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Research Article

A note on generating identities for multiplicative arithmetic functions

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Abstract

In this article, it is shown that many of the identities involving multiplicative arithmetic functions are special cases of a more general formula. The approach employed in this article avoids using classical techniques, including Dirichlet's convolution.

Keywords: arithmetic function; sum of divisors; identities.

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1. Introduction

In the theory of arithmetic functions, there are many curious identities. Consider for example the following classic ones [4]:

$$\sum_{d|n} |\mu(d)| = 2^{\omega(n)} \quad \text{and} \quad \sum_{d|n} \varphi(d) = n. \tag{1}$$

The second identity goes back to Gauss itself. Identities involving arithmetic functions are usually considered and proved independently, as they use properties of the functions included in the identity itself. Other useful tools used in their proofs are Dirichlet convolution, auxiliary arithmetic functions, reducing proofs to prime numbers and so on.

Consider for instance Theorem 2.3 from [1], which is the following identity:

$$\varphi(n) = \sum_{d|n} \mu(d)\frac{n}{d}. \tag{2}$$

The proof of (2) goes as follows: using the definition, rewrite φ as

$$\varphi(n) = \sum_{k=1}^{n} \left[\frac{1}{(n,k)}\right],$$

use a different result to write $[1/(n,k)] = \sum_{d|(n,k)} \mu(d)$, change the order of summation and simplify to the desired form. The formula itself looks simple, yet the proof seems to rely on a few (simple) tricks. The question that could be asked here is whether the proof can be simplified with a method that could be applied to other identities involving arithmetic functions.

The goal of this article is to establish a simple and useful formula valid for arbitrary multiplicative arithmetic functions. This formula then becomes, in some way, a generator for the just mentioned and many more identities of such kind. The usual formula that we are about to see is of the following form:

$$\sum_{d|n} f(n) = (*), \tag{3}$$

where f is some arithmetic function and $(*)$ is the closed form of the left-hand side. We highlight that our proofs are direct and our method skips consideration of arithmetic functions as a commutative ring with unity, where the role of multiplication is taken by the Dirichlet convolution. In other words, our approach is purely number-theoretical and does not resolve to any algebraic structure of arithmetic functions.

Notice that whenever we have a formula of type (3), Möbious inversion formula [1,4] immediately allows us to write a new formula; for example, the provided examples in (1) turn into

$$|\mu(n)| = \sum_{d|n} \mu\left(\frac{n}{d}\right) 2^{\omega(d)} \quad \text{and} \quad \varphi(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) d.$$

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We will not write these formulas explicitly unless they are directly mentioned and proved without the inversion formula. The reader can always transform one identity into another one using the inversion formula.

Throughout the remaining part of this article, let n be a positive integer with the canonical factorization

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where each p_i is a prime number and $\alpha_i \geq 1$ for each $i = 1, \dots, k$. Let μ, φ, ψ, λ, J_m, and ω denote the Möbious, Euler, Dedekind, Liouville lambda, Jordan totient, and prime omega functions, respectively. Denote by $\sigma_m(n)$ the sum of m-th powers of divisors of n (in particular, $\sigma_0(n)$ denotes the number of divisors of n). We refer to [1, 4] or other classic number theory books for definitions, examples, and properties of arithmetic functions.

2. Main identity

Let $g\colon \mathbb{N} \to \mathbb{C}$ be any (not necessarily completely) multiplicative arithmetic function; that is, $g(ab) = g(a)g(b)$ for co-prime a and b. Recall that if g is multiplicative, then

$$g(n) = \prod_{i=1}^{\omega(n)} g(p_i^{\alpha_i}).$$

Notice the slight abuse of notation: we write $\omega(n)$ instead of k so that the number of prime divisors is not explicit and in fact becomes a variable in the identity. We now state our basic result.

Theorem 2.1. *If g is a multiplicative arithmetic function and n is any integer greater than 1, then*

$$\prod_{i=1}^{\omega(n)} \sum_{k=0}^{\alpha_i} g(p_i^k) = \sum_{d|n} g(d). \tag{4}$$

Proof. Fix $n > 1$ and define the function

$$G(n) = \sum_{d|n} g(d).$$

Notice that G is also multiplicative (see [4, Theorem 265]). This implies

$$G(n) = \prod_{i=1}^{\omega(n)} g(p_i^{a_i}) = \prod_{i=1}^{\omega(n)} \sum_{d|p_i^{a_i}} g(d) = \prod_{i=1}^{\omega(n)} \sum_{k=0}^{\alpha_i} g(p_i^k).$$

On the other hand, we have

$$G(n) = \sum_{d|n} g(n)$$

and the proof is finished. □

The identity (4) might not be very attractive on its own. This is however our generator for a variety of identities. We note that we can observe similar formulas in some of the classic number theory books, but they were never considered as a piece of universal machinery in the way we consider them.

3. Applications

In this section, we present applications of Theorem 2.1 to obtain many identities. Let us note that most of these identities can be found in [1, 4, 7].

First, we note intermediate formulas (see [1, Theorem 2.18]).

Corollary 3.1. *If h is a multiplicative arithmetic function, then*

$$\sum_{d|n} \mu(d)h(d) = \prod_{p|n} (1 - h(p))$$

and

$$\sum_{d|n} |\mu(d)|h(d) = \prod_{p|n} (1 + h(p)).$$

Proof. The identities follow immediately from (4) after noting that $\mu(p^k) = 0$ for any prime p and integer $k > 1$ (also recall that $h(1) = 1$ for any multiplicative arithmetic function). □

We note that $\mu^2 = |\mu|$ and hence some of the results can be expressed in two ways. We keep $|\mu|$ whenever it is possible.

Corollary 3.2. *The following hold for $n > 1$:*

$$\sum_{d|n} \mu(d) = 0, \tag{5}$$

$$\sum_{d|n} |\mu(d)| = 2^{\omega(n)}, \tag{6}$$

$$\sum_{d|n} |\mu(d)| m^{\omega(d)} = (m+1)^{\omega(n)}, \tag{7}$$

$$\sum_{d|n} d\mu(d) = \prod_{i=1}^{\omega(n)} (1 - p_i) = \varphi^{-1}(n), \tag{8}$$

$$\sum_{d|n} \varphi(d) = n, \tag{9}$$

$$\sum_{d|n} J_m(d) = n^m, \tag{10}$$

$$\sum_{d|n} d\varphi(d) = \prod_{i=1}^{\omega(n)} \frac{1 + p_i^{2\alpha_i+1}}{1 + p_i}, \tag{11}$$

$$\sum_{d|n} \mu(d)\frac{n}{d} = \varphi(n) \tag{12}$$

$$\sum_{d|n} \sigma_0(d) = \prod_{i=1}^{\omega(n)} \binom{a_i + 2}{2}, \tag{13}$$

$$\sum_{d|n} \frac{1}{d} = \frac{\sigma_1(n)}{n}, \tag{14}$$

$$\sum_{d|n} \sigma_1(d) = \prod_{i=1}^{\omega(n)} \frac{p_i^{\alpha_i+2} - p_i - (\alpha_i + 1)(p_i - 1)}{(p_i - 1)^2}. \tag{15}$$

$$\sum_{d|n} \frac{|\mu(d)|}{\varphi(d)} = \frac{n}{\varphi(n)}, \tag{16}$$

$$\sum_{d|n} \frac{|\mu(d)|}{J_m(d)} = \frac{n^m}{J_m(n)}, \tag{17}$$

$$\sum_{d|n} \mu(d) \left(\frac{n}{d}\right)^m = J_m(n), \tag{18}$$

$$\sum_{d|n} \lambda(d) = \begin{cases} 1, & n \textit{ is a perfect square}, \\ 0, & \textit{otherwise}. \end{cases} \tag{19}$$

Proof. Identities (5)–(8) follow from Corollary 3.1 with $h(n) = 1$, $h(n) = 1$, $h(n) = m^{\omega(n)}$ and $h(n) = n$, respectively. Let $h = J_m$. Then

$$\sum_{k=0}^{\alpha_i} J_m(p_i^k) = 1 + (p_i^m - 1) + (p_i^{2m} - p_i^m) + \cdots + (p^{\alpha_i m} - p^{(\alpha_i - 1)m}) = (p_i^{\alpha_i})^m$$

which implies (9) and (10).

To obtain (11) we take $h(n) = n\varphi(n)$ and notice that

$$\sum_{k=0}^{\alpha_i} p_i^k \varphi(p_i^k) = 1 + \sum_{k=1}^{\alpha_i} p_i^k (p_i^k - p_i^{k-1}) = \frac{1 + p_i^{2\alpha_i+1}}{1 + p_i}.$$

To see (13) we take $h = \sigma_0$ and notice that

$$\sum_{k=0}^{\alpha_i} \sigma_0(p_i^k) = 1 + 2 + \cdots + (\alpha_i + 1) = \binom{\alpha_i + 2}{2}.$$

Identity (14) is obtained by taking $g(n) = \frac{1}{n}$ and noticing that

$$\sum_{k=0}^{\alpha_i} \frac{1}{p_i^k} = \frac{p_i^{\alpha_i+1} - 1}{p_i^{\alpha_i}(p_i - 1)}$$

and using the well-known formula

$$\sigma_1(n) = \prod_{i=1}^{\omega(n)} \frac{p^{\alpha_i+1} - 1}{p_i - 1}.$$

Next, we have

$$\sum_{k=0}^{\alpha_i} \sigma_1(p_i^k) = 1 + (1 + p_i) + (1 + p_i + p_i^2) + \cdots + (1 + p_i + \cdots + p_i^{\alpha_i}) = \frac{p_i^{\alpha_i+2} - p_i - (\alpha_i + 1)(p_i - 1)}{(p_i - 1)^2}$$

which implies (15).

For (17) we apply Corollary 3.1 with $h = \frac{|\mu|}{J_m}$ and use

$$J_m(n) = n^m \prod_{p|n} (1 - \tfrac{1}{p^m})$$

to obtain

$$\sum_{d|n} \frac{|\mu(d)|}{J_m(d)} = \prod_{i=1}^{\omega(n)} \left[1 + \frac{1}{p_i^m - 1}\right] = \frac{1}{\prod_{i=1}^{\omega(n)} \left(1 - \frac{1}{p_i^m}\right)} = \frac{n^m}{J_m(n)}.$$

Note also that (17) implies (16) due to $\varphi = J_1$.

To prove (18) we can either apply Möbious inversion formula to (10) or apply our techniques. Notice that (18) is equivalent to

$$\sum_{d|n} \frac{\mu(d)}{d^m} = \frac{J_m(n)}{n^m}.$$

Then (compare with the proof of (17)),

$$\sum_{d|n} \frac{\mu(d)}{d^m} = \prod_{i=1}^{\omega(n)} \left(1 - \frac{1}{p_i^m}\right) = \frac{J_m(n)}{n^m}.$$

Finally, to prove (19) we notice that

$$\sum_{k=0}^{\alpha_i} \lambda(p_i^k) = \begin{cases} 1, & \alpha_i \text{ is even}, \\ 0, & \alpha_i \text{ is odd}. \end{cases}$$

Thus the final product does not vanish if and only if each α_i is even, that is, if n is a perfect square. □

We note that the proof of identity (14) might be a bit artificial in our reasoning. The immediate proof of that follows from the equality

$$\sum_{d|n} \frac{1}{d} = \frac{1}{n} \sum_{d|n} \frac{n}{d} = \frac{1}{n} \sum_{d|n} d = \frac{1}{n} \sigma_1(n).$$

Identity (11) appeared in [5]. Identity (16) is proven in [2] but the (very detailed and descriptive) proof is almost 3 pages long.

We now show the second part of the identities. We omit the proof and also indicate that one can generate more identities with our approach, thus the list provided in this article is not exhausted.

Corollary 3.3. *The following hold for $n > 1$:*

$$\sum_{d|n} \mu(d)\varphi(d) = \prod_{p|n}(2-p),$$

$$\sum_{d|n} |\mu(d)|\varphi(d) = \prod_{p|n} p,$$

$$\sum_{d|n} |\mu(d)|J_m(d) = \prod_{p|n} p^m,$$

$$\sum_{d|n} \mu(d)\sigma_1(d) = (-1)^{\omega(n)} \prod_{d|n} p,$$

$$\sum_{d|n} \mu(d)\sigma_0(d) = (-1)^{\omega(n)},$$

$$\sum_{d|n} \mu(d)\lambda(d) = 2^{\omega(n)},$$

$$\sum_{d|n} \frac{\mu(d)}{\varphi(d)} = \prod_{p|n} \frac{p-2}{p-1},$$

$$\sum_{d|n} \frac{\mu(d)}{\sigma_1(d)} = \frac{n}{\psi(n)},$$

$$\sum_{d|n} \frac{\mu(d)}{\sigma_0(d)} = 2^{-\omega(n)},$$

$$\sum_{d|n} \frac{|\mu(d)|}{d} = \frac{\psi(n)}{n}.$$

Proof. Apply Corollary 3.1. The details are similar to the proof of the identity (16). □

Combining some of the identities obtained in this section we can prove the identities given in the next corollary.

Corollary 3.4. *The following hold for $n > 1$.*

$$\left(\sum_{d|n} \frac{\mu(d)}{\sigma_0(d)}\right)\left(\sum_{d|n} |\mu(d)|\right) = 1,$$

$$\left(\sum_{d|n} \mu(d)\sigma_1(d)\right)\left(\sum_{d|n} \mu(d)\sigma_0(d)\right) = \sum_{d|n} |\mu(d)|\varphi(d).$$

We now demonstrate how to apply our main result to obtain an identity, counting the number of squares dividing a given number.

Corollary 3.5. *The following identity holds:*

$$\sum_{d^2|n} 1 = \prod_{i=1}^{\omega(n)} \left(1 + \left[\frac{\alpha_i}{2}\right]\right),$$

where $[x]$ denotes the integer part of x.

Proof. Notice that

$$g(n) = \left[\sqrt{n}\right] - \left[\sqrt{n-1}\right] = \begin{cases} 1, & n \text{ is a square}, \\ 0, & \text{otherwise}, \end{cases}$$

is a multiplicative function.

It is now clear that

$$\sum_{d^2|n} 1 = \sum_{d|n}\left(\left[\sqrt{d}\right]-\left[\sqrt{d-1}\right]\right) = \prod_{i=1}^{\omega(n)}\sum_{k=0}^{\alpha_i}\left(\left[\sqrt{p_i^k}\right]-\left[\sqrt{p_i^k-1}\right]\right) = \prod_{i=1}^{\omega(n)}\left(1+\left[\frac{\alpha_i}{2}\right]\right).$$

□

For the next observation, let $A_d(n)$ (respectively, $H_d(n)$) denote the arithmetic mean (respectively, harmonic mean) of the divisors of n.

Corollary 3.6. *For any n, the identity $A_d(n)\cdot H_d(n) = n$ holds.*

Proof. Using (14) we have

$$A_d(n)\cdot H_d(n) = \frac{\sum_{d|n} d}{\sigma_0(n)}\cdot\frac{\sigma_0(n)}{\sum_{d|n}\frac{1}{d}} = \frac{\sigma_1(n)}{\frac{\sigma_1(n)}{n}} = n.$$

□

The identity given in the next corollary appeared in [6].

Corollary 3.7. *For any $n > 1$, it holds that*

$$\sigma_0^2(n) = \sum_{c|n}\sum_{b|c}\sum_{a|b}\mu^2(a).$$

Proof. By (6), we have

$$\sum_{a|b}\mu^2(a) = 2^{\omega(b)}.$$

We use Theorem 3.1 of [3] to get

$$\sum_{b|c} 2^{\omega(b)} = \prod_{i=1}^{\omega(c)}(1+2\nu_{p_i}(b)) = \prod_{i=1}^{\omega(n)}(1+2\nu_{p_i}(b)),$$

where $\nu_{p_i}(b)$ stands for p_i-adic valuation of b. To finish the proof, first recall that $\sum_{k=0}^{m}(2k+1) = (m+1)^2$ and then,

$$\begin{aligned}\sum_{c|n}\prod_{i=1}^{\omega(n)}(1+2\nu_{p_i}(b)) &= \sum_{0\le i_j\le\alpha_j}(1+2i_1)\cdots(1+2i_{\omega(n)})\\ &= \prod_{i=1}^{\omega(n)}\sum_{j=0}^{\alpha_i}(1+2j) = \prod_{i=1}^{\omega(n)}(1+\alpha_i)^2 = \sigma_0^2(n),\end{aligned}$$

where the first equality follows by identifying each $c|n$ with the sequence of exponents $(i_1,\ldots,i_{\omega(n)})$ so that $c = p_i^{i_1}\cdots p_{\omega(n)}^{i_{\omega(n)}}$. The second equality involves changing the order of addition and multiplication. □

By the proof of Corollary 3.7, we obtain also the following well-known identity:

$$\sum_{d|n}\sigma_0(d^2) = \sigma_0^2(n).$$

4. Liouville identity

In the last part of this article, we prove the classic identity given in the next theorem, due to Liouville from 1857. Notice that for example in [7] the proof again uses Dirichlet convolution.

Theorem 4.1. *The following identity holds:*

$$\sum_{d|n}\sigma_0^3(d) = \left(\sum_{d|n}\sigma_0(n)\right)^2.$$

Proof. Recall the classic identity $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$. We use identity (13) to obtain

$$\left(\sum_{d|n} \sigma_0(d)\right)^2 = \prod_{i=1}^{\omega(n)} \binom{\alpha_i+2}{2}^2 = \prod_{i=1}^{\omega(n)} \left(\sum_{k=1}^{\alpha_i+1} k\right)^2 = \prod_{i=1}^{\omega(n)} \sum_{k=0}^{\alpha_i} (1+k)^3.$$

On the other hand, using (4) with $g = \sigma_0^3$, we get

$$\sum_{d|n} \sigma_0^3(d) = \prod_{i=1}^{\omega(n)} \sum_{k=0}^{\alpha_i} \sigma_0^3(p_i^k) = \prod_{i=1}^{\omega(n)} \sum_{k=0}^{\alpha_i} (1+k)^3,$$

which concludes the proof. □

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