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Research Article

Two new fibonomial difference sequence spaces and related dual properties

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Abstract

This paper introduces two new fibonomial difference sequence spaces that are inspired by the Fibonacci calculus (or Golden calculus). It is shown that both of these spaces are complete linear metric spaces. Also, it is demonstrated that one of these two spaces is linearly isomorphic to the set of all bounded sequences and the other one is linearly isomorphic to the set of all sequences constituting $p-$absolutely convergent series. Furthermore, the Schauder basis and the $\alpha-$, $\beta-$, $\gamma-$duals of these spaces are determined.

Keywords: fibonomial sequence spaces; Schauder basis; matrix domain; $\alpha-$, $\beta-$, $\gamma-$duals.

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1. Introduction

The theory of matrix transformations deals with discovering necessary and sufficient conditions on the entries of a matrix to map a sequence space X into a sequence space Y. Indeed, this is a natural generalization of the problem of characterizing all summability methods given by infinite matrices that preserve convergence. Details about the matrix transformations and summability theory, and the domain of triangular matrices in normed sequence spaces, can be found in [4].

A sequence space is a linear subspace of the set of all real-valued sequences ω. The set ℓ_∞ of all bounded sequences, the set c of all convergent sequences, the set c_0 of all convergent-to-zero sequences and the set ℓ_p of all sequences constituting $p-$absolutely convergent series are well-known examples of sequence spaces. These are Banach spaces with the following norms

$$\|x\|_{\ell_\infty} = \|x\|_c = \|x\|_{c_0} = \sup_{k\in\mathbb{N}} |x_k|$$

and

$$\|x\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p}.$$

For any sequence space $X \in \{\ell_\infty, c, c_0\}$, the following difference space was introduced by Kizmaz [12]:

$$X(\Delta) = \{x_k \in \omega : (\Delta x_k) \in X\},$$

where $\Delta x_k = x_k - x_{k+1}$ for each k in the set of positive integers $\mathbb{N}$.

Let $T = (t_{nk})$ be an infinite matrix with real entries t_{nk} and T_n be the sequence in the nth row of the matrix T for each $n \in \mathbb{N}$. The $T-$transform of a sequence $x = (x_k) \in \omega$ is the sequence Tx obtained by the usual matrix product and its entries are stated as

$$(Tx)_n = \sum_k t_{nk} x_k$$

provided that the series is convergent for each $n \in \mathbb{N}$. The matrix T is said to be a matrix mapping from a sequence space X to a sequence space Y if the sequence Tx exists and $Tx \in Y$ for all $x \in X$. The collection of all infinite matrices from X to Y is denoted by (X, Y). The multiplier space of X and Y is the set $S(X, Y)$ defined by $S(X, Y) = \{u \in \omega : zu \in Y \text{ for all } z \in X\}$. Using this notation, the $\alpha-$dual, $\beta-$dual and $\gamma-$dual of a sequence space X are defined by

$$X^\alpha = S(X, \ell_1), \quad X^\beta = S(X, cs) \text{ and } X^\gamma = S(X, bs),$$

where cs and bs correspond to the spaces of sequences with convergent and bounded series, respectively.

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Recall that the domain of the infinite matrix T in the space X is denoted by $X_T = \{x \in \omega : Tx \in X\}$. During the past two decades, many researchers have been interested in the concept of domains of special triangular matrices. One of them is due to Euler means of order r, denoted by $E^r = (e^r_{nk})$, where

$$e^r_{nk} = \begin{cases} \binom{n}{k} (1-r)^{n-k} r^k, & \text{if } 0 \le k \le n; \\ 0, & \text{if } k > n; \end{cases}$$

for all $k, n \in \mathbb{N}$. The Euler sequence spaces e^r_p and e^r_∞ were defined by Altay, Başar and Mursaleen [2]:

$$e^r_p = \left\{ x = (x_k) \in \omega : \sum_n \left| \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k \right|^p < \infty \right\}$$

and

$$e^r_\infty = \left\{ x = (x_k) \in \omega : \sup_n \left| \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k \right| < \infty \right\}.$$

Altay and Polat [3] presented the following sequence spaces:

$$e^r_0 (\bigtriangledown) = \left\{ x = (x_k) \in \omega : \bigtriangledown (x_k) \in e^r_p \right\},$$

$$e^r_c (\bigtriangledown) = \left\{ x = (x_k) \in \omega : \bigtriangledown (x_k) \in e^r_c \right\}$$

and

$$e^r_\infty (\bigtriangledown) = \left\{ x = (x_k) \in \omega : \bigtriangledown (x_k) \in e^r_\infty \right\},$$

where $\bigtriangledown (x_k) = x_k - x_{k-1}$ for each $k \in \mathbb{N}$ and any term with a negative subscript equals naught.

The approach of constructing new sequence spaces by employing the Euler matrix via the matrix domain of a particular limitation method has recently been considered in [1, 10, 11, 18, 21] and some topological and geometric properties have been investigated. Bişgin [5, 6] offered another type of generalization of the Euler sequence spaces. For instance, he gave the binomial sequence spaces

$$b^{r,s}_p = \left\{ x = (x_k) \in \omega : \sum_n \left| \frac{1}{(s+r)^n} \binom{n}{k} r^k s^{n-k} x_k \right|^p < \infty \right\}$$

and

$$b^{r,s}_\infty = \left\{ x = (x_k) \in \omega : \sup_n \left| \frac{1}{(s+r)^n} \binom{n}{k} r^k s^{n-k} x_k \right| < \infty \right\},$$

by means of the binomial matrix $B^{r,s} = (b^{r,s}_{nk})$, with

$$b^{r,s}_{nk} = \begin{cases} \frac{1}{(s+r)^n} \binom{n}{k} r^k s^{n-k}, & \text{if } 0 \le k \le n; \\ 0, & \text{if } k > n; \end{cases}$$

and computed some special duals and Schauder basis, and revealed their geometric features. Several types of binomial sequence spaces and their generalizations can be found in the studies [7, 15–17, 22, 24, 25].

Now, recall the concept of a fibonomial coefficient, which is an analog of a binomial coefficient. Fibonomial coefficients are defined by means of Fibonacci numbers as

$$\binom{n}{k}_F = \frac{F_n!}{F_k! F_{n-k}!}, \quad \text{for } n \ge k \ge 0,$$

where

$$F_n! = F_n F_{n-1} \dots F_1$$

denotes the $F-$factorial with the Fibonacci sequence $(F_n)_{n \ge 0}$, given by $F_{n+2} = F_{n+1} + F_n$ such that $F_0 = 0$ and $F_1 = 1$. Note that

$$\binom{n}{0}_F = 1 \quad \text{and} \quad \binom{n}{k}_F = 0 \quad \text{for } n < k.$$

The followings are some properties (cf. [14]) for fibonomial coefficients:

$$\binom{n}{k}_F = \binom{n}{n-k}_F,$$
$$\binom{n}{k}_F \binom{k}{i}_F = \binom{n}{i}_F \binom{n-i}{k-i}_F,$$
$$(x+y)_F^n = \sum_{k=0}^{n} \binom{n}{k}_F x^k y^{n-k} \quad \text{(Fibonomial Theorem)}.$$

For additional properties and applications of Fibonacci calculus (or Golden calculus), the interested readers are referred to the studies [13, 19, 20].

Quite recently, Dağlı and Yaying [8, 9] introduced the fibonomial matrix $B^{r,s,F} = \left(b_{nk}^{r,s,F}\right)$, with

$$b_{nk}^{r,s,F} = \begin{cases} \frac{1}{(s+r)_F^n} \binom{n}{k}_F r^k s^{n-k}, & \text{if } 0 \le k \le n; \\ 0, & \text{if } k > n; \end{cases}$$

where s and r are nonzero real numbers such that $s + r \neq 0$. It was shown that the matrix $B^{r,s,F} = \left(b_{nk}^{r,s,F}\right)$ satisfies the following regularity conditions for $rs > 0$:

(i) $\left\|B^{r,s,F}\right\| < \infty$,

(ii) $\lim_{n\to\infty} b_{nk}^{r,s,F} = 0$ for each k,

(iii) $\lim_{n\to\infty} \sum_k b_{nk}^{r,s,F} = 1$.

Here and henceforth, we assume that $rs > 0$ unless otherwise stated.

By taking into consideration the fibonomial matrix, the following fibonomial sequence spaces were defined and it was shown that these spaces are linearly isomorphic to ℓ_p and ℓ_∞:

$$b_p^{r,s,F} = \left\{ x = (x_k) \in \omega : \sum_n \left| \frac{1}{(s+r)_F^n} \binom{n}{k}_F r^k s^{n-k} x_k \right|^p < \infty \right\}$$

and

$$b_\infty^{r,s,F} = \left\{ x = (x_k) \in \omega : \sup_n \left| \frac{1}{(s+r)_F^n} \binom{n}{k}_F r^k s^{n-k} x_k \right| < \infty \right\}.$$

Besides, the Schauder basis and the $\alpha-$, $\beta-$, $\gamma-$duals of these spaces were constructed and some matrix classes were characterized. Finally, some geometric structures of the space $b_p^{r,s,F}$ were investigated.

In the present paper, two new normed difference spaces $b_p^{r,s,F}(\triangledown)$ and $b_\infty^{r,s,F}(\triangledown)$ of the fibonomial sequence are considered whose $B^{r,s,F}(\triangledown)-$transforms (see (1), below) are in the spaces ℓ_p and ℓ_∞, respectively. Also, it is shown that these new spaces are linearly isomorphic to the spaces ℓ_p and ℓ_∞, respectively. Moreover, the Schauder basis and the $\alpha-$dual, $\beta-$dual and $\gamma-$dual for these spaces are determined.

2. Fibonomial difference sequence spaces

In this section, two new normed difference spaces $b_p^{r,s,F}(\triangledown)$ and $b_\infty^{r,s,F}(\triangledown)$ are introduced and it is demonstrated that these spaces are linearly isomorphic to the spaces ℓ_p and ℓ_∞. The fibonomial difference sequence spaces $b_p^{r,s,F}(\triangledown)$ and $b_\infty^{r,s,F}(\triangledown)$ are defined as

$$b_p^{r,s,F}(\triangledown) = \left\{ x = (x_k) \in \omega : \triangledown(x_k) \in b_p^{r,s,F} \right\}$$

and

$$b_\infty^{r,s,F}(\triangledown) = \left\{ x = (x_k) \in \omega : \triangledown(x_k) \in b_\infty^{r,s,F} \right\}.$$

Consider the transformation

$$y_n = \left(B^{r,s,F} \triangledown (x_k)\right)_n = \frac{1}{(s+r)_F^n} \sum_{k=0}^{n} \binom{n}{k}_F r^k s^{n-k} \triangledown (x_k), \tag{1}$$

which we call the $B^{r,s,F}(\triangledown)-$transform of the sequence $x = (x_k)$.

Theorem 2.1. *The fibonomial difference sequence spaces* $b_p^{r,s,F}(\triangledown)$ *and* $b_\infty^{r,s,F}(\triangledown)$ *are complete linear metric spaces with the norms*

$$f_{b_p^{r,s,F}(\triangledown)}(x) = \|y\|_{\ell_p} = \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{1/p}$$

and

$$f_{b_\infty^{r,s,F}(\triangledown)}(x) = \|y\|_{\ell_\infty} = \sup_{n\in\mathbb{N}} |y_n|,$$

where $1 \le p < \infty$ *and* y_n *is the* $B^{r,s,F}(\triangledown)$*−transform of a sequence* x.

Proof. Only the proof for the space $b_p^{r,s,F}(\triangledown)$ is given because the proof for the other space is similar to this one. The linearity of the space follows from the routine verification. For $\alpha \in \mathbb{R}$, it is clear that $f_{b_p^{r,s,F}(\triangledown)}(\alpha x) = |\alpha| f_{b_p^{r,s,F}(\triangledown)}(x)$ and $f_{b_p^{r,s,F}(\triangledown)}(x) = 0$ if and only if $x = \theta$ for all $x \in b_p^{r,s,F}(\triangledown)$ with the zero element θ in $b_p^{r,s,F}$. If $x, z \in b_p^{r,s,F}(\triangledown)$ then

$$\begin{aligned} f_{b_p^{r,s,F}(\triangledown)}(x+z) &= \left(\sum_n \left|\left(B^{r,s,F}\left[\triangledown(x_k+z_k)\right]\right)_n\right|^p\right)^{1/p} \\ &\le \left(\sum_n \left|\left(B^{r,s,F}\left[\triangledown(x_k)\right]\right)_n\right|^p\right)^{1/p} + \left(\sum_n \left|\left(B^{r,s,F}\left[\triangledown(z_k)\right]\right)_n\right|^p\right)^{1/p} \\ &= f_{b_p^{r,s,F}(\triangledown)}(x) + f_{b_p^{r,s,F}(\triangledown)}(z), \end{aligned}$$

which concludes that $f_{b_p^{r,s,F}(\triangledown)}$ is a norm on the space $b_p^{r,s,F}(\triangledown)$. Now, for $x_m = (x_{m_k})_{k=1}^{\infty} \in b_p^{r,s,F}(\triangledown)$ (each $m \in \mathbb{N}$), if we take x_m as a Cauchy sequence in $b_p^{r,s,F}$, then, for a given $\varepsilon > 0$, there exists an integer $m_0(\varepsilon) \in \mathbb{N}$ such that $f_{b_p^{r,s,F}(\triangledown)}(x_m - x_l) < \varepsilon$, for all $m, l \ge m_0(\varepsilon)$. Hence, for $m, l \ge m_0(\varepsilon)$ and each $k \in \mathbb{N}$, one has

$$\left|\left(B^{r,s,F}\left[\triangledown(x_{m_k} - x_{l_k})\right]\right)_n\right| \le \left(\sum_n \left|\left(B^{r,s,F}\left[\triangledown(x_{m_k} - x_{l_k})\right]\right)_n\right|^p\right)^{1/p} < \varepsilon,$$

which means that $\left(B^{r,s,F}\left[\triangledown(x_{m_k})\right]\right)_{m=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. From the completeness of the set of real numbers, it follows that $B^{r,s,F}\left[\triangledown(x_{m_k})\right] \to B^{r,s,F}\left[\triangledown(x_k)\right]$ as $m \to \infty$ for each $k \in \mathbb{N}$. Thus, observe that

$$\sum_{n=0}^{j} \left|\left(B^{r,s,F}\left[\triangledown(x_{m_k} - x_{l_k})\right]\right)_n\right| \le f_{b_p^{r,s,F}(\triangledown)}(x_m - x_l) < \varepsilon, \tag{2}$$

for $m > m_0(\varepsilon)$. Letting i and $l \to \infty$ yields that $f_{b_p^{r,s,F}(\triangledown)}(x_m - x) \to 0$ due to (2). So, it can be readily written that

$$f_{b_p^{r,s,F}(\triangledown)}(x) = f_{b_p^{r,s,F}(\triangledown)}(x_m - x) + f_{b_p^{r,s,F}(\triangledown)}(x_m),$$

i.e. $x \in b_p^{r,s,F}(\triangledown)$. Consequently, the space $b_p^{r,s,F}(\triangledown)$ is complete. □

Theorem 2.2. *The fibonomial difference sequence spaces* $b_p^{r,s,F}(\triangledown)$ *and* $b_\infty^{r,s,F}(\triangledown)$ *are linearly isomorphic to* ℓ_p *and* ℓ_∞*, respectively.*

Proof. We prove the theorem for the space $b_p^{r,s,F}(\triangledown)$ since the proof of the other space follows analogously. We provide the existence of a linear transformation between the spaces $b_p^{r,s,F}(\triangledown)$ and ℓ_p that is injective, subjective, and norm-preserving. For any $x \in b_p^{r,s,F}(\triangledown)$, let $L : b_p^{r,s,F}(\triangledown) \to \ell_p$ be a transformation such that $L(x) = B^{r,s,F} \triangledown (x_k)$. The linearity of L is obvious due to the linearity of a matrix transformation. The transformation L is injective by the following fact: if $L(x) = \theta$ then $x = \theta$. For any sequence $y = (y_n) \in \ell_p$, denote the sequence $x = (x_k)$ for $k \in \mathbb{N}$ by

$$x_k = \sum_{i=0}^{k} (s+r)_F^i \sum_{j=i}^{k} \binom{j}{i}_F r^{-j} (-s)^{j-i} y_i, \tag{3}$$

then, we have

$$\begin{aligned} f_{b_p^{r,s,F}(\triangledown)}(x) &= \left\|\left(B^{r,s,F} \triangledown (x_k)\right)_n\right\|_{\ell_p} \\ &= \left(\sum_{n=1}^{\infty} \left|\frac{1}{(s+r)_F^n} \sum_{k=0}^{n} \binom{n}{k}_F r^k s^{n-k} \triangledown (x_k)\right|^p\right)^{1/p} \\ &= \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{1/p} = \|y\|_{\ell_p} = \|L(x)\|_{\ell_p} < \infty. \end{aligned}$$

Thus, L is norm-preserving and $x \in b_p^{r,s,F}$. Consequently, L is surjective, which concludes the proof. □

3. The Schauder basis and the $\alpha-$, $\beta-$, $\gamma-$duals

The aim of this section is to calculate the Schauder basis and the $\alpha-$, $\beta-$, $\gamma-$duals of the fibonomial difference sequence spaces $b_p^{r,s,F}(\bigtriangledown)$ and $b_\infty^{r,s,F}(\bigtriangledown)$.

First, we recall the definition of the Schauder basis. The sequence (δ_n) is said to be the Schauder basis for the space X if for any $x \in X$, there exists a unique sequence of scalars τ_n such that

$$\left\| x - \sum_{k=0}^{n} \tau_k \delta_k \right\| \to 0, \quad \text{as } n \to \infty,$$

for a normed space $(X, \|.\|)$ and a sequence δ_n in X. Then, we write

$$x = \sum_{k=0}^{\infty} \tau_k \delta_k.$$

Theorem 3.1. *Let $\mu_k(r,s,F) = \left\{B^{r,s,F} \bigtriangledown (x_j)\right\}_k$ be given for all $k \in \mathbb{N}$. Let the sequence $s^{(k)}(r,s,F) = \left\{s_j^{(k)}(r,s,F)\right\}_{j\in\mathbb{N}}$ be denoted as the elements of the fibinomial difference sequence space $b_p^{r,s,F}(\bigtriangledown)$ by*

$$s_j^{(k)}(r,s,F) = \begin{cases} (s+r)_F^k \sum\limits_{v=k}^{j} \binom{v}{k}_F r^{-v}(-s)^{v-k}, & \text{if } 0 \le k \le j; \\ 0, & \text{if } k > j. \end{cases}$$

Then, the sequence $\left\{s^{(0)}(r,s,F), s^{(1)}(r,s,F), \dots\right\}$ is a basis for the space $b_p^{r,s,F}(\bigtriangledown)$ and any $x = (x_j)$ in $b_p^{r,s,F}(\bigtriangledown)$ is uniquely determined as

$$x = \sum_k \mu_k(r,s,F)\, s^{(k)}(r,s,F),$$

where $1 \le p < \infty$.

Proof. Given any $x = (x_k) \in b_p^{r,s,F}(\bigtriangledown)$ for $1 \le p < \infty$. For every non-negative integer m, consider

$$x^{[m]} = \sum_{k=0}^{m} \mu_k(r,s,F)\, s^{(k)}(r,s,F).$$

By virtue of the linearity of $B^{r,s,F}(\bigtriangledown)$, we have

$$B^{r,s,F}\left(\bigtriangledown x_j^{[m]}\right) = \sum_{k=0}^{m} \mu_k(r,s,F)\, B^{r,s,F}\left(\bigtriangledown s_j^{(k)}(r,s,F)\right) = \sum_{k=0}^{m} \mu_k(r,s,F)\, e^{(k)}$$

and

$$\left[B^{r,s,F}\left(\bigtriangledown\left(x_j - x_j^{[m]}\right)\right)\right]_k = \begin{cases} \left[B^{r,s,F}(\bigtriangledown x_j)\right]_k, & \text{if } k > m; \\ 0, & \text{if } 0 \le k \le m; \end{cases}$$

for all $k, m \in \mathbb{N}$. Now, for any given $\varepsilon > 0$, there exists a non-negative integer m_0 such that

$$\sum_{k=m_0+1}^{\infty} \left|\left[B^{r,s,F}(\bigtriangledown x_j)\right]_k\right|^p \le \left(\frac{\varepsilon}{2}\right)^p$$

for all $k \ge m_0$. So, we have

$$\begin{aligned} f_{b_p^{r,s,F}(\bigtriangledown)}\left(x - x^{[m]}\right) &= \left(\sum_{k=m+1}^{\infty} \left|\left[B^{r,s,F}(\bigtriangledown x_j)\right]_k\right|^p\right)^{1/p} \\ &\le \left(\sum_{k=m_0+1}^{\infty} \left|\left[B^{r,s,F}(\bigtriangledown x_j)\right]_k\right|^p\right)^{1/p} \\ &\le \frac{\varepsilon}{2} < \varepsilon, \quad \text{for all } m \ge m_0, \end{aligned}$$

which implies that

$$x = \sum_k \mu_k(r,s,F)\, s^{(k)}(r,s,F),$$

as desired.

For the uniqueness of the considered representation, suppose that

$$x = \sum_k \mu'_k (r,s,F)\, s^{(k)} (r,s,F)$$

is another representation of x. Then, it is readily seen for every $n \in \mathbb{N}$ that

$$\begin{aligned}\left[B^{r,s,F} (\triangledown x_j)\right]_k &= \sum_k \mu'_k (r,s,F) \left[B^{r,s,F} \left(\triangledown s^{(k)} (r,s,F)\right)\right]_k \\ &= \sum_k \mu'_k (r,s,F) (e_k)_k \\ &= \mu'_k (r,s,F),\end{aligned}$$

which contradicts the representation $\left[B^{r,s,F} (\triangledown x_j)\right]_k = \mu_k (r,s,F)$ for every $k \in \mathbb{N}$. So, the proof is completed. □

Corollary 3.1. *The fibonomial difference sequence space $b_p^{r,s,F} (\triangledown)$ is separable for $1 \leq p < \infty$.*

Let us continue with the following lemma, which is useful for identifying the $\alpha-$dual, $\beta-$dual and $\gamma-$dual of the fibonomial difference sequence spaces $b_p^{r,s,F} (\triangledown)$ and $b_\infty^{r,s,F} (\triangledown)$. Denote by $\mathcal{F}$ the family of all finite subsets of $\mathbb{N}$, and $\frac{1}{p} + \frac{1}{q} = 1$ for $1 < p \leq \infty$.

Lemma 3.1 (see [23]). *$T = (t_{nk}) \in (\ell_1, \ell_1)$ if and only if*

$$\sup_{k\in\mathbb{N}} \sum_n |t_{nk}| < \infty.$$

$T = (t_{nk}) \in (\ell_1, \ell_\infty)$ if and only if

$$\sup_{n,k\in\mathbb{N}} |t_{nk}| < \infty. \tag{4}$$

$T = (t_{nk}) \in (\ell_1, c)$ if and only if (4) holds and

$$\lim_{n\to\infty} t_{nk} \text{ exists} \tag{5}$$

for each $k \in \mathbb{N}$. $T = (t_{nk}) \in (\ell_p, \ell_\infty)$ if and only if

$$\sup_n \sum_k |t_{nk}|^q < \infty, \tag{6}$$

where $1 < p < \infty$. $T = (t_{nk}) \in (\ell_p, c)$ if and only if (5) and (6) hold for $1 < p < \infty$. $T = (t_{nk}) \in (\ell_p, \ell_1)$ if and only if

$$\sup_{K\in\mathcal{F}} \sum_k \left| \sum_{n\in K}^{\infty} t_{nk} \right|^q < \infty \quad \text{for } 1 < p < \infty.$$

$T = (t_{nk}) \in (\ell_\infty, \ell_\infty) = (c, \ell_\infty)$ if and only if (6) holds for $q = 1$. $T = (t_{nk}) \in (\ell_\infty, c)$ if and only if (5) holds and

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} |t_{nk}| = \sum_{k=0}^{\infty} \left| \lim_{n\to\infty} t_{nk} \right|. \tag{7}$$

Theorem 3.2. *Define the sets*

$$\xi_1^{r,s,F} = \left\{ b = (b_n) \in \omega : \sup_{K\in\mathcal{F}} \sum_k \left| \sum_{n\in K} (s+r)_F^k \sum_{j=k}^n \binom{j}{k}_F (-s)^{j-k} r^{-j} b_n \right|^q < \infty \right\}$$

and

$$\xi_2^{r,s,F} = \left\{ b = (b_n) \in \omega : \sup_{k\in\mathbb{N}} \sum_n \left| (s+r)_F^k \sum_{j=k}^n \binom{j}{k}_F (-s)^{j-k} r^{-j} b_n \right| < \infty \right\}.$$

Then, $\{b_p^{r,s,F} (\triangledown)\}^\alpha = \xi_1^{r,s,F}$, where $1 < p \leq \infty$ and $\left\{b_1^{r,s,F} (\triangledown)\right\}^\alpha = \xi_2^{r,s,F}$.

Proof. For any $b = (b_n) \in \omega$, one can write from (3) that

$$b_n x_n = \sum_{k=0}^n (s+r)_F^k \sum_{j=k}^n \binom{n}{k}_F (-s)^{j-k} r^{-j} b_n y_k = \left(G^{r,s,F} y\right)_n \quad \text{for all } n \in \mathbb{N}.$$

Here, $G^{r,s,F}=\left(g^{r,s,F}_{nk}\right)$ is defined by

$$g^{r,s,F}_{nk}=\begin{cases}(s+r)^k_F\sum\limits_{j=k}^{n}\binom{n}{k}_F(-s)^{j-k}r^{-j}b_n, & \text{if } 0\le k\le n;\\ 0, & \text{if } k>n.\end{cases}$$

So, we have $bx=(b_nx_n)\in\ell_1$ whenever $x=(x_k)\in b^{r,s,F}_p(\bigtriangledown)$ or $x=(x_k)\in b^{r,s,F}_p$ if and only if $G^{r,s,F}y\in\ell_1$ whenever $y=(y_k)\in\ell_1$ or $y=(y_k)\in\ell_p$, respectively, for $1<p\le\infty$. Consequently, we deduce that $b=(b_n)\in\left\{b^{r,s,F}_1(\bigtriangledown)\right\}^\alpha$ or $b=(b_n)\in\left\{b^{r,s,F}_p(\bigtriangledown)\right\}^\alpha$ if and only if $G^{r,s,F}\in(\ell_1,\ell_1)$ or $G^{r,s,F}\in(\ell_p,\ell_1)$, respectively, for $1<p\le\infty$. So, if we gather the related parts of Lemma 3.1 and these facts, we arrive at the desired conclusion. □

Theorem 3.3. *Define the sets*

$$\xi^{r,s,F}_3=\left\{b=(b_k)\in\omega:\lim_{n\to\infty}(s+r)^k_F\sum_{i=k}^{n}\sum_{j=k}^{i}\binom{j}{k}_F(-s)^{j-k}r^{-j}b_i \text{ exists for each } k\in\mathbb{N}\right\},$$

$$\xi^{r,s,F}_4=\left\{b=(b_k)\in\omega:\sup_{n,k}\left|(s+r)^k_F\sum_{i=k}^{n}\sum_{j=k}^{i}\binom{j}{k}_F(-s)^{j-k}r^{-j}b_i\right|<\infty\right\},$$

$$\xi^{r,s,F}_5=\left\{b=(b_k)\in\omega:\lim_{n\to\infty}\sum_k\left|(s+r)^k_F\sum_{i=k}^{n}\sum_{j=k}^{i}\binom{j}{k}_F(-s)^{j-k}r^{-j}b_i\right|=\sum_k\left|\lim_{n\to\infty}(s+r)^k_F\sum_{i=k}^{n}\sum_{j=k}^{i}\binom{j}{k}_F(-s)^{j-k}r^{-j}b_i\right|\right\},$$

$$\xi^{r,s,F}_6=\left\{b=(b_k)\in\omega:\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\left|(s+r)^k_F\sum_{i=k}^{n}\sum_{j=k}^{i}\binom{j}{k}_F(-s)^{j-k}r^{-j}b_i\right|^q<\infty\right\},$$

for $1<q<\infty$, *and*

$$\xi^{r,s,F}_7=\left\{b=(b_k)\in\omega:\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\left|(s+r)^k_F\sum_{i=k}^{n}\sum_{j=k}^{i}\binom{j}{k}_F(-s)^{j-k}r^{-j}b_i\right|<\infty\right\}.$$

Then, $\left\{b^{r,s,F}_1(\bigtriangledown)\right\}^\beta=\xi^{r,s,F}_3\cap\xi^{r,s,F}_4$, $\left\{b^{r,s,F}_p(\bigtriangledown)\right\}^\beta=\xi^{r,s,F}_3\cap\xi^{r,s,F}_6$ *for* $1<p<\infty$, $\left\{b^{r,s,F}_\infty(\bigtriangledown)\right\}^\beta=\xi^{r,s,F}_3\cap\xi^{r,s,F}_5$, $\left\{b^{r,s,F}_1(\bigtriangledown)\right\}^\gamma=\xi^{r,s,F}_4$, $\left\{b^{r,s,F}_p(\bigtriangledown)\right\}^\gamma=\xi^{r,s,F}_6$ *for* $1<p<\infty$, *and* $\left\{b^{r,s,F}_\infty(\bigtriangledown)\right\}^\gamma=\xi^{r,s,F}_7$.

Proof. In order to avoid unnecessary repetitions of similar expressions, only the proof of $\left\{b^{r,s,F}_p(\bigtriangledown)\right\}^\beta=\xi^{r,s,F}_3\cap\xi^{r,s,F}_6$ for $1<p<\infty$ is given. For any $b=(b_n)\in\omega$, it follows from (3) that

$$\begin{aligned}\sum_{k=0}^{n}b_kx_k&=\sum_{k=0}^{n}\left(\sum_{i=0}^{k}(s+r)^i_F\sum_{j=i}^{k}\binom{j}{i}_F(-s)^{j-i}r^{-j}y_i\right)b_k\\&=\sum_{k=0}^{n}\left((s+r)^k_F\sum_{i=k}^{n}\sum_{j=k}^{i}\binom{j}{k}_F(-s)^{j-k}r^{-j}b_i\right)y_k\\&=\left(M^{r,s,F}y\right)_n,\end{aligned}$$

for all $n\in\mathbb{N}$. Here, $M^{r,s,F}=\left(m^{r,s,F}_{nk}\right)$ denotes the matrix defined by

$$m^{r,s,F}_{nk}=\begin{cases}(s+r)^k_F\sum\limits_{i=k}^{n}\sum\limits_{j=k}^{i}\binom{j}{k}_F(-s)^{j-k}r^{-j}b_i, & \text{if } 0\le k\le n;\\ 0, & \text{if } k>n;\end{cases}$$

for all $n,k\in\mathbb{N}$. So, $bx=(b_nx_n)\in cs$ whenever $x=(x_k)\in b^{r,s,F}_p(\bigtriangledown)$ if and only if $M^{r,s,F}y\in c$ whenever $y=(y_k)\in\ell_p$ for $1<p<\infty$, which yields that $b=(b_k)\in\left\{b^{r,s,F}_p\right\}^\beta$ if and only if $M^{r,s,F}\in(\ell_p,c)$, where $1<p<\infty$. By combining these observations and the related parts of Lemma 3.1, we have $\left\{b^{r,s,F}_p\right\}^\beta=\xi^{r,s,F}_3\cap\xi^{r,s,F}_6$ for $1<p<\infty$. □

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