrices associated with ordinary hyper-Leonardo numbers. Furthermore, we have deduced some bounds for spectral norms of Kronecker and Hadamard products of these matrices. Exploring spectral norms of k-circulant matrices associated with incomplete Leonardo p-numbers seems to be an interesting future work related to the present study.

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