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Research Article

Dispersed Dyck paths revisited

Helmut Prodinger[1,2,*]

[1] *Department of Mathematics, University of Stellenbosch, Stellenbosch, South Africa*
[2] *NITheCS (National Institute for Theoretical and Computational Sciences), South Africa*

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Abstract

Dispersed Dyck paths are Dyck paths with possible flat steps on level 0. Questions about dispersed Dyck paths, from the Encyclopedia of Integer Sequences, are revisited and augmented in a systematic way that uses generating functions and the kernel method.

Keywords: dispersed Dyck paths; prefix; ascent; descent; valley kernel method.

2020 Mathematics Subject Classification: 05A15.

1. Introduction

Dispersed Dyck paths consist of up-steps $U = (1, 1)$ and down-steps $D = (1, -1)$, never go below the x-axis, and can have horizontal steps on the x-axis, $(n, 0) \to (n + 1, 0)$. According to [7], it seems that such Dyck paths were invented/suggested by Emeric Deutsch. All sequences that are explicitly cited are from [7], using the local identifiers $A\#\#\#\#\#\#$.

As in various examples discussed in the past [3, 5, 6], the kernel method will be used to set up appropriate generating functions. There are three (or four) bivariate (or trivariate) generating functions $F(u), G(u), H(u)$ (also depending on z) related to the nature of the last step of the prefix of a dispersed Dyck path. To solve the system, a 'bad' factor $u - r_2$ must be divided out from numerator and denominator, after which one can plug in $u = 0$ and identify the unknown quantities $f_1 = F(0)$, etc. Various questions (from the encyclopedia of integer sequences [7]) about such paths will be revisited; our method is fairly automatic and purely refers to generating functions. To be more specific, these include 1-ascents, 1-descents, valleys on level 0, and occurrences of $UUDD$. Also, we obtain expression for paths that do not have to go back to the x-axis, rather finish at a level j, or, more generally, on *any* level. This is either achieved by looking at the coefficient of u^j, say, or, setting $u = 1$. The generating functions of interest usually have 3 variables: z for the length (number of steps), u for the final height, and t for an additional parameter of interest.

2. Counting 1-ascents

A 1-ascent is an ascent consisting of exactly 1 up step. The paper [2] contains some analysis, but without generating functions. The paper [1] discusses d-ascents for $d \geq 2$ as well, but the quadratic equations that are so common in the context of Dyck paths, are then of higher order, and the results are consequently of an asymptotic nature.

We distinguish 3 states, together with the current level. Down-steps are unproblematic, but when after them an up-step arrives, it might be the only one or further up-steps follow. A graph describes all the possible scenarios, as in [5]. It has 3 layers of states, and sequences f_i, $i \geq 1$, g_i, h_i, $i \geq 0$, in that order. These quantities all depend on the variable z and describe generating functions of paths leading to a particular state. Since on level 0, flat (horizontal) steps are also allowed, the quantities f_0, g_0, and h_0 are somewhat special and will be treated as parameters. Compare [4] for this technique. We will deal with trivariate generating functions

$$F(u) = \sum_{i\geq 1} f_i u^{i-1},\ G(u) = \sum_{i\geq 1} g_i u^{i-1},\ H(u) = \sum_{i\geq 1} h_i u^{i-1},$$

the other variables z (counting the length) and t (counting the 1-ascents) are not explicitly mentioned. The recursions can be read off, by considering the last step made,

$$f_i = zf_{i+1} + ztg_{i+1} + zh_{i+1},\ i \geq 1,\ f_0 = 1 + zf_0 + zf_1 + ztg_1,$$

*E-mail address: hproding@sun.ac.za

$$g_{i+1} = zf_i, \ i \geq 1, \ g_1 = zf_0,$$
$$h_{i+1} = zg_i + zh_i, \ i \geq 0.$$

Translating these into the trivariate generating functions, we get

$$\sum_{i\geq 1} u^i f_i = \sum_{i\geq 1} u^i z f_{i+1} + \sum_{i\geq 1} u^i z t g_{i+1} + \sum_{i\geq 1} u^i z h_{i+1},$$
$$uF(u) = zF(u) - zf_1 + ztG(u) - ztg_1 + zH(u),$$
$$G(u) - g_1 = zuF(u), \quad H(u) = zuG(u) + zuH(u).$$

The system can be solved, but still depends on the initial values f_1, g_1, h_1:

$$F(u) = \frac{z(-f_1 + zuf_1 + zug_1)}{-z^3u^2 + u - zu^2 - z + z^2u - z^2tu + z^3tu^2},$$
$$G(u) = \frac{-z^2uf_1 - z^2tug_1 - zg_1 + ug_1 + z^3u^2f_1 + z^3tu^2g_1 + z^2ug_1 - zu^2g_1}{-z^3u^2 + u - zu^2 - z + z^2u - z^2tu + z^3tu^2},$$
$$H(u) = -\frac{zu(z^2uf_1 + z^2tug_1 + zg_1 - ug_1)}{-z^3u^2 + u - zu^2 - z + z^2u - z^2tu + z^3tu^2}.$$

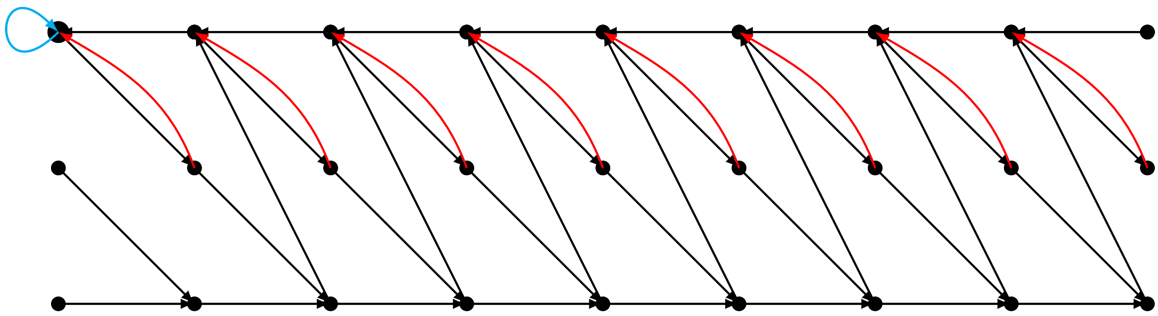

Figure 2.1: Three layers of states, labelled f, g, h, in that order. The state f_0 is responsible for dispersed Dyck paths, and all others to prefixes of them.

The naive approach would be to plug in $u = 0$ and identity the initial values, but this doesn't work, and requires some preparations. The denominator will be factored,

$$-z^3u^2 + u - zu^2 - z + z^2u - z^2tu + z^3tu^2 = z(-z^2 - 1 + z^2t)(u - r_1)(u - r_2)$$

with

$$r_1 = \frac{1 + z^2(1-t) + W}{2z(1 + z^2(1-t))}, \quad r_2 = \frac{1 + z^2(1-t) - W}{2z(1 + z^2(1-t))}$$

and

$$W := \sqrt{1 - 2(t+1)z^2 - (t+3)(1-t)z^4}\,.$$

Dividing out the factor $u - r_2$ from numerator and denominator ('kernel method'), we find

$$F(u) = \frac{z^2(f_1 + g_1)}{(-z^2 - 1 + z^2t)(r_2z + zu - 1)},$$
$$G(u) = \frac{r_2z^3f_1 + r_2z^3tg_1 - r_2zg_1 - z^2f_1 - z^2tg_1 + g_1 + z^2g_1 + uz^3f_1 + uz^3tg_1 - zug_1}{(-z^2 - 1 + z^2t)(r_2z + zu - 1)},$$
$$H(u) = -\frac{z(r_2z^2f_1 + r_2z^2tg_1 - r_2g_1 + z^2uf_1 + z^2tug_1 + zg_1 - ug_1)}{(-z^2 - 1 + z^2t)(r_2z + zu - 1)},$$

and now $u = 0$ is possible, with

$$f_1 = \frac{z^2(f_1 + g_1)}{(-z^2 - 1 + z^2t)(r_2z - 1)}, \quad g_1 = zf_0, \quad h_1 = 0.$$

But, we have

$$f_0 = 1 + zf_0 + zf_1 + ztg_1,$$

and both, f_1 and g_1 can be expressed in terms of f_0:

$$f_1 = \frac{z^2 g_1}{1 - z^2 t - r_2 z - r_2 z^3 + r_2 z^3 t}.$$

The ultimate solution is now

$$f_0 = \frac{-1 + 2z - z^2 + z^2 t + W}{2z(z^2 + 1 - 2z - z^3 + z^3 t - z^2 t)},$$

with the series expansion

$$f_0 = 1 + z + (t+1)z^2 + (2t+1)z^3 + (t^2 + 3t + 2)z^4 + (3t^2 + 4t + 3)z^5 + (t^3 + 6t^2 + 8t + 5)z^6 + \cdots.$$

For $t = 0$, we find the enumeration of dispersed Dyck paths without 1-ascents (A191385)

$$\begin{aligned} f_0\big|_{t=0} &= \frac{-1 + 2z - z^2 + \sqrt{1 - 2z^2 - 3z^4}}{2z(1 - 2z + z^2 - z^3)} \\ &= 1 + z + z^2 + z^3 + 2z^4 + 3z^5 + 5z^6 + 7z^7 + 12z^8 + 18z^9 + 31z^{10} + 47z^{11} + 81z^{12} + \cdots. \end{aligned}$$

For $t = 1$, we find

$$f_0\big|_{t=1} = -\frac{1}{2z} + \frac{\sqrt{1 - 4z^2}}{2z(1 - 2z)} = \sum_{n\ge0} \binom{2n}{n} z^{2n} + \sum_{n\ge0} \binom{2n+1}{n} z^{2n+1},$$

which is the enumeration of dispersed Dyck paths of length n: $\binom{n}{\lfloor n/2 \rfloor}$. The number of 1-ascents in paths of length 5 can be deduced from $(3t^2 + 4t + 3)$: it is $3 \cdot 2 + 4$, and in general, we must differentate f_0 w.r.t. t, followed by $t = 1$.

$$\begin{aligned} \frac{\partial f_0}{\partial z}\bigg|_{t=1} &= \frac{z^2(1 - 4z^2 + \sqrt{1 - 4z^2})}{2(1 - 2z)(1 - 4z^2)} \\ &= \frac{z^2}{2(1 - 2z)} + \frac{z^2}{2(1 - 4z^2)^{3/2}} + \frac{z^3}{(1 - 4z^2)^{3/2}} \\ &= \sum_{n\ge2} 2^{n-3} z^n + \sum_{n\ge1} \frac{(2n-1)!}{2(n-1)!(n-1)!} z^{2n} + \sum_{n\ge1} \frac{(2n-1)!}{(n-1)!(n-1)!} z^{2n+1}. \end{aligned}$$

The coefficients of z^n form the sequence A045621. Furthermore, we get

$$\begin{aligned} F(u) &= \frac{z^2 r_2}{1 - zr_2 - zu} f_0, \\ G(u) &= \frac{z(1 - zu)(1 - zr_2)}{1 - zr_2 - zu} f_0, \\ H(u) &= \frac{uz^2(1 - zr_2)}{1 - zr_2 - zu} f_0. \end{aligned}$$

From this, it is easy to find f_j, g_j, h_j since we only have to expand $1/(1 - zr_2 - zu)$ in powers of u, which is basically a geometric series. We will not write this out. However, we will sum *all* these quantities, as it describes all dispersed Dyck paths with open end (partial paths, sometimes called meander). We have to be careful and, when a path ends in a state from the second layer ('g'), the last up step is a 1-ascent, and we need to attach an extra factor t. The resulting formula is surprisingly simple:

$$\frac{\partial}{\partial t}\Big(f_0 + F(1) + tG(1) + H(1)\Big)\bigg|_{t=1} = \frac{z(1 - z)^2}{(1 - 2z)^2} = z + \sum_{n\ge2} (n + 2)2^{n-3} z^n.$$

3. Counting 1-descents

In this section, the number of 1-descents will be counted. The definition is similar; a down-step rendered by up-steps or standing at the end of the dispersed Dyck path. The same letters as in the previous section will be used, but now with a different meaning. The figure is again self-explanatory: Here are the relevant recursions:

$$f_{i+1} = zf_i + ztg_i + zh_i,\ i \ge 1,\ f_0 = \frac{1}{1 - z},$$

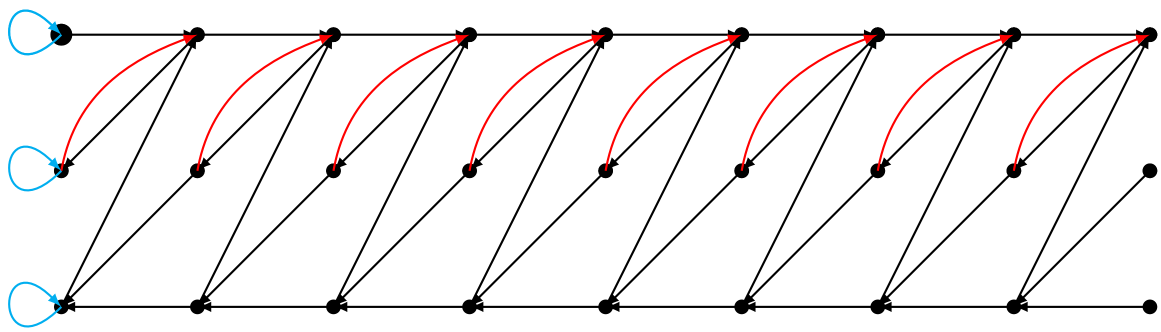

Figure 3.1: Three layers of states, labelled f, g, h, in that order.

$$g_i = zf_{i+1},\ i \ge 1,\ g_0 = zg_0 + zf_1,\ g_0 = \frac{z}{1-z}f_1$$

$$h_i = zg_{i+1} + zh_{i+1},\ i \ge 1,\ h_0 = zh_0 + zg_1 + zh_1,\ h_0 = \frac{z}{1-z}(g_1 + h_1),$$

and

$$\begin{aligned} f_1 = zf_0 + ztg_0 + zh_0 &= \frac{z}{1-z} + \frac{z^2t}{1-z}f_1 + \frac{z^2}{1-z}(g_1 + h_1) \\ &= \frac{z(1 + zg_1 + zh_1)}{1 - z - z^2t}. \end{aligned}$$

So, the kernel method will give us g_1 and h_1, and f_1 follows. The recursions will be translated into trivariate generating functions, as before:

$$\sum_{i\ge1} u^i f_{i+1} = \sum_{i\ge1} u^i zf_i + \sum_{i\ge1} u^i ztg_i + \sum_{i\ge1} u^i zh_i,$$
$$F(u) - f_1 = uzF(u) + uztG(u) + uzH(u);$$

$$\sum_{i\ge1} u^i g_i = \sum_{i\ge1} u^i zf_{i+1},$$
$$uG(u) = zF(u) - zf_1;$$

$$\sum_{i\ge1} u^i h_i = \sum_{i\ge1} u^i zg_{i+1} + \sum_{i\ge1} u^i zh_{i+1},$$
$$uH(u) = zG(u) - zg_1 + zH(u) - zh_1.$$

We do not write the solutions for $F(u), G(u), H(u)$, only the relevant denominator

$$z^3 - u + u^2z + uz^2t + z - uz^2 - z^3t = z(u - r_1)(u - r_2)$$

with

$$r_1 = \frac{1 + z^2 - z^2t + W}{2z}, \quad r_2 = \frac{1 + z^2 - z^2t - W}{2z},$$

and

$$W = \sqrt{1 - 2z^2t - 2z^2 + z^4t^2 + 2z^4t - 3z^4}.$$

Then, we divide out the factor $u - r_2$ in the usual way, and get, after setting $u = 0$,

$$g_1 = \frac{zr_2}{1 + z^2 - z^2t}f_1, \quad g_1 + h_1 = \frac{r_2 - z}{z}f_1,$$
$$f_1 = \frac{z}{1 - z + z^2 - z^2t - zr_2}.$$

This provides also the values f_0, g_0, h_0, and

$$f_0 + tg_0 + h_0 = \frac{1 - r_2}{1 - 2z + z^2 - z^3 - z^2t + z^3t},$$

which is the generating function of dispersed Dyck paths returning to the 0-level. Again, to count the contributions of the 1-descents, we compute

$$\begin{aligned}\frac{\partial}{\partial z}(f_0 + tg_0 + h_0)\Big|_{t=1} &= \frac{z^2(1 - 4z^2 + \sqrt{1 - 4z^2})}{2(1 - 2z)(1 - 4z^2)}\\ &= \frac{z^2}{2(1 - 2z)} + \frac{z^2}{2(1 - 4z^2)^{3/2}} + \frac{z^3}{(1 - 4z^2)^{3/2}}\end{aligned}$$

as before. This is natural, as reading from right to left turns 1-upsteps to 1-downsteps, and vice versa. Considering *all* paths, regardless where they end,

$$\frac{\partial}{\partial z}(f_0 + F(1) + tg_0 + tG(1) + h_0 + H(1))\Big|_{t=1} = \frac{z^2}{2(1 - 2z)^2} + \frac{z^2\sqrt{1 - 4z^2}}{2(1 - 2z)^2},$$

which is surprisingly simple, with a simple series expansion,

$$\sum_{n\ge 2}(n - 1)2^{n-3}z^n + \sum_{n\ge 1}\frac{2(2n - 2)!}{(n - 1)!(n - 1)!}z^{2n} + \sum_{n\ge 1}\frac{(2n - 2)!(4n - 3)}{2(n - 1)!(n - 1)!}z^{2n+1}.$$

4. Counting valleys on level 0

We do now exactly what the title of the section says.

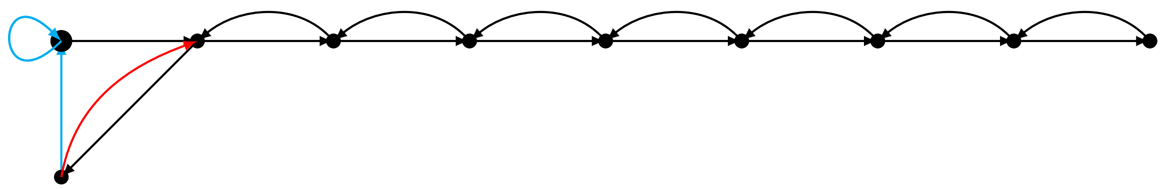

Figure 4.1: Two layers of states, labelled f, g, in that order.

Here are the usual recursions,

$$f_0 = 1 + zf_0 + zg_0,\ g_0 = zf_1,\ f_1 = ztg_0 + zf_0 + zf_2,\ f_i = zf_{i-1} + zf_{i+1},\ i \ge 2.$$

$$f_0 = \frac{1}{1 - z} + \frac{z}{1 - z}g_0 = \frac{1}{1 - z} + \frac{z^2}{1 - z}f_1, \quad tf_0 - t - ztf_0 = ztg_0$$

$$\begin{aligned}\sum_{i\ge 2}u^{i-1}f_i &= \sum_{i\ge 2}u^{i-1}zf_{i-1} + \sum_{i\ge 2}u^{i-1}zf_{i+1}\\ F(u) - f_1 &= uzF(u) + \frac{z}{u}(F(u) - f_1) - zf_2,\\ F(u) &= uzF(u) + \frac{z}{u}(F(u) - f_1) + ztg_0 + zf_0,\\ F(u) &= uzF(u) + \frac{z}{u}(F(u) - f_1) + z^2tf_1 + zf_0.\end{aligned}$$

We have $f_0 = \frac{1+z^2f_1}{1-z}$, and from the kernel method, after dividing out

$$u - r_2, \quad \text{with} \quad r_2 := \frac{1 - \sqrt{1 - 4z^2}}{2z},$$

$$f_1 = \frac{z}{1 - z - z^2t + z^3(t - 1) + z(z - 1)r_2}.$$

Then,

$$f_1\big|_{t=0} = \frac{2}{2-3z+z\sqrt{1-z^2}}$$
$$= 1+z+z^2+2z^3+3z^4+5z^5+8z^6+14z^7+23z^8+41z^9+69z^{10}+125z^{11}+\cdots,$$

which is the sequence A191388 (no valleys on level 0). Furthermore,

$$F(u) = \frac{f_1}{1-ur_2} \Longrightarrow [u^{j-1}]F(u) = f_j = f_1 r_2^{j-1}.$$

Finally,

$$\frac{\partial f_0}{\partial t}\bigg|_{t=1} = \frac{1-3z^2+(z^2-1)\sqrt{1-4z^2}}{2z(1-2z)}$$
$$= z^5+2z^6+7z^7+14z^8+37z^9+74z^{10}+176z^{11}+352z^{12}+\cdots,$$

which is the sequence A191389 (number of valleys on level 0). By setting $u=1$ in $g_0+F(u)$, we can address the number of such partial dispersed Dyck paths according to no valleys or number of valleys, respectively. Since these formulæ are easy to obtain and not too attractive, we do not display them here.

5. Counting occurrences of $UUDD$

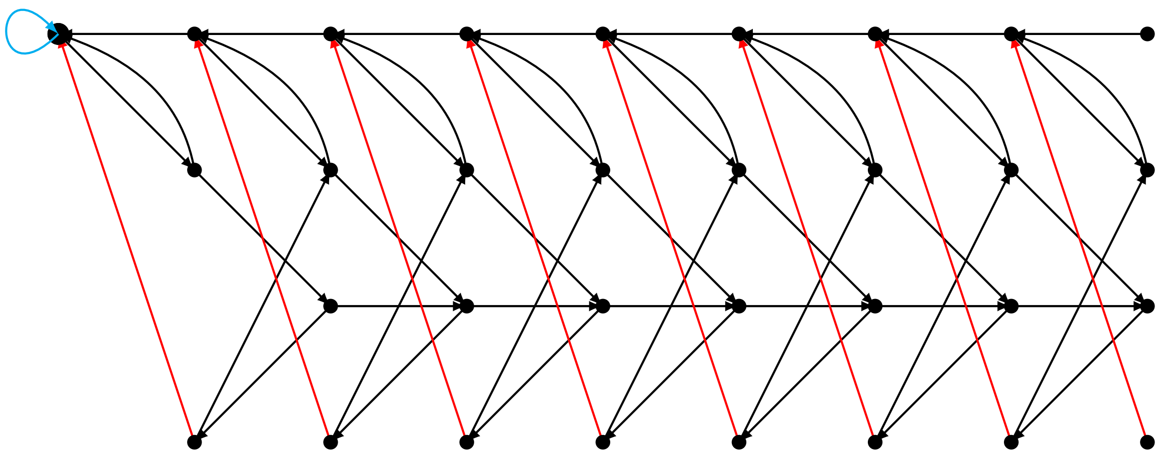

Figure 5.1: Four layers of states, labelled f, g, h, k, in that order.

The recursions are

$$f_i = zf_{i+1}+zg_{i+1}+ztk_{i+1},\ i\ge 1,\ f_0 = 1+zf_0+zf_1+zg_1+ztk_1,$$
$$g_{i+1} = zf_i+zk_i,\ i\ge 1,\ g_1 = zf_0,$$
$$h_{i+1} = zg_i+zh_i,\ i\ge 2,\ h_2 = zg_1,$$
$$k_i = zh_{i+1},\ i\ge 1,\ k_1 = zh_2 = z^2g_1.$$

The following generating functions will be used:

$$F(u) = \sum_{i\ge1} f_i u^{i-1},$$
$$G(u) = \sum_{i\ge1} g_i u^{i-1},$$
$$H(u) = \sum_{i\ge2} h_i u^{i-2},$$
$$K(u) = \sum_{i\ge1} k_i u^{i-1}.$$

Summing and simplifying,

$$\sum_{i\ge1}u^if_i=\sum_{i\ge1}u^izf_{i+1}+\sum_{i\ge1}u^izg_{i+1}+\sum_{i\ge1}u^iztk_{i+1},$$

$$uF(u)=zF(u)-zf_1+zG(u)-zg_1+ztK(u)-ztk_1,$$

$$\sum_{i\ge1}u^ig_{i+1}=\sum_{i\ge1}u^izf_i+\sum_{i\ge1}u^izk_i,$$

$$G(u)-g_1=zuF(u)+zuK(u),$$

$$\sum_{i\ge2}u^{i-1}h_{i+1}=\sum_{i\ge2}u^{i-1}zg_i+\sum_{i\ge2}u^{i-1}zh_i,$$

$$H(u)-h_2=zG(u)-zg_1+zuH(u),\ H(u)=\frac{z}{1-zu}G(u),$$

$$\sum_{i\ge1}u^{i-1}k_i=\sum_{i\ge1}u^{i-1}zh_{i+1},$$

$$K(u)=zH(u).$$

It is beneficial to reduce the system to just one equation, for $F(u)$, say. Note also that

$$f_1=-\frac{1+zf_0+z^2f_0+z^4tf_0-f_0}{z},\ g_1=zf_0,\ h_2=z^2f_0,\ k_1=z^3f_0.$$

Then

$$F(u)=-\frac{-z^3u-z^4f_0u-zu-z^2f_0u+zuf_0+1+zf_0+z^2f_0+z^4tf_0-f_0}{-z^4u+z+z^4tu+zu^2-u}.$$

Dividing out, as part of the kernel method,

$$u-r_2,\quad \text{with}\quad r_2=\frac{1+z^4-z^4t-\sqrt{z^8-2z^8t+2z^4+z^8t^2-2z^4t+1-4z^2}}{2z}$$

and setting $u=0$ leads to

$$-\frac{1+zf_0+z^2f_0+z^4tf_0-f_0}{z}=f_1=\frac{z\left(z^2+z^3f_0+1+zf_0-f_0\right)}{-z^4+z^4t-1+r_2z}$$

from which f_0 can be computed:

$$f_0=\frac{-z^4-1+z^4t+2z+\sqrt{z^8-2z^8t+2z^4+z^8t^2-2z^4t+1-4z^2}}{2z\left(-z^4t+1-2z+z^4\right)}$$
$$=1+z+2z^2+3z^3+(t+5)z^4+(8+2t)z^5+(14+6t)z^6+(23+12t)z^7+\cdots.$$

The special cases are as follows:

$$f_0\Big|_{t=0}=\frac{2z+\sqrt{z^8+2z^4+1-4z^2}-z^4-1}{2z\left(+1-2z+z^4\right)}$$
$$=1+z+2z^2+3z^3+5z^4+8z^5+14z^6+23z^7+41z^8+69z^9+124z^{10}+\cdots,$$

which enumerates dispersed Dyck paths without $UUDD$ (sequence A191794) and

$$\frac{\partial f_0}{\partial t}\Big|_{t=1}=\frac{z^4}{(1-2z)\sqrt{1-4z^2}}$$
$$=z^4+2z^5+6z^6+12z^7+30z^8+60z^9+140z^{10}+280z^{11}+630z^{12}+\cdots,$$

which enumerates the number of occurrences of $UUDD$ in dispersed Dyck paths (sequence A100071).

Of course, dispersed Dyck paths with arbitrary endpoints can also be discussed using $f_0+F(1)+G(1)+H(1)+K(1)$, but we are not going to display any formula; they are easy to obtain.

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