Contributions to Mathematics
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Contrib. Math. **9** (2024) 18–25
DOI: 10.47443/cm.2024.006

Research Article

Some properties of eccentrical graphs

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(Received: 19 January 2024. Received in revised form: 22 February 2024. Accepted: 27 February 2024. Published online: 28 February 2024.)

Abstract

In this paper, the concept of the eccentrical graph of a graph is introduced. Let G be a connected graph with the vertex set $V(G)$. The eccentrical graph of G is the graph $\epsilon(G)$ with the vertex set $V(\epsilon(G)) = V(G)$ and two vertices $v_i, v_j \in V(\epsilon(G))$ are adjacent in $\epsilon(G)$ if and only if the distance between them is $\min\{e(v_i), e(v_j)\}$, where $e(v_i)$ is the eccentricity of v_i. A sufficient condition for the eccentrical graph of a connected graph to be connected is given. It is proved that the eccentrical graph of every tree is connected and its diameter does not exceed 3. The extremum values of the greatest eigenvalue of eccentrical graphs of trees and connected graphs of fixed order are also studied. Furthermore, spectra of eccentrical graphs of various classes of graphs are computed.

Keywords: eccentrical graph; connected graph; ϵ-spectrum; spectral radius.

2020 Mathematics Subject Classification: 05C50.

1. Introduction

Let $G = (V(G), E(G))$ be a finite simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$. The notion $v_i \sim v_j$ is used to indicate that vertices v_i and v_j are adjacent, and the edge between them is denoted by v_iv_j or e_{ij}. The degree of the vertex v_i of G is denoted by $deg(v_i|G)$. The *adjacency matrix* of G is an $n \times n$ matrix, denoted by $A(G)$, whose rows and columns are indexed by the vertex set of G and its entries are defined by

$$A(G)_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases}$$

The *distance* between the vertices $v_i, v_j \in V(G)$, denoted by $d(v_i, v_j)$, is defined as the smallest value among the lengths (i.e., the number of edges) of the paths between the vertices v_i and v_j. The *distance matrix* of a connected graph G, denoted by $D(G)$, or simply by D, is the $n \times n$ matrix whose $(i, j)^{th}$-entry is equal to $d(v_i, v_j)$, where $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, n$. For other terminologies and notations not defined here, the readers are referred to [2]. The adjacency matrix and the distance matrix of a graph are well-studied matrices in the field of spectral graph theory. Details about the study of these matrices and other matrices associated with graphs can be found in [1, 3, 4].

The *eccentricity* $e(v_i)$ of the vertex v_i is defined as $e(v_i) = \max\{d(v_i, v_j) : v_j \in V(G)\}$. The eccentrical graph $\epsilon(G)$ of a connected graph G is the graph with the vertex set $V(\epsilon(G)) = V(G)$ and $v_i \sim v_j$ in $\epsilon(G)$ if and only if $d(v_i, v_j) = \min\{e(v_i), e(v_j)\}$ (see Figure 1.1 for two examples). Thus, the $(i, j)^{th}$-entry of the adjacency matrix of the eccentrical graph $\epsilon(G)$, denoted by $A_\epsilon(G)$, is given as follows:

$$A_\epsilon(G)_{ij} = \begin{cases} 1 & \text{if } d(v_i, v_j) = \min\{e(v_i), e(v_j)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\epsilon_1, \epsilon_2, \ldots \epsilon_n$ denote the eigenvalues of the matrix $A_\epsilon(G)$. Since, the matrix$A_\epsilon(G)$ is symmetric, all the ϵ-eigenvalues of G are real. If $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$ are the all distinct ϵ-eigenvalues of G satisfying $\epsilon_1 > \epsilon_2 > \ldots > \epsilon_k$, then the ϵ-spectrum of G is denoted by

$$Spec_\epsilon(G) = \left\{ \begin{array}{cccc} \epsilon_1 & \epsilon_2 & \ldots & \epsilon_k \\ m_1 & m_2 & \ldots & m_k \end{array} \right\},$$

where m_i is the algebraic multiplicity of the eigenvalue ϵ_i for $1 \leq i \leq k$. The eigenvalue ϵ_1 is known as the spectral radius of the eccentrical graph $\epsilon(G)$.

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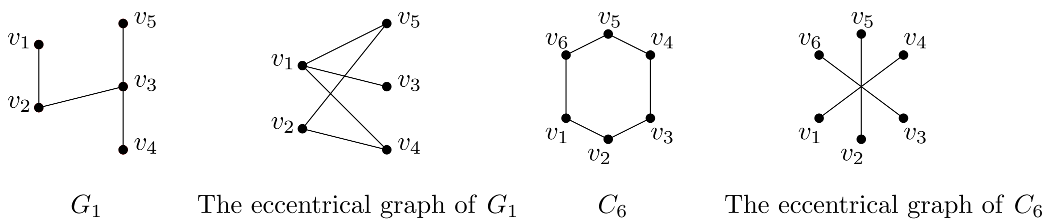

Figure 1.1: Two graphs and their eccentrical graphs.

The complement $\overline{G}$ of a graph G is the graph whose vertex set is the same as that of G and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ such that $n_i = |V(G_i)|$ for $i = 1, 2$. The union of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph whose vertex set is $V(G_1) \cup V(G_2)$ and the edge set is $E(G_1) \cup E(G_2)$. The join of G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph obtained from $G_1 \cup G_2$ by making every vertex of G_1 adjacent to all vertices of G_2. The join operation of two graphs is also known as the complete product of two graphs. The corona of G_1 and G_2, denoted by $G_1 \circ G_2$, is defined as the graph obtained by taking one copy of G_1 and n_1 copies of G_2, and then making the i-th vertex of G_1 adjacent to every vertex in the i-th copy of G_2.

This article is organized as follows. In Section 2, a sufficient condition for the eccentrical graph of a connected graph to be connected is given. It is also proved in Section 2 that the eccentrical graph of every tree is connected and its diameter does not exceed 3. In Section 3, the extremum values of the spectral radius of eccentrical graphs of trees and connected graphs of fixed order are investigated. Spectra of eccentrical graphs of various classes of graphs are also computed in Section 3.

2. The eccentrical graph of a tree

For a symmetric matrix M of order n, its matrix graph G^M is the graph whose vertices are $1, 2, \ldots, n$, and two distinct vertices i, j are adjacent if and only if $M_{ij} \neq 0$, where M_{ij} is the $(i,j)^{th}$-entry of M. It is well-known that G^M is connected if and only if M is irreducible [7]. By Figure 1.1, the eccentrical graph of a connected graph may be disconnected.

Theorem 2.1. *Let G be a connected graph of order n. Let P_{uv} be a path of longest length in G with end vertices u and v. If for every vertex $w \in V(G)$, it holds that $e(w) = \max\{d(w,u), d(w,v)\}$, then the eccentrical graph of G is connected.*

Proof. For any $s, t \in V(G)$, we have $e(s) = \max\{d(s,u), d(s,v)\}$ and $e(t) = \max\{d(t,u), d(t,v)\}$. By the definition of the eccentrical graph, at least one of the two edges $su(tu)$ and $sv(tv)$ belongs to $E(\epsilon(G))$, and also $uv \in E(\epsilon(G))$. Therefore, the eccentrical graph of G is connected. $\square$

Theorem 2.2. *If T is a tree of order n, then the eccentrical graph $\epsilon(T)$ of T is connected and the diameter of $\epsilon(T)$ does not exceed 3.*

Proof. Let P_{uv} be a path of the longest length in T with end vertices u and v. Consider a vertex $w \in V(T)$ such that $e(w) \neq d(w,u)$ and $e(w) \neq d(w,v)$. Then there is a vertex $s \in V(T)$ such that $d(w,s) > d(w,u)$ and $d(w,s) > d(w,v)$.

Case 1: $w \in P_{uv}$.
In this case, either the path P_{vs} or the path P_{us} has length larger than the length of the path P_{uv} in T, which is not possible.

Case 2: $w \notin P_{uv}$.

Subcase 2.1: $P_{uv} \cap P_{ws} \neq \Phi$.
Let $V(P_{uv}) \cap V(P_{ws}) = \{v_i, \ldots, v_j\}$ and $d(v_i, v_j) = d$, then $d(w,s) = d(w,v_i) + d(v_i,v_j) + d(v_j,s)$, $d(w,u) = d(w,v_i) + d(v_i,w)$ and $d(w,v) = d(w,v_i) + d(v_i,v_j) + d(v_j,v)$. Since $d(w,s) > d(w,u)$ and $d(w,s) > d(w,v)$, we have that $d(v_j,s) > d(v_j,v)$ and $d(u,s) > d(u,v)$, which is a contradiction.

Subcase 2.2: $P_{uv} \cap P_{ws} = \Phi$.
In this case, let

$$d = \min_{v_i \in V(P_{uv}), u_i \in V(P_{ws})} \{d(v_i,u_i)\} = d(v',u') \quad (\text{where } v' \in V(P_{uv}) \text{ and } u' \in V(P_{ws}))$$

such that $d(v',u') = d > 0$, then $d(w,s) = d(w,u') + d(u',s)$, $d(w,u) = d(w,u') + d(u',v') + d(v',u)$ and

$$d(w,v) = d(w,u') + d(u',v') + d(v',v).$$

Since $d(w,s) > d(w,u)$ and $d(w,s) > d(w,v)$, we have that $d(u',s) > d(u',v') + d(v',v)$ and

$$d(u,s) = d(u,v') + d(v',u') + d(u',s) > d(u,v') + 2d(v',u') + d(v',v) > d(u,v),$$

which is again a contradiction.

Therefore, by the above discussion and Theorem 2.1, the eccentrical graph $\epsilon(T)$ is connected. Also, by the definition of the eccentrical graph, the diameter of $\epsilon(T)$ does not exceed 3. □

3. The ϵ-spectrum of graphs

3.1. The spectral radius of the ϵ-spectrum of graphs

The following theorem is known as the interlacing theorem.

Lemma 3.1 (see [7]). *Suppose that $A \in \mathbb{R}^{n\times n}$ is symmetric. Let $B \in \mathbb{R}^{m\times m}$, with $m < n$, be a principal submatrix of A (submatrix whose rows and columns are indexed by the same index set $\{i_1, \ldots, i_m\}$, for some m). Suppose that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A such that $\lambda_1 \le \ldots \le \lambda_n$ and $\beta_1, \ldots, \beta_m$ are the eigenvalues of B satisfying $\beta_1 \le \ldots \le \beta_m$. Then, $\lambda_k \le \beta_k \le \lambda_{k+n-m}$ for $k = 1, \ldots, m$, and if $m = n-1$, then $\lambda_1 \le \beta_1 \le \lambda_2 \le \beta_2 \le \ldots \le \beta_{n-1} \le \lambda_n$.*

By Lemma 3.1, we have the next result.

Lemma 3.2 (see [4]). *Let G be a graph with n vertices and m edges, where $m > 1$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the eigenvalues of the adjacency matrix $A(G)$ such that $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n$. Then $1 \le \lambda_1 \le n-1$.*

Let $\lambda(G)$ be the radius adjacency spectrum of a graph G.

Theorem 3.1. *The following statements hold.*

(a). *Let C_n be the cycle of order n.*

(i). *If $n = 2k$, then $Spec_\epsilon(C_{2k}) = \left\{ \begin{array}{cc} 1 & -1 \\ k & k \end{array} \right\}$;*

(ii). *If $n = 2k+1$, then the eigenvalues of the matrix $A_\epsilon(C_{2k+1})$ are $\epsilon_i = 2\cos\frac{2\pi i}{2k+1}$ where $i = 1, 2, \ldots, 2k+1$.*

(b). *Let $K_{n-k} \vee kK_1$ be the complete product of K_{n-k} with $kK_1 (0 \le k \le n-1)$, then*

$$Spec_\epsilon(K_{n-k} \vee kK_1) = \left\{ \begin{array}{cc} n-1 & -1 \\ 1 & n-1 \end{array} \right\}.$$

Proof. (a). By the definition of the eccentrical graph, we have that $\epsilon(C_{2k}) \cong kK_2$ and $\epsilon(C_{2k+1}) \cong C_{2k+1}$. Hence

$$Spec_\epsilon(C_{2k}) = \left\{ \begin{array}{cc} 1 & -1 \\ k & k \end{array} \right\}$$

and the eigenvalues of the matrix $A_\epsilon(C_{2k+1})$ are $\epsilon_i = 2\cos\frac{2\pi i}{2k+1}$ where $i = 1, 2, \ldots, 2k+1$.

(b). By the definition of the eccentrical graph, we have that $\epsilon(K_{n-k} \vee kK_1) \cong K_n$ and hence

$$Spec_\epsilon(K_{n-k} \vee kK_1) = \left\{ \begin{array}{cc} n-1 & -1 \\ 1 & n-1 \end{array} \right\}.$$

□

Let $BS(n_1, n_2)$ be the double star with $n = n_1 + n_2 + 2$ vertices. Denote by $BBS(n)$ the balance double star with n vertices.

Theorem 3.2. *Let T be a tree with n vertices, where $n \ge 7$. Let $\epsilon_1(T), \epsilon_2(T), \ldots, \epsilon_n(T)$ denote the eigenvalues of the matrix $A_\epsilon(T)$ such that $\epsilon_1(T) \ge \epsilon_2(T) \ge \ldots \ge \epsilon_n(T)$. Then $\lambda(BBS(n)) \le \epsilon_1(T) \le n-1$, with the left equality if $T \cong T^*$ (see Figure 3.1) and with the right equality if $T \cong K_{1,n-1}$.*

Proof. By Lemma 3.1 and Lemma 3.2, we have $\epsilon_1(T) \le n-1$ and if $T \cong K_{1,n-1}$ then $\epsilon_1(T) = n-1$.

By Theorem 2.2, the diameter of $\epsilon(T)$ is at most 3. If the diameter of $\epsilon(T)$ is at most 2, we have $\epsilon_1(T) \ge \sqrt{n-1}$. If the diameter of $\epsilon(T)$ is 3, then by the definition of the eccentrical graph and Theorem 2.2, there is a double star $BS(n_1, n_2)$ with $n = n_1 + n_2 + 2$ such that $BS(n_1, n_2)$ is a subgraph of $\epsilon(T)$. Hence, $\epsilon_1(T) \ge \lambda(BS(n_1, n_2)) \ge \lambda(BBS(n))$. If $T \cong T^*$ (see Figure 3.1), by the definition of the eccentrical graph we have $\epsilon(T^*) \cong BBS(n)$. □

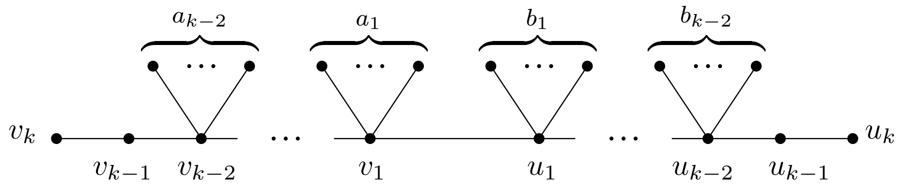

Figure 3.1: The tree T^*. Here $|\sum_{i=1}^{i=k-2} a_i - \sum_{i=1}^{i=k-2} b_i| \leq 1$ and $k \geq 3$.

Theorem 3.3. *Let G be a connected graph with n vertices. Let $\epsilon_1(G), \epsilon_2(G), \ldots, \epsilon_n(G)$ denote the eigenvalues of the matrix $A_\epsilon(G)$ such that $\epsilon_1(G) \geq \epsilon_2(G) \geq \ldots \geq \epsilon_n(G)$. Then the following statements hold:*

(a). *The inequality $\epsilon_1(G) \leq n-1$ holds, where the equality holds if $G \cong K_{n-k} \vee kK_1 (0 \leq k \leq n-1)$.*

(b). *When n is even, then $\epsilon_1(G) \geq 1$ with equality if $G \cong C_n$.*

Proof. Let d be the diameter of the graph G. Let $P(v_1, v_d)$ be a path of the longest length in G.

By Theorem 3.1 and Lemma 3.2, we have $\epsilon_1(G) \leq n-1$ with equality if $G \cong K_{n-k} \vee kK_1 (0 \leq k \leq n-1)$.

By the definition of the eccentrical graph, The vertices v_1 and v_d are adjacent in $\epsilon(G)$. By Lemma 3.1, we have that $\epsilon_1(G) \geq 1$. □

Conjecture 3.1. *Let G be a connected graph with n vertices. Let $\epsilon_1(G), \epsilon_2(G), \ldots, \epsilon_n(G)$ denote the eigenvalues of the matrix $A_\epsilon(G)$ such that $\epsilon_1(G) \geq \epsilon_2(G) \geq \ldots \geq \epsilon_n(G)$. When n is odd, then $\epsilon_1(G) \geq 2$ with equality if $G \cong C_n$.*

3.2. The ϵ-spectrum of some classes of graphs

Let A be an $n \times n$ matrix partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square matrices. If A_{11} is nonsingular, then the Schur complement of A_{11} in A is defined as $A_{22} - A_{21}A_{11}^{-1}A_{12}$. For Schur complements, we have $\det A = (\det A_{11}) \det(A_{22} - A_{21}A_{11}^{-1}A_{12})$. Similarly, if A_{22} is nonsingular, then the Schur complement of A_{22} in A is $A_{11} - A_{12}A_{22}^{-1}A_{21}$, and we have $\det A = (\det A_{22}) \det(A_{11} - A_{12}A_{22}^{-1}A_{21})$.

Lemma 3.3 (see [5, 9]). *Let $B = \begin{bmatrix} B_0 & B_1 \\ B_1 & B_0 \end{bmatrix}$ be a symmetric 2×2 block matrix such that B_0 and B_1 are square matrices of the same order. Then, the spectrum of B is the union of the spectra of $B_0 + B_1$ and $B_0 - B_1$.*

Lemma 3.4 (see [9]). *Let B be a square matrix of order n. If each column sum of B is equal to some eigenvalue (say α) of B, then*

$$J_{1\times n}(\lambda I - B)^{-1} J_{n\times 1} = \frac{n}{\lambda - \alpha}.$$

The following result is about the spectrum of a special kind of block matrices.

Lemma 3.5. *Let A be an $(n+1) \times (n+1)$ matrix of the form*

$$A = \begin{bmatrix} 0 & J_{1\times n} \\ J_{n\times 1} & J_n \end{bmatrix}.$$

Then

$$\sigma(A) = \left\{ \begin{array}{cc} 0 & \dfrac{n \pm \sqrt{n^2 + 4n}}{2} \\ (n-1) & 1 \end{array} \right\}.$$

Proof. The characteristic polynomial of A is given by

$$\det(\lambda I_{n+1} - A) = \det \begin{bmatrix} \lambda & -J_{1\times n} \\ -J_{n\times 1} & \lambda I_n - J_n \end{bmatrix}.$$

By Schur complement formula and Lemma 3.4, we have

$$\begin{aligned}\det(\lambda I_{n+1}-A) &= \det(\lambda I_n - J_n)\det\left[\lambda - J_{1\times n}(\lambda I_n - J_n)^{-1}J_{1\times n}\right]\\ &= \lambda^{n-1}(\lambda-n)\det\left[\lambda - \frac{n}{\lambda-n}\right]\\ &= \lambda^{n-1}(\lambda^2-n\lambda-n),\end{aligned}$$

which gives the required result. □

If A and B are matrices of order $m\times n$ and $p\times q$, respectively, then the Kronecker product of the matrices A and B, denoted by $A\otimes B$, is the $mp\times nq$ block matrix $[a_{ij}B]$.

Lemma 3.6 (see [6]). *Let $A\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{m\times m}$. If $\sigma(A)=\{\lambda_1,\ldots,\lambda_n\}$ and $\sigma(B)=\{\mu_1,\ldots,\mu_m\}$ are the spectra of A and B, respectively, then $\sigma(A\otimes B)=\{\lambda_i\mu_j : i=1,\ldots,n;\ j=1,\ldots,m\}$.*

Next, we compute ϵ-eigenvalues of some classes of graphs. First, we compute the ϵ-spectrum of the corona of any connected graph G with the complete graph on n vertices.

Theorem 3.4. *If K_n is the complete graph on n vertices and G is any connected graph on m vertices, then*

$$Spec_\epsilon(K_n\circ G)=\left\{\begin{array}{ccccc}0 & -\lambda_1 & -\lambda_2 & \lambda_1(n-1) & \lambda_2(n-1)\\ n(m-1) & n-1 & n-1 & 1 & 1\end{array}\right\},$$

where λ_1 and λ_2 are the roots of $x^2-mx-m=0$.

Proof. Let K_n be the complete graph on n vertices and let G be any connected graph on m vertices. Then, the graph $K_n\circ G$ consists of n vertices of the complete graph K_n which are labeled using the index set $\{1,2,\ldots,n\}$, and n disjoint copies $G_1,G_2,\ldots,G_n$ of G. Choose an arbitrary ordering $g_1,g_2,\ldots,g_m$ of the vertices of G, and label the vertices of G_i corresponding to g_k by the indices $i+nk$ (see [8]). Under this labeling, the adjacency matrix of the eccentrical graph of $K_n\circ G$ is given by $A_\epsilon(K_n\circ G)=A\otimes B$, where

$$A=\begin{bmatrix}0 & J_{1\times m}\\ J_{m\times 1} & J_m\end{bmatrix}\quad\text{and}\quad B=J_n-I_n.$$

By Lemma 3.5, we have

$$\sigma(A)=\left\{\begin{array}{cc}0 & \dfrac{3m\pm\sqrt{m^2+4m}}{2}\\ (m-1) & 1\end{array}\right\}$$

and

$$\sigma(B)=\left\{\begin{array}{cc}-1 & (n-1)\\ (n-1) & 1\end{array}\right\}.$$

Now, by Lemma 3.6, the spectrum of $A\otimes B$ is

$$\sigma(A\otimes B)=\left\{\begin{array}{ccccc}0 & -\lambda_1 & -\lambda_2 & \lambda_1(n-1) & \lambda_2(n-1)\\ n(m-1) & n-1 & n-1 & 1 & 1\end{array}\right\},$$

and hence

$$Spec_\epsilon(K_n\circ G)=\left\{\begin{array}{ccccc}0 & -\lambda_1 & -\lambda_2 & \lambda_1(n-1) & \lambda_2(n-1)\\ n(m-1) & n-1 & n-1 & 1 & 1\end{array}\right\}.$$

□

Next, we consider the ϵ-spectrum of the complete product of two graphs.

Theorem 3.5. *Let G_1 and G_2 be any two non-complete connected graphs. If the eigenvalues of the adjacency matrices of $\overline{G_1}$ and $\overline{G_2}$ are known, then the ϵ-spectrum of $G_1\vee G_2$ is the union of the spectra of $A(\overline{G_1})$ and $A(\overline{G_2})$.*

Proof. By the definition of the eccentrical graph, we have that

$$A_\epsilon(G_1\vee G_2)=\begin{bmatrix}A(\overline{G_1}) & 0\\ 0 & A(\overline{G_2})\end{bmatrix}.$$

Therefore, the ϵ-spectrum of $G_1\vee G_2$ is the union of the spectra of $A(\overline{G_1})$ and $A(\overline{G_2})$.. □

Lemma 3.7 (see [4]). *Let G_i be a connected r_i-regular graph with n_i vertices, where $i=1,2$. The characteristic polynomial of the complete product of G_1 and G_2 is*

$$P_{G_1\vee G_2}(\lambda)=\frac{P_{G_1}(\lambda)P_{G_2}(\lambda)}{(\lambda-r_1)(\lambda-r_2)}\left[(\lambda-r_1)(\lambda-r_2)-n_1n_2\right].$$

The following result gives the ϵ-spectrum of the complete product of a connected regular graph G with the complete graph K_m.

Theorem 3.6. *Let G be a connected r-regular graph with n vertices. Let $r, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the adjacency matrix of G.*

(a). *If $r < n-1$, then*

$$Spec_\epsilon(G \vee K_m) = \left\{ \begin{array}{ccccccc} \mu_1 & \mu_2 & -(\lambda_2+1) & \ldots & -(\lambda_n+1) & -1 \\ 1 & 1 & 1 & \ldots & 1 & m-1 \end{array} \right\},$$

where μ_1 and μ_2 are the roots of $x^2-(n+m-r-2)x+(n-r-1)(m-1)-nm=0$.

(b). *If $r = n-1$, then*

$$Spec_\epsilon(G \vee K_m) = \left\{ \begin{array}{cc} n+m-1 & -1 \\ 1 & n+m-1 \end{array} \right\}.$$

Proof. (a). If $r < n-1$, then by the definition of the eccentrical graph, we have that $\epsilon(G \vee K_m) \cong \overline{G} \vee K_m$ and hence

$$A_\epsilon(G \vee K_m) = \left[\begin{array}{cc} A(\overline{G}) & J_{n\times m} \\ J_{m\times n} & J_{m\times m}-I_m \end{array} \right],$$

By Lemma 3.7, we have

$$Spec_\epsilon(G \vee K_m) = \left\{ \begin{array}{cccccc} \mu_1 & \mu_2 & -(\lambda_2+1) & \ldots & -(\lambda_n+1) & -1 \\ 1 & 1 & 1 & \ldots & 1 & m-1 \end{array} \right\}.$$

where μ_1 and μ_2 are the roots of $x^2-(n+m-r-2)x+(n-r-1)(m-1)-nm=0$.

(b). If $r = n-1$, then by the definition of the eccentrical graph, we have $\epsilon(G \vee K_m) \cong K_{n+m}$ and hence

$$Spec_\epsilon(G \vee K_m) = \left\{ \begin{array}{cc} n+m-1 & -1 \\ 1 & n+m-1 \end{array} \right\}.$$

□

Theorem 3.7. *Let G be a connected r-regular graph with n vertices. Let $r, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the adjacency matrix of G.*

(a). *If $r < n-1$, then*

$$Spec_\epsilon(G \vee 2K_m) = \left\{ \begin{array}{ccccccc} n-r-1 & m & 0 & -(\lambda_2+1) & \ldots & -(\lambda_n+1) & -m \\ 1 & 1 & 2m-2 & 1 & \ldots & 1 & 1 \end{array} \right\};$$

(b). *If $r = n-1$, then*

$$Spec_\epsilon(G \vee 2K_m) = \left\{ \begin{array}{ccccc} \mu_1 & \mu_2 & 0 & -1 & -m \\ 1 & 1 & 2m-2 & n-1 & 1 \end{array} \right\}.$$

where μ_1 and μ_2 are the roots of $x^2-(n+m-1)x-m(n+1)=0$.

Proof. (a). If $r < n-1$, by the definition of the eccentrical graph, we have

$$A_\epsilon(G \vee 2K_m) = \left[\begin{array}{cc} A(\overline{G}) & 0 \\ 0 & B \end{array} \right],$$

where

$$B = \left[\begin{array}{cc} 0 & J_{m\times m} \\ J_{m\times m} & 0 \end{array} \right].$$

Also, we have

$$Spec(\overline{G}) = \left\{ \begin{array}{cccc} n-r-1 & -(\lambda_2+1) & \ldots & -(\lambda_n+1) \\ 1 & 1 & \ldots & 1 \end{array} \right\}$$

and

$$Spec(B) = \left\{ \begin{array}{ccc} m & 0 & -m \\ 1 & 2m-2 & 1 \end{array} \right\}.$$

Therefore,

$$Spec_\epsilon(G \vee 2K_m) = \left\{ \begin{array}{ccccccc} n-r-1 & m & 0 & -(\lambda_2+1) & \ldots & -(\lambda_n+1) & -m \\ 1 & 1 & 2m-2 & 1 & \ldots & 1 & 1 \end{array} \right\}.$$

(b). If $r = n-1$, then by the definition of the eccentrical graph, we have

$$A_\epsilon(G \vee 2K_m) = \begin{bmatrix} J_{n\times n} - I_n & J_{n\times 2m} \\ J_{2m\times n} & B \end{bmatrix},$$

where

$$B = \begin{bmatrix} 0 & J_{m\times m} \\ J_{m\times m} & 0 \end{bmatrix}.$$

By Lemma 3.4, we have

$$J_{n\times 2m}(\lambda I_n - B)^{-1} J_{2m\times n} = \frac{2m}{\lambda - m} J_{n\times n}.$$

By Schur complement formula, we have

$$\begin{aligned}
\det(\lambda I_{n+2m} - A_\epsilon(G \vee 2K_m)) &= \det(\lambda I_{2m} - B)\det(\lambda I_n + I_n - J_n - J_{n\times 2m}(\lambda I_n - B)^{-1} J_{2m\times n}) \\
&= \det(\lambda I_{2m} - B)\det\left(\lambda I_n + I_n - \left(1 + \frac{2m}{\lambda - m}\right) J_n\right) \\
&= (\lambda - m)(\lambda + m)\left(\lambda - n + 1 - \frac{2mn}{\lambda - m}\right)\lambda^{2m-1}(\lambda+1)^{n-1} \\
&= (\lambda + m)[\lambda^2 - (m+n-1)\lambda - m(n+1)]\lambda^{2m-1}(\lambda+1)^{n-1},
\end{aligned}$$

which yields the required result. □

In the next theorem, we compute the ϵ-spectrum of the cocktail-party graph.

Theorem 3.8. *If $CP(n)$ is the cocktail-party graph on $2n$ vertices, then*

$$Spec_\epsilon(CP(n)) = \left\{ \begin{array}{cc} 1 & -1 \\ n & n \end{array} \right\}.$$

Proof. Let K_{2n} be the complete graph on $2n$ vertices. We delete n disjoint edges from the K_{2n} to obtain $CP(n)$ as follows: First label K_{2n} using the indices $1, 2, \ldots, 2n$ in clockwise direction, and then delete the edges e_{ij} for $i = 1, \ldots, n$ and $j = n+1, n+2, \ldots, 2n$, only when $i \equiv j \pmod{n}$. The adjacency matrix of eccentrical graph $\epsilon(CP(n))$ is given by

$$A_\epsilon\big(CP(n)\big) = \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ I_{n\times n} & 0_{n\times n} \end{bmatrix}.$$

Therefore, by Lemma 3.3, we have

$$Spec_\epsilon(CP(n)) = \left\{ \begin{array}{cc} 1 & -1 \\ n & n \end{array} \right\}.$$

□

Theorem 3.9. *Let $K_{n_1,\ldots,n_k}$ be the complete k-partite graph such that $\sum_{i=1}^{k} n_i = n$ where $n_i \geq 1$ and $k \leq n-1$. Then*

$$Spec_\epsilon(K_{n_1,\ldots,n_k}) = \left\{ \begin{array}{ccccc} (-1)^n & n_1 - 1 & n_2 - 1 & \ldots & n_k - 1 \\ n-k & 1 & 1 & \ldots & 1 \end{array} \right\}.$$

Proof. The adjacency matrix of the eccentrical graph $\epsilon(K_{n_1,\ldots,n_k})$ is given by

$$A_\epsilon(K_{n_1,\ldots,n_k}) = \begin{bmatrix} J_{n_1} - I_{n_1} & 0 & \ldots & 0 \\ 0 & J_{n_2} - I_{n_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & J_{n_k} - I_{n_k} \end{bmatrix}.$$

Hence, the spectrum of $\epsilon(K_{n_1,\ldots,n_k})$ is the union of eigenvalues of $J_{n_1} - I_{n_1}$, $J_{n_2} - I_{n_2}$, $\cdots$, $J_{n_k} - I_{n_k}$. □

Acknowledgments

The author is very grateful to the referees for their valuable comments and suggestions. This work is supported by the National Natural Science Foundation of China (No. 12201634) and the Hunan Provincial Natural Science Foundation of China (No. 2020JJ4423).

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