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Review Article

An instructive treatment of the Brézis–Browder ordering and maximality principles*

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Abstract

In this paper, the widely used ordering and maximality principles of Haim Brézis and Felix Browder, together with their proofs, are slightly generalized in content and greatly improved in format. For this, the concept of preorders is used instead of the concept of partial orders. Also, the more convenient notation S is used instead of "$\leq$". Moreover, the importance of the supremal composition and a theorem on the existence of a particular increasing sequence is highlighted. Furthermore, instead of an ordering principle, a maximality principle is proved first and it is shown that the arguments applied by Brézis and Browder allow some more general results, even in much more attractive formulations.

Keywords: preorder relations; maximality principles; fixed points.

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1. Introduction

In [11, Section 1], by considering an ordered set X and using the notation $S(x) = \{ y \in X;\ y \geq x \}$ for all $x \in X$, Brézis and Browder proved the following theorem and its corollaries.

Theorem 1.1. *Let* $\phi : X \to \mathbb{R}$ *be a function satisfying*

(1) $x \leq y$ *implies* $\phi(x) \leq \phi(y)$*;*

(2) *for any increasing sequence* $\{x_n\}$ *in* X *such that* $\phi(x_n) \leq C < \infty$ *for all* n*, there exists some* $y \in X$ *such that* $x_n \leq y$ *for all* n*;*

(3) *for every* $x \in X$ *there exists* $u \in X$ *such that* $x \leq u$ *and* $\phi(x) < \phi(u)$.

Then for each $x \in X$*,* $\phi\big(S(x)\big)$ *is unbounded.*

Corollary 1.1. *Let* $\phi : X \to \mathbb{R}$ *be a function, bounded above, and satisfying* (1)*. Assume*

(4) *for any increasing sequence* $\{x_n\}$ *in* X*, there exists some* $y \in X$ *such that* $x_n \leq y$ *for all* n.

Then, for each $a \in X$*, there exists some* $\bar{a} \in X$ *such that* $a \leq \bar{a}$ *and* $\phi\big(S(\bar{a})\big) = \phi(\bar{a})$.

In particular, if we strengthen assumption (1) *to*

(1') $x \leq y$ *and* $x \neq y$ *imply* $\phi(x) < \phi(y)$*,*

then for each $a \in X$ *there exists* $\bar{a} \in X$ *such that* $a \leq \bar{a}$ *and* $\bar{a}$ *is maximal* (*i.e.,* $S(\bar{a}) = \{\bar{a}\}$).

Corollary 1.2. *Let* $\phi : X \to \mathbb{R}$ *be a function satisfying* (1) *and*

(2') *for any increasing sequence* $\{x_n\}$ *in* X *such that* $\phi(x_n) \leq C < \infty$ *for all* n*, there exists some* $y \in X$ *such that* $x_n \leq y$ *for all* n *and* $\phi(x_n) \to \phi(y)$ *as* $n \to \infty$*;*

(3') *for every* $x \in X$ *and for every* $\varepsilon > 0$*, there exists* $x' \in X$ *such that* $x \leq x'$ *and* $\phi(x) < \phi(x') < \phi(x) + \varepsilon$.

Then for each $x \in X$*,* $\phi\big(S(x)\big) = [\,\phi(x),\ +\infty\,)$.

*Theorem 5.1 of this paper, together with a theorem on the existence of monotone subsequences [36], was presented by the first author at The Twenty-First Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities, Brenna, Poland, 2022 (see [35]).

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Theorem 1.1 is usually called the Brézis–Browder ordering principle. Corollary 1.1 is usually called the Brézis–Browder maximal or maximality principle. However, Corollary 1.2 seems not to have a special name. The formulations of Theorem 1.1, Corollary 1.1, and Corollary 1.2 together with their proofs are not completely satisfactory for us in several respects. Concerning them, we make the following remarks.

(a) The authors seem to have used the term "ordered set" for a "partially ordered set". Such a terminological inconvenience frequently occurs in recent works too.

(b) They used the notations "$\leq$" and "S" for the same partial order relation on X. Therefore, their formulations are, in a certain sense, mixed-type, not consequent.

(c) Instead of a partial order relation, one may more naturally consider a preorder (reflexive and transitive) relation on X. Namely, these relations allow almost the same statements.

(d) In contrast to Birkhoff [5, p. 20], preorders are frequently called quasi-orders [53]; the term "pseudo-orders" is also used [47]. Kuttler [43, p. 529] even used the term "partial order" for "preorder". Moreover, Qiu [53] used the term "pre-order" for a transitive relation.

(e) Having in mind relators (families of relations) [63], instead of the notation "$\leq$", it is much better to use the notation "S" for a preorder relation on X.

(f) The importance of preorders lies mainly in the fact that, all minimal structures, generalized topologies and stacks on X can be derived from preorder relators [2, 51, 67]. Therefore, they should not be studied separately.

(g) The arguments applied in their proofs allow greater generality in the formulation of their results; even in much more elegant forms than those in Theorem 1.1 and its corollaries.

(h) It seems more convenient to prove initially the first statement of Corollary 1.1, as a theorem, and then Theorem 1.1 as a consequence of this theorem. In Corollary 1.1, it is better to write $\phi[S(\bar{a})] = \{\phi(\bar{a})\}$.

(i) Instead of a metric, it is more convenient to consider a quasi-pseudo-metric on X [28, p. 3]; such metrics allow almost the same results.

(j) If S is a preorder relation on X, then the function d, defined by $d(x, y) = 0$ if $(x, y) \in S$ and $d(x, y) = 1$ if $(x, y) \notin S$, is a quasi-pseudo-metric on X. Preorders and quasi-pseudo-metrics can also be well motivated in physics; for example, see [42].

(k) Each topology on X can be derived from a family of quasi-pseudo-metrics [52, Section 11.2], and also from a single, semigroup-valued quasi-pseudo-metric [41].

(l) The function ρ and the sequence $(x_n)_{n=1}^{\infty}$, defined in the proof of [11, Theorem 1], and the relation "$\leq$", defined in the proof of [11, Corollary 4], deserve some more detailed and separate considerations.

(m) For instance, instead of the corresponding definitions of [11], for any $x \in X$, we define

$$\rho(x) = \sup\big((\phi \circ S)(x)\big) = \sup\big(\phi[S(x)]\big) \qquad \text{and} \qquad S(x) = \big\{y \in X : \quad d(x, y) \leq \phi(y) - \phi(x)\big\}.$$

(n) Note that ρ is just the supremal composition of the function ϕ and the relation S. Its dual, the infimal composition, plays an important role in linear analysis [56, Theorem 2.1].

(o) If $a \in X$ such that $\rho(a) < +\infty$, then we shall show that there exists an increasing sequence $(x_n)_{n=1}^{\infty}$ in the preordered set $X(S)$, with $x_1 = a$, such that $\phi(x_{n+k}) \leq \rho(x_{n+1}) \leq \rho(x_n)$ for all $n, k \in \mathbb{N}$ and

$$\lim_{n \to \infty} \phi(x_n) = \lim_{n \to \infty} \rho(x_n) = \inf_{n \in \mathbb{N}} \rho(x_n).$$

According to Altman [1], Turinici [85], Park [47] and the second author [69], it would also be much better to consider the more natural expression $\Phi(x, y)$ instead of $\phi(y) - \phi(x)$. That is, to consider a function Φ of X^2 to $\mathbb{R}$ or $\overline{\mathbb{R}}$ instead of a function ϕ of X to $\mathbb{R}$ or $\overline{\mathbb{R}}$.

In the present paper, we are interested in some improved reformulations and straightforward generalizations of Theorem 1.1, Corollary 1.1, and Corollary 1.2. For most of the applications, it is enough to use only some particular cases of the second statement of Corollary 1.1. This is, in contrast to *Zorn's lemma* (see [4, p. 43] and [43, p. 532]), a quite elementary maximality principle which can still be used to prove several famous theorems of nonlinear analysis, such as *Ekeland's variational principle* [25,26], *Caristi's fixed point theorem* [16] and *Daneš' drop theorem* [20] for instance; see also [11,45,66].

The second author's interest in the Brézis–Browder principles was raised by Corneliu Ursescu, who proved [93] advanced versions of the Lagrange-type increment inequality [58] with such a principle. The present paper is a continuation of several former similar instructive papers of the second author and his collaborators [10,33,65,68,71,73]. The main purpose of all these papers was to transform the results of analysis into some more easily readable relational forms. In this respect, both S and d are to be replaced by suitable relators.

2. Relations and functions

A subset R of a product set $X \times Y$ is called a *relation* on X to Y. In particular, a relation R on X to itself is called a relation on X. The relation $\Delta_X = \{(x, x) : \ x \in X\}$ is called the *identity relation* of X. If R is a relation on X to Y, then for any $x \in X$ and $A \subseteq X$ the sets $R(x) = \{y \in Y : \ (x, y) \in R\}$ and $R[A] = \bigcup_{a \in A} R(a)$ are called the *images* or *neighbourhoods* of x and A under R, respectively. If $(x, y) \in R$, then instead of $y \in R(x)$, we may also write $x R y$. However, instead of $R[A]$, we cannot write $R(A)$. Namely, it may occur that, in addition to $A \subseteq X$, we also have $A \in X$. Now, the sets $D_R = \{x \in X : \ R(x) \neq \emptyset\}$ and $R[X]$ may be called the *domain* and *range* of R, respectively. If $D_R = X$, then we say that R is a *relation of* X *to* Y, or that R is a *non-partial relation on* X *to* Y.

In particular, a relation f on X to Y is called a *function* if for every $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ instead of $f(x) = \{y\}$. Moreover, a function $\star$ of X to itself is called a *unary operation* on X. While, a function $*$ of X^2 to X is called a *binary operation* on X. For any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*(x, y)$. If R is a relation on X to Y, then a function f of D_R to Y is called a *selection function* of R if $f(x) \in R(x)$ for all $x \in D_R$. Thus, by the Axiom of Choice [38], we can see that every relation is the union of its selection functions.

For a relation R on X to Y, we may naturally define two *set-valued functions* φ_R of X to $\mathcal{P}(Y)$ and Φ_R of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ such that $\varphi_R(x) = R(x)$ for all $x \in X$ and $\Phi_R(A) = R[A]$ for all $A \subseteq X$. Functions of X to $\mathcal{P}(Y)$ can be naturally identified with relations on X to Y. While, functions of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ are more powerful objects than relations on X to Y. In [77], they were briefly called *corelations* on X to Y. However, if U is a relation on $\mathcal{P}(X)$ to Y and V is a relation on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, then it is better to say that U is a *super relation* and V is a *hyper relation* on X to Y [54,80]. Thus, closures (proximities) [83] are super (hyper) relations.

If R is a relation on X to Y, then we have $R = \bigcup_{x \in X} \big(\{x\} \times R(x)\big)$. Therefore, the values $R(x)$, where $x \in X$, uniquely determine R. Thus, a relation R on X to Y can also be naturally defined by specifying $R(x)$ for all $x \in X$. For instance, the *complement* R^c and the *inverse* R^{-1} of R can be defined such that $R^c(x) = R(x)^c$ for all $x \in X$ and $R^{-1}(y) = \{x \in X : \ y \in R(x)\}$ for all $y \in Y$. We also have $R^c = \big(X \times Y\big) \setminus R$ and $R^{-1} = \{(y, x) : \ (x, y) \in R\}$. Moreover, if S is a relation on Y to Z, then the *composition relation* $S \circ R$ can be defined such that $(S \circ R)(x) = S[R(x)]$ for all $x \in X$. Thus, it can be easily seen that $(S \circ R)[A] = S\big[R[A]\big]$ also holds for all $A \subseteq X$. While, if S is a relation on Z to W, then the *box product* $R \boxtimes S$ can be defined such that $(R \boxtimes S)(x, z) = R(x) \times S(z)$ for all $x \in X$ and $z \in Z$. Thus, it can be shown that $(R \boxtimes S)[A] = S \circ A \circ R^{-1}$ for all $A \subseteq X \times Z$ (see [70,78]). Hence, by taking $A = \{(x, z)\}$, and $A = \Delta_Y$ if $Y = Z$, one can at once see that the box and composition products are actually equivalent tools. However, the box product can be immediately defined for any family of relations.

Now, a relation R on X, i.e., a subset R of X^2, may be briefly defined to be *reflexive* and *transitive* if under the plausible notations $R^0 = \Delta_X$ and $R^2 = R \circ R$, we have $R^0 \subseteq R$ and $R^2 \subseteq R$, respectively. Moreover, R may be briefly defined to be *symmetric* and *antisymmetric* if $R^{-1} \subseteq R$ and $R \cap R^{-1} \subseteq R^0$, respectively. Also, R may be briefly defined to be *total* and *directive* if $X^2 \subseteq R \cup R^{-1}$ and $X^2 \subseteq R^{-1} \circ R$, respectively. Furthermore, a reflexive and transitive (symmetric) relation is called a *preorder (tolerance) relation*. A symmetric (antisymmetric) preorder relation is called an *equivalence (partial order) relation*. In addition to the above well-known basic properties, several further remarkable relational properties were also studied in [59] with the help of the self closure and interior relations $R^- = R^{-1} \circ R$ and $R^\circ = R^{c-c} = \big(R^{-1} \circ R^c\big)^c$.

For a relation R on X, by using $R^0 = \Delta_X$, we define $R^n = R \circ R^{n-1}$ for $n \in \mathbb{N}$. We also define $R^\infty = \bigcup_{n=0}^\infty R^n$. Thus, R^∞ is the smallest preorder relation on X containing R (see [30]). For $A \subseteq X$, the *Pervin relation* defined as $R_A = A^2 \cup (A^c \times X)$ is an important preorder on X [51]. Also, for a *pseudo-metric* d on X, the *Weil surrounding* $B_r = \{(x, y) \in X^2 : \ d(x, y) < r\}$, with $r > 0$, is an important tolerance on X [96]. Note that $S_A = R_A \cap R_A^{-1} = R_A \cap R_{A^c} = A^2 \cup (A^c)^2$ is already an equivalence relation on X. More generally, if $\mathcal{A}$ is a *cover (partition)* of X, then

$S_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$ is a tolerance (equivalence) relation on X. Now, as an important generalization of the Pervin relation R_A, for any $A \subseteq X$ and $B \subseteq Y$, we may also naturally consider the *Hunsaker-Lindgren relation* $R_{(A,B)} = (A \times B) \cup (A^c \times Y)$ (see [34]). However, in general, this is not a preorder relation.

3. Ordered sets and increasing functions

If R is a relation on X to Y, then the ordered pair $(X, Y)(R) = \big((X, Y), R\big)$ is called a *context space* [29], a *relational space*, or a *simple relator space* [48]; for some generalizations, see [54,63,81]. In such a space $(X, Y)(R)$, for any $B \subseteq Y$, we define $\operatorname{Int}_R(B) = \{A \subseteq X : \ R[A] \subseteq B\}$, and in particular $\operatorname{int}_R(B) = \{x \in X : \ \{x\} \in \operatorname{Int}_R(B)\}$. We also define $\mathcal{E}_R = \{B \subseteq Y : \ \operatorname{int}_R(B) \neq \emptyset\}$. Moreover, we define $\operatorname{Lb}_R(B) = \{A \subseteq X : \ A \times B \subseteq R\}$, and in particular $\operatorname{lb}_R(B) = \{x \in X : \ \{x\} \in \operatorname{Lb}_R(B)\}$. However, these tools are not independent of the former ones. Namely, by [63], we have $\operatorname{Lb}_R = \operatorname{Int}_{R^c} \circ \mathcal{C}_Y$.

If in particular R is a relation on X, then having in mind a widely used terminology of Birkhoff [5] the ordered pair $X(R) = (X, R)$ may be naturally called a *goset* (generalized ordered set) [74]. The goset $X(R)$ is, for instance, called *reflexive* if R is a reflexive relation on X. Moreover, the goset $X(R)$ is, for instance, called *total* if R is a total relation on X. Furthermore, the goset $X(R)$ is called a *proset* (preordered set) if R is a preorder on X. Moreover, $X(R)$ is called a *poset* (partially ordered set) if R is a partial order on X. Quite similarly, the abbreviations *toset* (totally ordered set) and *woset* (well-ordered set) of Rudeanu [55] can also be well used.

Every set X is a poset with the identity relation Δ_X. Moreover, X is a proset with the *universal relation* X^2. The power set $\mathcal{P}(X) = \{A : \ A \subseteq X\}$ of X is a poset with the ordinary set inclusion "$\subseteq$". Several definitions on posets can be applied to gosets as well. For instance, if $X(R)$ is a goset, then for any $Y \subseteq X$ the goset $Y(R \cap Y^2)$ is called a *subgoset* of $X(R)$. While, the goset $X(R^{-1})$ is called the *dual* of $X(R)$.

For any $A \subseteq X(R)$, we may naturally define $\min_R(A) = A \cap \operatorname{lb}_R(A)$ and $\sup_R(A) = \min_R\big(\operatorname{ub}_R(A)\big)$. However, if R is antisymmetric, then each of these sets, and their duals, is either empty or a singleton. In particular, the goset $X(R)$ may, for instance, be naturally called *sup-complete* if $\sup_R(A) \neq \emptyset$ for all $A \subseteq X$. Thus, by using that $\inf_R(A) = \sup_R\big(\operatorname{lb}_R(A)\big)$, we can see that if $X(R)$ is sup-complete, then it is also *inf-complete* [7,8].

Now, a function f of one goset $X(R)$ to another $Y(S)$ may be naturally called *increasing* if $u\,R\,v$ implies $f(u)\,S\,f(v)$ for all $u, v \in X$. Also, f is defined to be *decreasing* if it is increasing as a function of $X(R)$ to the dual of $Y(S)$. Moreover, the function f may be called *strictly increasing* if $u\,R\,v$ and $u \neq v$ imply $f(u)\,S\,f(v)$ and $f(u) \neq f(v)$. Furthermore, f is defined to be *strictly decreasing* if it is strictly increasing as a function of $X(R)$ to the dual of $Y(S)$. The relationships between increasing and strictly increasing functions were established in [74].

Now, instead of "increasing" we may also naturally say "continuous". Namely, f is increasing if and only if $v \in R(u)$ implies $f(v) \in S\big(f(u)\big)$. That is, if v is in the R–neighbourhood of u, then $f(v)$ is in the S–neighbourhood of $f(u)$. This property can be reformulated in several equivalent forms. For instance, it can be shown that f is increasing if and only if $f \circ R \subseteq S \circ f$, $R \subseteq f^{-1} \circ S \circ f$, $(f \boxtimes f)[R] \subseteq S$, or $R \subseteq (f \boxtimes f)^{-1}[S]$ (see [76]).

In [74], several additional properties of increasing functions were also proved. For instance, it was proved that if f is an increasing function of one complete poset $X(R)$ to another $Y(S)$, then $\sup_S\big(f[A]\big) \leq f\big(\sup_R(A)\big)$ for all $A \subseteq X$. More importantly, it was also proved that if f is a function of one goset $X(R)$ to another $Y(S)$ such that $f[\sup_R(A)] \subseteq \operatorname{lb}_S\big(\operatorname{ub}_S\big(f[A]\big)\big)$ for all $A \subseteq X$, then $\max_R\big(\operatorname{Int}_f(y)\big) = \sup_R\big(\operatorname{Int}_f(y)\big)$ for all $y \in Y$ (see [64,79]).

Finally, we note that if f is a function of a set X to a goset $Y(S)$, then according to [79], for any $x \in X$ and $y \in Y$, we may also naturally define $\operatorname{Ord}_f(x) = \big\{v \in X : \ f(x)\,S\,f(v)\big\}$ and $\operatorname{Int}_f(y) = \big\{u \in X : \ f(u)\,S\,y\big\}$. Thus, it can be easily seen that Ord_f is the largest relation on X making f to be increasing. Moreover, we have $\operatorname{Ord}_f = \big(\operatorname{Int}_f \circ f\big)^{-1}$ and $\operatorname{Int}_f(y) = f^{-1}\big[\operatorname{lb}_S\big(\{y\}\big)\big]$ for all $y \in Y$.

4. A supremal composition

In the sequel, by following Brézis and Browder [11] as closely as possible, we shall use the following notation.

Notation 4.1. We assume that:

(1) X **is a nonvoid set;**

(2) φ **is a function of** X **to** $\mathbb{R}$**;**

(3) S **is a preorder relation on** X**;**

(4) $\rho(x) = \sup\big((\varphi \circ S)(x)\big)$ **for all** $x \in X$**.**

Here, the supremum is to be taken in the complete ordered set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ instead of $\mathbb{R}$. (Precise instructive treatments of these sets can found in [6,27].)

Remark 4.1. For any $x \in X$, we have

(1) $x \in S(x)$; **(2)** $S(x) = S[S(x)]$; **(3)** $\rho(x) = \sup_{y \in S(x)} \varphi(y)$.

To prove assertion (2) of Remark 4.1, note that $\Delta_X \subseteq S$ and $S \circ S \subseteq S$, and thus

$$S(x) = \Delta_X[S(x)] \subseteq S[S(x)] = (S \circ S)(x) \subseteq S(x).$$

Now, by using the properties of the supremum, we prove the next result.

Theorem 4.1. *ρ is a decreasing function of the proset $X(S)$ to $]-\infty, +\infty]$ such that $\varphi \leq \rho$.*

Proof. For any $x \in X$, it is sufficient to prove that the following assertions hold:

(1) $-\infty < \varphi(x) \leq \rho(x) \leq +\infty$; (2) $x S y \implies \rho(y) \leq \rho(x)$.

From Remark 4.1, it follows that $\varphi(x) \in \varphi[S(x)]$. Therefore, $\varphi[S(x)]$ is a nonvoid subset of $\mathbb{R}$, and thus also of $\overline{\mathbb{R}}$. Hence, by the corresponding properties of the supremum in $\overline{\mathbb{R}}$, we infer that $-\infty < \rho(x) \leq +\infty$. Since $\rho(x) \in \mathrm{ub}(\varphi[S(x)])$, we also note that $\varphi(x) \leq \rho(x)$. Therefore, assertion (1) is true. On the other hand, if $x S y$, then $y \in S(x)$. Hence, by using Remark 4.1, we infer that

$$S(y) \subseteq S[S(x)] \subseteq S(x), \quad \text{and thus} \quad \varphi[S(y)] \subseteq \varphi[S(x)].$$

Now, by using the fact that $\rho(x) \in \mathrm{ub}(\varphi[S(x)])$, we have $\rho(x) \in \mathrm{ub}(\varphi[S(y)])$. Moreover, by using the fact that $\rho(y) \in \mathrm{lb}(\mathrm{ub}(\varphi[S(y)]))$, we also have $\rho(y) \leq \rho(x)$. Therefore, assertion (2) is also true. □

The following result is an immediate consequence of Theorem 4.1.

Corollary 4.1. *If φ is an increasing function of the proset $X(S)$ to $\mathbb{R}$, then $\rho - \varphi$ is a decreasing function of the proset $X(S)$ to $]-\infty, +\infty]$.*

Proof. If $x, y \in X$ such that $x S y$, then by Theorem 4.1, we have $\rho(y) \leq \rho(x)$. Also, by the assumption, we have $\varphi(x) \leq \varphi(y)$. Hence, we infer that $-\varphi(y) \leq -\varphi(x)$. Therefore, by the additivity property of the inequality in $]-\infty, +\infty]$, we also have $\rho(y) - \varphi(y) \leq \rho(x) - \varphi(x)$. □

Remark 4.2. Having in mind the usual definition that $\mathrm{res}(A) = \mathrm{cl}(A) \setminus A$, the nonnegative function $\rho - \varphi$ may be naturally referred to as the *residue* of φ. It is important to note that, analogously to the *infimal composition* considered in [56], the *supremal composition* ρ may also have some further useful properties depending upon some additional properties of X, φ and S. Actually, to prove duals of the forthcoming theorems, an infimal composition has to be used instead of the present supremal one [18]. Infimal compositions are very similar tools to the *infimal convolutions* considered in [31], for instance.

5. An increasing sequence in $X(S)$

By improving the inductive proof of [11, Theorem 1] due to Brézis and Browder, we establish the following fundamental existence theorem.

Theorem 5.1. *If $a \in X$ such that $\rho(a) < +\infty$, then there exists an increasing sequence $(x_n)_{n=1}^{\infty}$ in the proset $X(S)$, with $x_1 = a$, such that*

(1) $(\rho(x_n))_{n=1}^{\infty}$ *is decreasing;*

(2) $\lim_{n \to \infty} \varphi(x_n) = \lim_{n \to \infty} \rho(x_n) = \inf_{n \in \mathbb{N}} \rho(x_n)$; **(3)** $\varphi(x_k) \leq \rho(x_n) \leq \rho(a)$ *for all* $n \in \mathbb{N}$ *and* $n \leq k \in \mathbb{N}$.

Proof. Define $x_1 = a$. Then, because of the finiteness and the definition of $\rho(x_1)$, we have

$$\rho(x_1) - 1 < \rho(x_1) = \sup\big(\varphi[S(x_1)]\big) = \min\big(\mathrm{ub}(\varphi[S(x_1)])\big) \in \mathrm{lb}\big(\mathrm{ub}(\varphi[S(x_1)])\big)$$

Therefore, $\rho(x_1) - 1 \notin \mathrm{ub}(\varphi[S(x_1)])$. Thus, there exists $x_2 \in S(x_1)$ such that

$$\rho(x_1) - 1 < \varphi(x_2).$$

Also, by Theorem 4.1, we state that $\rho(x_2) \leq \rho(x_1) < +\infty$.

Now, because of the finiteness and the definition of $\rho(x_2)$, we quite similarly infer that there exists $x_3 \in S(x_2)$ such that

$$\rho(x_2) - 2^{-1} < \varphi(x_3).$$

Moreover, by Theorem 4.1, we state that $\rho(x_3) \leq \rho(x_2) < +\infty$.

Hence, by induction, it is clear that there exists an increasing sequence $(x_n)_{n=1}^{\infty}$ in the proset $X(S)$ such that $\rho(x_n)$ is finite,

$$\rho(x_n) - n^{-1} < \varphi(x_{n+1})$$

and $\rho(x_{n+1}) \leq \rho(x_n)$ for all $n \in \mathbb{N}$. Thus, assertion (1) is true.

If $n \in \mathbb{N}$, then by the definition of $\rho(x_n)$ we have

$$\rho(x_n) = \sup\big(\varphi[S(x_n)]\big) = \min\big(\text{ub}\,(\varphi[S(x_n)])\big) \in \text{ub}\,(\varphi[S(x_n)]).$$

Since $x_{n+1} \in S(x_n)$, thus $\varphi(x_{n+1}) \in \varphi[S(x_n)]$, and we infer that

$$\varphi(x_{n+1}) \leq \rho(x_n).$$

Therefore, we also state that

$$0 \leq \rho(x_n) - \varphi(x_{n+1}) < n^{-1}$$

for all $n \in \mathbb{N}$, and thus

$$\lim_{n\to\infty}\big(\rho(x_n) - \varphi(x_{n+1})\big) = 0.$$

Moreover, by assertion (1) and a basic theorem on decreasing sequences, we state that

$$\lim_{n\to\infty} \rho(x_n) = \inf_{n\in\mathbb{N}} \rho(x_n),$$

which can however be $-\infty$. Hence, since

$$\varphi(x_{n+1}) = \rho(x_n) - \big(\rho(x_n) - \varphi(x_{n+1})\big)$$

for all $n \in \mathbb{N}$, we infer that

$$\lim_{n\to\infty} \varphi(x_{n+1}) = \lim_{n\to\infty} \rho(x_n).$$

Thus, since $\lim_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} \varphi(x_{n+1})$, assertion (2) is also true.

Now, to complete the proof, we need only note that, analogously to the inequality $\varphi(x_{n+1}) \leq \rho(x_n)$, assertion (3) can also be easily derived from the definition of $\rho(x_n)$ and assertion (1). Note that, analogously to assertion (1), assertion (3) also needs the transitivity of S. □

Remark 5.1. If $a \in X$ such that $\rho(a) < +\infty$ and φ is increasing on $S(a)$, then in addition to the assertions of Theorem 5.1, we also state that

(4) $(\varphi(x_n))_{n=1}^{\infty}$ is increasing;

(5) $\lim_{n\to\infty} \varphi(x_n) = \sup_{n\in\mathbb{N}} \varphi(x_n)$.

Therefore, in this case, the limits considered in assertion (2) of Theorem 5.1 are actually finite.

6. φ – Maximal elements

Motivated by Turinici [87, 88], we introduce the following:

Definition 6.1. An element x of the proset $X(S)$ is called

(1) *φ–maximal* if $x\,S\,y \implies \varphi(y) \leq \varphi(x)$ for all $y \in X$;

(2) *strongly φ–maximal* if $x\,S\,y \implies \varphi(y) = \varphi(x)$ for all $y \in X$.

Remark 6.1. Following [53, 61], the element x is called

(1) maximal if $x\,S\,y \implies y\,S\,x$ for all $y \in X$;

(2) strongly maximal if $x\,S\,y \implies y = x$ for all $y \in X$.

Thus, x is maximal if and only if S is *symmetric at* x, i.e., $S(x) \subseteq S^{-1}(x)$. Also, x is strongly maximal if and only if $S(x) \subseteq \{x\}$. Note that, by the *reflexivity of* S *at* x, we have $x \in S(x)$, and thus $\{x\} \subseteq S(x)$. Moreover, by the reflexivity of S on X, we state that "strongly maximal" implies "maximal". While, if S is antisymmetric, then we state that "maximal" implies "strongly maximal". Therefore, in a poset, the two notions coincide. In this respect, it is also worth noticing that S is symmetric if and only if every element of $X(S)$ is maximal. Moreover, x is a strongly maximal element of $X(S)$ if and only if $x\,S\,y$ does not hold for all $y \in X$ with $y \neq x$.

Now, we are prepared to prove the next result.

Theorem 6.1. *The following assertions hold:*

(1) *if x is a strongly maximal element of $X(S)$, then x is a strongly φ–maximal element of $X(S)$;*

(2) *if x is a maximal element of $X(S)$ and φ is increasing on $S(x)$, then x is a strongly φ–maximal element of $X(S)$.*

Proof. Because of the reflexivity of S, for any $x \in X$ we have $x \in S(x)$. Moreover, for any $y \in X$, with $x\,S\,y$, we have $y \in S(x)$. Therefore, if the conditions of assertion (2) hold, then for any $y \in X$ we have

$$\big(\, x\,S\,y \implies \varphi(x) \leq \varphi(y) \,\big) \qquad \text{and} \qquad \big(\, x\,S\,y \implies y\,S\,x \implies \varphi(y) \leq \varphi(x) \,\big).$$

Hence, we see that $x\,S\,y$ implies $\varphi(y) = \varphi(x)$, and thus the conclusion of assertion (2) also holds. □

Remark 6.2. From the above proof, we also observe that if x is a φ–maximal element of $X(S)$ and φ is increasing on $S(x)$, then x is a strongly φ–maximal element of $X(S)$. Therefore, if φ is increasing on X, then the two notions introduced in Definition 6.1 coincide.

The next result gives a partial converse to Theorem 6.1.

Theorem 6.2. *The following assertions hold:*

(1) *if x is a strongly φ–maximal element of $X(S)$ and φ is either injective or strictly increasing on $S(x)$, then x is a strongly maximal element of $X(S)$;*

(2) *if x is a φ–maximal element of $X(S)$, S is total on $S(x)$ and φ is either strictly increasing or injective and increasing on $S(x)$, then x is a maximal element of $X(S)$.*

Proof. If $x, y \in X$ such that $x\,S\,y$, then we have $x, y \in S(x)$ (as in the proof of Theorem 6.1). Moreover, if x is a φ–maximal element of $X(S)$, then $\varphi(y) \leq \varphi(x)$.

Now, if $y\,S\,x$ does not hold, then by the reflexivity and the totality of S on $S(x)$, we note that $x \neq y$ and $x\,S\,y$. Hence, if φ is strictly increasing on $S(x)$, we infer that $\varphi(x) < \varphi(y)$. Therefore, $\varphi(y) < \varphi(y)$, and thus $\varphi(y) \neq \varphi(y)$. This contradiction proves that $y \leq x$, and thus x is a maximal element of $X(S)$. □

Remark 6.3. By the corresponding statements of [74], we note that:

(1) if φ is injective and increasing, then φ is strictly increasing;

(2) if φ is strictly increasing and S is total, then φ is injective and φ^{-1} is also strictly increasing.

7. Characterizations of φ – maximal elements

In this section, we prove two theorems about the characterization of φ – maximal elements.

Theorem 7.1. *For any $x \in X(S)$, the following assertions are equivalent:*

(1) *x is φ–maximal;*

(2) *$\varphi[\,S(x)\,] \subseteq \,]-\infty, \varphi(x)\,]$;*

(3) *$\rho(x) \leq \varphi(x)$;*

(4) *$\rho(x) = \varphi(x)$.*

Proof. If assertion (1) holds, then

$$y \in S(x) \implies x\,S\,y \implies \varphi(y) \leq \varphi(x) \implies \varphi(y) \in \,]-\infty, \varphi(x)\,].$$

Therefore, assertion (2) also holds.

If assertion (2) holds, then $\varphi(y) \leq \varphi(x)$ for all $y \in S(x)$, and thus $\varphi(x) \in \mathrm{ub}\,(\,\varphi[\,S(x)\,]\,)$. Hence, by using the fact that

$$\rho(x) = \sup\,(\,\varphi[\,S(x)\,]\,) = \min\,\big(\,\mathrm{ub}\,(\varphi[\,S(x)\,])\,\big) \in \mathrm{lb}\,(\,\mathrm{ub}\,(\,\varphi[\,S(x)\,]\,)\,)\,,$$

we conclude that assertion (3) also holds.

If assertion (3) holds, then by Theorem 4.1, assertion (4) also holds; namely, by this theorem, the inequality $\varphi(x) \leq \rho(x)$ is always true.

Finally, if assertion (4) holds, then

$$\varphi(x) = \rho(x) = \sup(\varphi[S(x)]) = \min\big(\mathrm{ub}\big(\varphi[S(x)]\big)\big) \in \mathrm{ub}(\varphi[S(x)]) .$$

Therefore, for any $y \in X$,

$$x\,S\,y \implies y \in S(x) \implies \varphi(y) \in \varphi[S(x)] \implies \varphi(y) \leq \varphi(x) .$$

Thus, assertion (1) also holds. □

Theorem 7.2. *For any $x \in X(S)$, the following assertions are equivalent :*

(1) *x is strongly φ–maximal ;* **(2)** $\varphi[S(x)] \subseteq \{\varphi(x)\}$ *;* **(3)** $\varphi[S(x)] = \{\varphi(x)\}$.

Proof. If assertion (1) holds, then

$$y \in S(x) \implies x\,S\,y \implies \varphi(y) = \varphi(x) \implies \varphi(y) \in \{\varphi(x)\}.$$

Hence, assertion (2) also holds.

If assertion (2) holds, then by using the facts that $x \in S(x)$ and $\varphi(x) \in \varphi[S(x)]$, we conclude that assertion (3) also holds.

Finally, if assertion (3) holds, then for any $y \in X$

$$x\,S\,y \implies y \in S(x) \implies \varphi(y) \in \varphi[S(x)] \implies \varphi(y) \in \{\varphi(x)\} \implies \varphi(y) = \varphi(x) .$$

Therefore, assertion (1) also holds. □

Remark 7.1. By taking $O = \mathrm{Ord}_\varphi$, we note that :

(1) O is a total preorder relation on X;

(2) O is the largest relation on X making the function φ increasing;

(3) if x is a φ–maximal element of $X(S)$, then $S(x) \subseteq O^{-1}(x)$;

(4) if x is a strongly φ–maximal element of $X(S)$, then $S(x) \subseteq \big(O \cap O^{-1}\big)(x)$.

The induced preorder was formerly considered also by Wallace [95] and Patrone [49]. However, it is now more interesting for us that, for the relation S , Patrone would define the corresponding irreflexive relation T on X by $T(x) = S(x) \setminus S^{-1}(x)$ instead of $T(x) = S(x) \setminus \{x\}$.

8. Relationships between maximal elements and fixed points

Definition 8.1. If T is a relation on X, then an element x of X is called a

(1) *fixed point* of T if $x \in T(x)$; **(2)** *strong fixed point* of T if $\{x\} = T(x)$.

Remark 8.1. An element x of X is a fixed point of T if and only if T is reflexive at x . Moreover, x is a strong fixed point of T if and only if x is both a fixed point of T and a strongly maximal element of $X(T)$.
Our present terminology can only be motivated by the latter fact. In the existing literature, instead of "strong fixed point" the terms "stationary point", "invariant point" and "endpoint" are commonly used [3, 37].

Because of the assumed reflexivity of S , we have the next result.

Theorem 8.1. *For an element x of $X(S)$, the following assertions are equivalent :*

(1) *x is a strong fixed point of S ;* **(2)** *x is a strongly maximal element of $X(S)$.*

In addition to Definition 8.1, we also introduce the following:

Definition 8.2. A relation T on $X(S)$ is called

(1) *intensive* if for each $x \in X$ there exists $y \in T(x)$ such that $y\,S\,x$,

(2) *extensive* if for each $x \in X$ there exists $y \in T(x)$ such that $x\,S\,y$.

Remark 8.2. For a function f of the poset $X(S)$ to itself, besides "extensive", the terms "expansive", "progressive", "inflationary" and "noncontractive" are also frequently used.
More curiously, Davey and Pristley [21, p. 186] would even call a function f of a goset $X(S)$ to itself to be "increasing" if each $x \in X$ is a "post-fixpoint" of f in the sense that $x\,S\,f(x)$.

A simple reformulation of Property (2) in Definition 8.2 gives the next result.

Theorem 8.2. *For a relation T on $X(S)$, the following assertions are equivalent:*

(1) *T is extensive;* **(2)** *$S^{-1} \circ T$ is reflexive on X;* **(3)** *$S(x) \cap T(x) \neq \emptyset$ for all $x \in X$.*

Remark 8.3. An extensive relation on $X(S)$ is non-partial. Also, if T is a reflexive relation on $X(S)$, then because of the assumed reflexivity of S, the relation T is extensive.

The importance of strongly maximal elements is apparent from the following extension of a simple, but important observation of Brondsted [13, 14].

Theorem 8.3. *If T is an extensive relation on a goset $X(S)$, then each strongly maximal element x of $X(S)$ is a fixed point of T.*

Proof. Since T is extensive, there exists $y \in T(x)$ such that $x\,S\,y$. Hence, $x = y$ because x is strongly maximal. Therefore, $x \in T(x)$, and thus x is a fixed point of T. □

The next result is an immediate consequence of Theorem 8.3.

Corollary 8.1. *If f is a extensive function of a goset $X(S)$ to itself, then each strongly maximal element x of $X(S)$ is a fixed point of f.*

By modifying an argument of Khamsi [40], we now prove the following result, which gives a partial converse to the above theorem and also it does not need any particular property of the relation S.

Theorem 8.4. *If every extensive function of a goset $X(S)$ to itself has a fixed point, then $X(S)$ has a strongly maximal element.*

Proof. Assume, on the contrary, that each $x \in X$ is not a strongly maximal element of $X(S)$. Then, by Definition 6.1, for each $x \in X$, there exists $y \in X$ such that $x\,S\,y$, but $y \neq x$. Hence, we infer that $y \in S(x) \setminus \{x\}$. Define

$$T(x) = S(x) \setminus \{x\}$$

for all $x \in X$. Then, by the above observation, T is a non-partial relation on X. Thus, by the Axiom of Choice, there exists a function f of X to itself such that

$$f(x) \in T(x) = S(x) \setminus \{x\} \subseteq S(x)$$

for all $x \in X$. Hence, by Definition 8.2, f is an extensive function of $X(S)$. Thus, by the assumption of the theorem, there exists an $x \in X$ such that $f(x) = x$. Hence, $x = f(x) \in T(x) = S(x) \setminus \{x\}$. Therefore, $x \notin \{x\}$, and thus $x \neq x$; this contradiction proves the theorem. □

As a slight generalization of the dual of [40, Theorem 1] of Khamsi, we also state the next result.

Corollary 8.2. *For a goset $X(S)$, the following assertions are equivalent:*

(1) *$X(S)$ has a strongly maximal element;*

(2) *every extensive relation on $X(S)$ has a fixed point;*

(3) *every extensive function of $X(S)$ to itself has a fixed point.*

Remark 8.4. To further clarify the importance of extensive relations, we note that a *closure operation* on $X(S)$ is, by definition, an extensive function of $\mathcal{P}(X)$ to itself.
Moreover, a strictly increasing function f of a *well-ordered* set $X(S)$ to itself is extensive. For this, by [74, Theorem 79], we need only that S is antisymmetric and *min–complete* in the sense that $\min_S(A) \neq \emptyset$ whenever $\emptyset \neq A \subseteq X$.

9. Maximality principles

The following theorem is a generalization of the first part of Corollary 1.1 of Brézis and Browder.

Theorem 9.1. *Suppose that* $a \in X$ *such that*

(1) φ *is increasing and bounded above on* $S(a)$*;*

(2) *each increasing sequence* $(x_n)_{n=1}^{\infty}$ *in* $X(S)$*, with* $x_1 = a$*, is bounded above.*

Then, there exists $b \in S(a)$ *such that* b *is a strongly* φ*–maximal element of* $X(S)$*.*

Proof. By assumption (1), there exists $\beta \in \mathbb{R}$ such that $\beta \in \operatorname{ub}\big(\varphi[S(a)]\big)$. Hence, by using the fact that

$$\rho(a) = \sup\big(\varphi[S(a)]\big) = \min\big(\operatorname{ub}\big(\varphi[S(a)]\big)\big) \in \operatorname{lb}\big(\operatorname{ub}\big(\varphi[S(a)]\big)\big)\,,$$

we infer that $\rho(a) \leq \beta < +\infty$. Thus, by Theorem 5.1, there exists an increasing sequence $(x_n)_{n=1}^{\infty}$ in the proset $X(S)$, with $x_1 = a$, such that

$$\lim_{n\to\infty} \rho(x_n) = \lim_{n\to\infty} \varphi(x_n)\,.$$

Hence, by using assumption (2), we infer that there exists $b \in X$ such that

$$x_n S\, b \quad \text{for all} \quad n \in \mathbb{N}, \quad \text{and thus in particular} \quad a\, S\, b\,.$$

Moreover, by using assumption (1) and the fact that $a\, S\, x_n$ holds also for all $n \in \mathbb{N}$, we have

$$\varphi(x_n) \leq \varphi(b) \quad \text{for all} \quad n \in \mathbb{N}, \quad \text{and thus} \quad \lim_{n\to\infty} \varphi(x_n) \leq \varphi(b)\,.$$

While, by using Theorem 4.1, we quite similarly see that $\varphi(b) \leq \rho(b)$,

$$\rho(b) \leq \rho(x_n) \quad \text{for all} \quad n \in \mathbb{N}, \quad \text{and thus} \quad \rho(b) \leq \lim_{n\to\infty} \rho(x_n)\,.$$

Therefore, we actually have

$$\lim_{n\to\infty} \varphi(x_n) \leq \varphi(b) \leq \rho(b) \leq \lim_{n\to\infty} \rho(x_n)\,.$$

Hence, by using the equality of the above two limits, we infer that

$$\varphi(b) = \rho(b)$$

Now, by Theorem 7.1, we conclude that b is a φ–maximal element of $X(S)$. Thus, by Remark 6.2, b is also a strongly φ–maximal element of $X(S)$. □

Remark 9.1. If $(x_n)_{n=1}^{\infty}$ is an increasing sequence in $X(S)$, with $x_1 = a$, then we may also say that $(x_n)_{n=1}^{\infty}$ is an *orbit (trajectory)* of S starting at a.
Thus, Condition (2) of Theorem 9.1 can also be nicely expressed by saying that, each S–orbit $(x_n)_{n=1}^{\infty}$ in X, starting at a, is in some inverse image $S^{-1}(b)$ with $b \in X$.

From Theorem 9.1, by using Theorem 6.2, we derive the following generalization of the second part of Corollary 1.1 of Brézis and Browder.

Corollary 9.1. *Suppose that* $a \in X$ *such that*

(1) φ *is bounded above on* $S(a)$*;*

(2) φ *is either strictly increasing or injective and increasing on* $S(a)$*;*

(3) *each increasing sequence* $(x_n)_{n=1}^{\infty}$ *in* $X(S)$*, with* $x_1 = a$*, is bounded above.*

Then, there exists $b \in S(a)$ *such that* b *is a strongly maximal element of* $X(S)$*.*

Proof. From Theorem 9.1, we know that there exist $b \in S(a)$ such that b is a strongly φ–maximal element of $X(S)$. Hence, by Theorem 6.2, we conclude that b is also a strongly maximal element of $X(S)$. □

Remark 9.2. To obtain a particular fixed point of an extensive relation T on $X(S)$, Theorem 8.3 can be applied.

10. Some further consequences of Theorem 9.1

From Theorem 9.1, we also derive the following generalization of Theorem 1.1 of Brézis and Browder.

Theorem 10.1. *Suppose that $a \in X$ such that*

(1) *φ is an increasing on $S(a)$;*

(2) *for each $b \in S(a)$, there exists $c \in S(b)$ such that $\varphi(b) < \varphi(c)$;*

(3) *if $(x_n)_{n=1}^{\infty}$ is an increasing sequence in $X(S)$, with $x_1 = a$, such that the sequence $(\varphi(x_n))_{n=1}^{\infty}$ is bounded above, then the sequence $(x_n)_{n=1}^{\infty}$ is also bounded above.*

Then, the set $\varphi[S(a)]$ is not bounded above.

Proof. Assume, on the contrary, that φ is bounded above on $S(a)$. Then, for each increasing sequence $(x_n)_{n=1}^{\infty}$ in $X(S)$, with $x_1 = a$, the sequence $(\varphi(x_n))_{n=1}^{\infty}$ is bounded above in $\mathbb{R}$. Thus, by assumption (3) and Theorem 9.1, there exists $b \in S(a)$ such that b is a strongly φ–maximal element of $X(S)$. That is, for any $c \in S(b)$, we have $\varphi(c) = \varphi(b)$. Also, by assumption (2) there exists $c \in S(b)$ such that $\varphi(b) < \varphi(c)$, and thus $\varphi(c) \neq \varphi(b)$. This contradiction proves that $\rho(a) = +\infty$. □

By using Theorem 9.1, we also prove the following generalization of Corollary 1.2 of Brézis and Browder.

Theorem 10.2. *Suppose that $a \in X$ such that*

(1) *φ is increasing on $S(a)$;*

(2) *for each $b \in S(a)$ and $\varepsilon > 0$, there exists $c \in S(b)$ such that*

$$\varphi(b) < \varphi(c) < \varphi(b) + \varepsilon;$$

(3) *if $(x_n)_{n=1}^{\infty}$ is an increasing sequence in $X(S)$, with $x_1 = a$, such that the sequence $(\varphi(x_n))_{n=1}^{\infty}$ is bounded above, then there exists $y \in X$ such that*

$$y \in S(x_n) \quad \textit{for all} \quad n \in \mathbb{N} \qquad \textit{and} \qquad \lim_{n \to \infty} \varphi(x_n) = \varphi(y).$$

Then, $\varphi[S(a)] = [\varphi(a), +\infty[$.

Proof. Suppose that $\beta \in [\varphi(a), +\infty[$. Define

$$Y = \{ x \in S(a) : \quad \varphi(x) \leq \beta \}.$$

Take

$$T = S \cap Y^2 \qquad \text{and} \qquad \psi = \varphi \cap (Y \times \mathbb{R}).$$

Then, $a \in Y$. Also, T is a preorder relation on Y and ψ is an increasing function of the proset $Y(T)$ to $\mathbb{R}$ which is bounded above. Moreover, we note that if $(x_n)_{n=1}^{\infty}$ is an increasing sequence in $Y(T)$, with $x_1 = a$, then it is an increasing sequence in $X(S)$ such that the sequence $(\varphi(x_n))_{n=1}^{\infty}$ is bounded above. Thus, by assumption (3), there exists $y \in X$ such that

$$y \in S(x_n) \quad \text{for all} \quad n \in \mathbb{N} \qquad \text{and} \qquad \lim_{n \to \infty} \varphi(x_n) = \varphi(y).$$

Hence, by using the fact that $x_n \in Y$, and thus

$$x_n \in S(a) \qquad \text{and} \qquad \varphi(x_n) \leq \beta$$

for all $n \in \mathbb{N}$, we infer that

$$y \in S(a) \qquad \text{and} \qquad \varphi(y) = \lim_{n \to \infty} \varphi(x_n) \leq \beta,$$

and therefore $y \in Y$. Consequently, the sequence $(x_n)_{n=1}^{\infty}$ is bounded above in $Y(T)$ too.

Hence, it is clear that the counterparts of assumptions (1) and (2) of Theorem 9.1 hold for $Y(T)$ and ψ. Thus, by Theorem 9.1, there exists $b \in T(a)$ such that b is a strongly ψ–maximal element of $Y(T)$. That is, for any $c \in T(b)$, we have $\psi(c) = \psi(b)$. Since T and ψ are restrictions of S and φ, respectively, we infer that, for any $c \in S(b) \cap Y$, we have $\varphi(c) = \varphi(b)$.

Now, we show that $\varphi(b)=\beta$. Because of $b\in Y$, we have $b\in S(a)$ and $\varphi(b)\leq\beta$. Also, if $\varphi(b)<\beta$, and thus $\beta-\varphi(b)>0$, then by assumption (2) there exists $c\in S(b)$ such that

$$\varphi(b)<\varphi(c)<\varphi(b)+2^{-1}\big(\beta-\varphi(b)\big).$$

Hence, we infer that $c\in S(a)$ and

$$\varphi(c)<2^{-1}\big(\varphi(b)+\beta\big)\leq\beta,$$

and thus $c\in Y$. Therefore, $c\in S(b)\cap Y$ such that $\varphi(c)\neq\varphi(b)$. This contradiction proves that $\varphi(b)=\beta$. □

11. Possibilities for some immediate continuations

First of all, it would be useful to know what more can be stated when S is a specific preorder on X. For instance, it is reasonable to investigate the particular cases when:

(1) $S=R^{\infty}$ for some relation R on X;

(2) $S=A^2\cup\big(A^c\times X\big)$ for some $A\in\mathcal{P}(X)$;

(3) $S=\Delta_X\cup\bigcup_{n=1}^{\infty}\big(A_n\times A_n^c\big)$ for some increasing sequence $(A_n)_{n=1}^{\infty}$ in $\mathcal{P}(X)$;

(4) $S=\big\{(x,y)\in X^2:\quad d(x,y)\leq\varphi(y)-\varphi(x)\big\}$ for some quasi-pseudo-metric d on X;

(5) $S=\big\{(x,y)\in X^2:\quad y\in x+C\big\}$ for some submonoid C of an additively written monoid X;

(6) S is the divisibility relation on $\mathbb{Z}$;

(7) S is the subsequence relation on $X^{\mathbb{N}}$;

(8) S is the refines or divides relation for covers, relations and relators [57].

Note that if R and S satisfy (1), then our definitions can be applied to R instead of S; however, our statements can only be applied to S. Some similar investigations have already been done in [69,89].

The relation S considered in (2) is just the *Pervin preorder* [51]. The relation S considered in (3) is a dual of the *Cantor order* introduced by Park [46] (see also [32,39]). In this respect, it is also worth noticing that if R is only a reflexive relation on X and $A_n(x)=R^n(x)$ for all $x\in X$, then $(A_n(x))_{n=1}^{\infty}$ is an increasing sequence in $\mathcal{P}(X)$. Thus, definition (3) can be applied.

The relation S considered in (4) is the *Brøndsted preorder* [12]. Its particular cases $\varphi=0$ and $d=0$ are the *specialization* and *preference preorders*, respectively; see [19,95]. Also, note that φ is automatically increasing. However, we still need the important assumption that each increasing sequence in $X(S)$ is bounded above.

For this, some analytical conditions are needed which will allow us to treat certain forms of Ekeland's variational principle, Caristi's fixed point theorem and some other closely related basic theorems [17,44] in similar instructive ways.

Moreover, it would also be worth to carry on some similar investigations on *Ursescu's maximality principle* [93] and its applications for proving simpler forms of Lagrange-type inequalities [58,93]. Mean value theorems, via maximal elements techniques, were formerly also proved by Turinici [84,86], who published a great number of papers on generalizations and applications of ordering and maximal principles [90–92].

Finally, we note that the relation S considered in (5) is the *translation relation* defined on X by C [60]. The particular case, when X is a vector space and C is a *pointed convex cone* in X, has been frequently used in the literature. Moreover, analogously to [15,62,75], Galois connections with respect to the above particular preorders (1)–(8) can also be investigated.

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References

[1] M. Altman, A generalization of the Brézis-Browder principle on ordered sets, *Nonlinear Anal.* **6** (1982) 157–165.
[2] H. Arianpoor, Preorder relators and generalized topologies, *J. Lin. Topol. Algebra* **5** (2016) 271–277.
[3] J. P. Aubin, J. Siegel, Fixed points and stationary points of dissipative multivalued maps, *Proc. Amer. Math. Soc.* **6** (1980) 391–398.

[4] C. Berge, *Topological Spaces, Including a Treatment of Multi-Valued Functions Vector Spaces and Convexity*, Oliver and Boyd, Edinburgh, 1963.

[5] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloq. Publ. **25**, Providence, 1967

[6] Z. Boros, Á. Száz, The smallest denominator function and the Riemann function, *Acta Math. Acad. Paedagog. Nyházi.* **14** (1998) 1–17.

[7] Z. Boros, Á. Száz, Finite and conditional completeness properties of generalized ordered sets, *Rostock. Math. Kolloq.* **59** (2005) 75–86.

[8] Z. Boros, Á. Száz, Infimum and supremum completeness properties of ordered sets without axioms, *An. Ştiinţ Univ. Ovidius Constanţa Ser. Mat.* **16** (2008) 31–37.

[9] Z. Boros, M. Iqbal, Á. Száz, A relational improvement of a true particular case of Fierro's maximality theorem, *Flomat* **36** (2022) 7091–7101.

[10] Z. Boros, M. Iqbal, Á. Száz, An instructive counterexample to a maximality theorem of Raúl Fierro, Submitted.

[11] H. Brézis, F. E. Browder, A general principle on ordered sets in nonlinear functional analysis, *Adv. Math.* **21** (1976) 355–364.

[12] A. Brøndsted, On a lemma of Bishop and Phelps, *Pacific J. Math.* **55** (1974) 335–34.

[13] A. Brøndsted, Fixed points and partial orders, *Proc. Amer. Math. Soc.* **60** (1976) 365–366.

[14] A. Brøndsted, Common fixed points and partial order, *Proc. Amer. Math. Soc.* **77** (1979) 365–368.

[15] S. Buglyó, Á. Száz, A more important Galois connection between distance functions and inequality relations, *Sci. Ser. A Math. Sci.* **18** (2009) 17–38.

[16] J. Caristi,

Fixed points theorems for mappings satisfying inwardness conditions, *Trans. Amer. Math. Soc.* **215** (1976) 241–251.

[17] K. Chaira, A. Eladraoi, M. Kabil, S. Laziz, Caristi-Type Fixed Point Theorem over Száz principle in quasi-metric space with a graph, *J. Math.* **2019** (2019) #8956083.

[18] S. Cobzaş, Completeness in quasi-metric spaces and Ekeland variational principle, *Topology Appl.* **158** (2011) 1073–1084.

[19] S. Cobzaş, Ekeland, Takahashi and Caristi principles in quasi-pseudometric spaces, *Topology Appl.* **265** (2019) #106831.

[20] J. Danes, A geometric theorem useful in nonlinear functional analysis, *Boll. Un. Math. Ital.* **6** (1972) 369–375.

[21] B. A. Davey, H. A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, Cambridge, 2002.

[22] R. DeMarr, Common fixed points for isotone mappings, *Colloq. Math.* **8** (1964) 45–48.

[23] R. DeMarr, Partially ordered linear spaces and locally convex linear topological spaces, *Illinois J. Math.* **8** (1964) 601–606.

[24] R. DeMarr, Partially ordered spaces and metric spaces, *Amer. Math. Monthly* **72** (1965) 628–631.

[25] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.* **47** (1974) 324–353.

[26] I. Ekeland, Nonconvex minimization problems, *Bull. Amer. Math. Soc.* **1** (1979) 443–474.

[27] Á. Figula, Á. Száz, Graphical relationships between the infimum and intersection convolutions, *Math. Pannon.* **21** (2010) 23–35.

[28] P. Fletcher, W. F. Lindgren, *Quasi-Uniform Spaces*, Marcel Dekker, New York, 1982.

[29] B. Ganter, R. Wille, *Formal Concept Analysis*, Springer, Berlin, 1999.

[30] T. Glavosits, Generated preorders and equivalences, *Acta Acad. Paed. Agrienses Sect. Math.* **29** (2002) 95–103.

[31] T. Glavosits, Á. Száz, The generalized infimal convolution can be used to naturally prove some dominated monotone additive extension theorems, *Ann. Math. Sil.* **25** (2011) 67–100.

[32] A. Granaś, C. D. Horváth, On the order-theoretic Cantor theorem, *Taiwanese J. Math.* **4** (2000) 203–213.

[33] E. Gselmann, Á. Száz, An instructive treatment of a generalization of Găvruţă's stability theorem, *Sarajevo J. Math.* **6** (2010) 3–21.

[34] W. Hunsaker, W. Lindgren, Construction of quasi-uniformities, *Math. Ann.* **188** (1970) 39–42.

[35] M. Iqbal, On the existence of monotone subsequences, In: R. Łukasik (Ed.), *Report of Meeting, The Twenty-First Katowice-Debrecen Winter Seminar on Functional Equations and Inequalities*, Brenna, Poland, February 2-5, 2022, *Ann. Math. Sil.* **36** (2022) 96–97.

[36] M. Iqbal, Á. Száz, On the existence of monotone subsequences, *Math. Student* **92** (2023) 65–77.

[37] J. Jachymski, A stationary point theorem characterizing metric completeness, *Appl. Math. Lett.* **24** (2011) 169–171.

[38] T. J. Jech, *The Axiom of Choice*, North-Holland, Amsterdam, 1973.

[39] G. J. Jinag, Y. J. Cho, Cantor order and completeness, *Int. J. Pure Appl. Math.* **2** (2002) 391–396.

[40] M. A. Khamsi, Remarks on Caristi's fixed point theorem, *Nonlinear Anal.* **71** (2009) 227–231.

[41] R. Kopperman, All topologies come from generalized metrics, *Amer. Math. Monthly* **98** (1988) 89–97.

[42] H. P. A. Künzi, V. Kreinovich, Static space-times naturally lead to quasi-pseudometrics, *Theoret. Comput. Sci.* **405** (2008) 64–72.

[43] K. Kuttler, *Modern Analysis*, CRC Press, New York, 1998.

[44] S. Lazaiz, M. Aamri, O. Zakary, Caristi's fixed point theorem and Ekeland's variational principle for set valued mapping using LZ–functions, *Int. J. Comput. Appl.* **158** (2017) 1–6.

[45] N. Mizoguchi, A generalization of Brondsted's results and its applications, *Proc. Amer. Math. Soc.* **108** (1990) 707–714.

[46] S. Park, Partial orders and metric completeness, *Proc. Coll. Natur. Sci. Seoul Nat. Univ.* **12** (1987) 11–17.

[47] S. Park, On generalizations of the Ekeland-type variational principles, *Nonlinear Anal.* **39** (2000) 881–889.

[48] G. Pataki, Supplementary notes to the theory of simple relators, *Radovi Mat.* **9** (1999) 101–118.

[49] F. Patrone, Well-posed minimum problems for preorders, *Rend. Sem. Mat. Univ. Padova* **84** (1990) 109–121.

[50] A. Pelczar, On the invariant points of a transformation, *Ann. Polon. Math.* **11** (1961) 199–202.

[51] W. J. Pervin, Quasi-uniformization of topological spaces, *Math. Ann.* **147** (1962) 316–317.

[52] W. J. Pervin, *Foundations of General Topology*, Academic Press, New York, 1964.

[53] J. H. Qiu, A pre-order principle and set-valued Ekeland variational principle, *J. Math. Anal. Appl.* **419** (2014) 904–937.

[54] T. M. Rassias, Á. Száz, Ordinary, super and hyper relators can be used to treat the various generalized open sets in a unified way, In: N. J. Daras, T. M. Rassias (Eds.), *Approximation and Computation in Science and Engineering*, Springer, Cham, 2022, 709–782.

[55] S. Rudeanu, *Sets and Ordered Sets*, Bentham Science, Sharjah, 2012.

[56] Á. Száz, Projective generation of preseminormed spaces by linear relations, *Studia Sci. Math. Hungar.* **23** (1988) 297–313.

[57] Á. Száz, Relations refining and dividing each other, *Pure Math. Appl.* **6** (1995) 385–394.

[58] Á. Száz, A Lagrange-type increment inequality, *Ann. Math. Sil.* **10** (1996) 13–19.

[59] Á. Száz, Topological characterizations of relational properties, *Grazer Math. Ber.* **327** (1996) 37–52.

[60] Á. Száz, Translation relations, the building blocks of compatible relators, *Math. Montisnigri* **12** (2000) 135–156.

[61] Á. Száz, An Altman type generalization of the the Brézis-Browdwer ordering principle, *Math. Moravica* **5** (2001) 1–6.

[62] Á. Száz, A Galois connection between distance functions and inequality relations, *Math. Bohem.* **127** (2002) 437–448.

[63] Á. Száz, Upper and lower bounds in relator spaces, *Serdica Math. J.* **29** (2003) 239–270.

[64] Á. Száz, Supremum properties of Galois–type connections, *Comment. Math. Univ. Carolin.* **47** (2006) 569–583.

[65] Á. Száz, *An Instructive Treatment of Convergence, Closure and Orthogonality in Semi-Inner Product Spaces*, Tecnical Report 2006/2, Inst. Math. Univ. Debrecen, 2006.

[66] Á. Száz, Some easy to remember abstract forms of Ekeland's variational principle and Caristi's fixed point theorem, *Appl. Anal. Discrete Math.* **1** (2007) 335–339.

[67] Á. Száz, Minimal structures, generalized topologies, and ascending systems should not be studied without generalized uniformities. *Filomat* **21** (2007) 87–97.

[68] Á. Száz, An instructive treatment of a generalization of Hyers's stability theorem, In: T. M. Rassias, D. Andrica (Eds.), *Inequalities and Applications*, Cluj University Press, Cluj-Napoca, 2008, 245–271.

[69] Á. Száz, Altman type generalizations of ordering and maximality principles of Brézis, Browder and Brøndsted, *Adv. Stud. Contemp. Math.* **20** (2010) 595–620.

[70] Á. Száz, Inclusions for compositions and box products of relations, *J. Int. Math. Virt. Inst.* **3** (2013) 97–125.

[71] Á. Száz, An instructive treatment and some natural extensions of a set-valued function of Páles, *Math Pannon.* **24** (2013) 77–108.

[72] Á. Száz, A relational reformulation of the Phelps–Cardwell lemma, *Malaya J. Mat.* **2** (2014) 254–262.

[73] Á. Száz, *An Intructive Treatment of Singletons, Doubletons and Ordered Pairs*, Tecnical Report 2014/4, Inst. Math. Univ. Debrecen, 2014.

[74] Á. Száz, Basic tools, increasing functions, and closure operations in generalized ordered sets, In: P. M. Pardalos, T. M. Rassias (Eds.), *Contributions in Mathematics and Engineering*, Springer, New York, 2016, 551–616.

[75] Á. Száz, A natural Galois connection between generalized norms and metrics, *Acta Univ. Sapientiae Math.* **9** (2017) 360–373.

[76] Á. Száz, A unifying framework for studying continuity, increasingness, and Galois connections, *MathLab J.* **1** (2018) 154–173.

[77] Á. Száz, Corelations are more powerful tools than relations, In: T. M. Rassias (Ed.), *Applications of Nonlinear Analysis*, Springer, New York, 2018, 711–779.

[78] Á. Száz, The closure-interior Galois connection and its applications to relational equations and inclusions, *J. Int. Math. Virt. Inst.* **8** (2018) 181–224.

[79] Á. Száz, Galois and Pataki connections on generalized ordered sets, *Earthline J. Math. Sci.* **2** (2019) 283–323.

[80] Á. Száz, Super and hyper products of super relations, *Tatra Mt. Math. Publ.* **78** (2021) 85-118.

[81] Á. Száz, Non-conventional stacked three relator spaces, In: S. Acharjee, C. Özel, J. F. Peters, G. Bosi (Eds.), *Advances in Topology and its Structures: Foundations, Theory and Application*, John Wiley & Sons, Submitted.

[82] M. R. Tasković, The axiom of choice, fixed point theorems, and inductive ordered sets, *Proc. Amer. Math. Soc.* **116** (1992) 897–904.

[83] W. J. Thron, *Topological Structures*, Holt, Rinehart and Winston, New York, 1966.

[84] M. Turinici, Mean value theorems via maximal element techniques, *J. Math. Anal. Appl.* **88** (1982) 48–60.

[85] M. Turinici, A generalization of Altman's ordering principle, *Proc. Amer. Math. Soc.* **90** (1984) 128–132.

[86] M. Turinici, Mean value theorems on abstract metric spaces, *Math. Nachr.* **115** (1984) 21–31.

[87] M. Turinici, Pseudometric extensions of the Brézis–Browder ordering principles, *Math. Nachr.* **180** (1987) 91–103.

[88] M. Turinici, Pseudometric and cardinal extensions of the Brézis-Browder ordering principle, *Bull. Math. Soc. Sci. Math. Roumanie* **33** (1989) 265–268.

[89] M. Turinici, Relational Brézis-Browder principles, *Fixed Point Theory* **7** (2006) 111–126.

[90] M. Turinici, Brezis–Browder principles and applications, In: P. M. Padalos, T. M. Rassias, A. A. Khan (Eds.), *Nonlinear Analysis and Variational Problems*, Springer, New York, 2010, 153–197.

[91] M. Turinici, Sequential maximality principles, In: T. M. Rassias, P. Pardalos (Eds.), *Mathematics Without Boundaries*, Springer, New York, 2014, 515–548.

[92] M. Turinici, Maximal and variational principles in vector spaces, In: N. J. Daras, M. T. Rassias (Eds.), *Computation, Cryptography and Network Security*, Springer, Cham, 2015, 525–575.

[93] C. Ursescu, A mean value inequality for multifunctions, *Bull. Math. Soc. Sci. Math. Roumanie* **54** (2011) 297–313.

[94] I. Vályi, A general maximality principle and a fixed point theorem in uniform space, *Period. Math. Hung.* **16** (1985) 127–134.

[95] A. D. Wallace, Relations on topological spaces, In: J. Novák (Ed.), *General Topology and its Relations to Modern Analysis and Algebra: Proceedings of the symposium held in Prague in September 1961*, Academia Publishing House of the Czechoslovak Academy of Sciences, Prague, 1962, 356–360.

[96] A. Weil, *Sur les Espaces á Structure Uniforme et Sur la Topologie Générale*, Herman and Cie, Paris, 1937.

[97] E. S. Wolk, On the principle of dependent choices and some forms of Zorn's lemma, *Canad. Math. Bull.* **26** (1983) 365–367.