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*Research Article*

# Criteria for $h$-purity in QTAG-modules

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**Abstract**

A right module $M$ over an associative ring with unity is a quasi-torsion Abelian group-like-module (QTAG-module, for short) if every finitely generated submodule of any homomorphic image of $M$ is a direct sum of uniserial modules. The main goal of this article is to produce a concrete class of QTAG-modules in which every $h$-pure submodule is an isotype submodule.

**Keywords:** QTAG-modules; $h$-pure submodules; isotype submodules.

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## 1. Introduction, terminology and definitions

Let $R$ be any ring. A module $M$ is of finite length if it has a composition series; that is, a sequence of $(k+1)$ submodules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k = M$$

such that $M_r/M_{r-1}$ is a simple module for $1 \leq r \leq k$. If the length of a module $M$ is $k$, then we write $d(M) = k$.

The study of Abelian groups advanced significantly during the second half of the twentieth century. Many researchers with an interest in module theory have made attempts to generalize the theory of Abelian groups. In fact, the theory of Abelian groups is one of the primary motivations for conducting new research in module theory. Nearly all concepts in the theory of Abelian groups have been generalized for modules over Dedekind rings, prime rings, Noetherian rings, Artinian rings, hereditary Noetherian prime rings, etc. Along this direction, a particular approach was developed in the 1970s; over an arbitrary associative ring with unity, a class of modules was defined by Singh [15] using the following two conditions related to uniserial modules:

**(I).** Every finitely generated submodule of any homomorphic image of $M$ is a direct sum of uniserial modules.

**(II).** Given any two uniserial submodules $U$ and $V$ of a homomorphic image of $M$, for any submodule $W$ of $U$, any non-zero homomorphism $f : W \to V$ can be extended to a homomorphism $g : U \to V$, provided that the inequality $d(U/W) \leq d(V/f(W))$ holds.

In 1987, Singh [16] studied the modules satisfying only Condition (I) and named them as quasi-torsion Abelian groups-like-modules (QTAG-modules, for short). Numerous researchers have conducted studies on different notions as well as on properties of QTAG-modules, characterized their different submodules, and developed the theory of these modules by introducing several concepts. Unsurprisingly, a lot of these advancements are similar to the earlier developments made in the theory of torsion Abelian groups. The present paper is a natural generalization of the research conducted in [1] and contributes to the existing knowledge on the structure of QTAG-modules.

We consider only rings with unity. Also, the modules that we consider are unital QTAG-modules. The notations and terminology that we use in this paper are standard, which may be found in the books [2, 3]. A module $M$ over a ring $R$ is said to be uniserial if it has a unique decomposition series of finite length. A module $M$ is said to be uniform if the intersection of any two of its non-zero submodules is non-zero. An element $x$ in $M$ is called uniform if $xR$ is a non-zero uniform (hence uniserial) module. For any module $M$ with a unique decomposition series, $d(M)$ denotes its decomposition length. For any uniform element $x$ of $M$, its exponent $e(x)$ is the decomposition length $d(xR)$. For any $0 \neq x \in M$, the height of $x$ in $M$ is denoted by $H_M(x)$ and is defined by

$$H_M(x) = \sup\{d(yR/xR) : y \in M,\ x \in yR \text{ and } y \text{ uniform}\}.$$

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For $k \geq 0$, $H_k(M) = \{x \in M \mid H_M(x) \geq k\}$ denotes the submodule of $M$ generated by the elements of height at least $k$ and $H^k(M)$ denotes the submodule of $M$ generated by the elements of exponents at most $k$. The sum of all simple submodules of $M$ is called the socle of $M$, denoted by $Soc(M)$. For any $k \geq 0$, $Soc^k(M)$ is defined inductively as follows:

$$Soc^0(M) = 0 \quad \text{and} \quad Soc^{k+1}(M)/Soc^k(M) = Soc\left(M/Soc^k(M)\right).$$

Now, we state some definitions given in [6,7]. A module $M$ is said to be elementary if $H^{k-1}(0) \leq H_1(M)$ provided that $H^k(0) \nsubseteq H_1(M)$, and $H^k(0) = M$, for every $k \geq 0$. A module $M$ is said to be bounded if there exists an integer $k$ such that $H_M(x) \leq k$ for all uniform elements $x \in M$. Also, $M$ is called $h$-divisible if $H_1(M) = M$ and it is called $h$-reduced if it does not contain any $h$-divisible submodule; in other words, it is free from the elements of infinite height. A submodule $N$ of $M$ is $h$-pure in $M$ if $N \cap H_k(M) = H_k(N)$ for every integer $k \geq 0$. A submodule $N$ of $M$ is $h$-neat in $M$ if $N \cap H_1(M) = H_1(N)$. A submodule $N \subseteq M$ is said to be high if it is a complement of $M^1$ i.e., $M = N \oplus M^1$, where $M^1$ is the submodule of $M$ generated by uniform elements of $M$ of infinite height.

A submodule $B$ of $M$ is said to be a basic submodule of $M$ if $B$ is an $h$-pure submodule of $M$, $B$ is a direct sum of uniserial modules and $M/B$ is $h$-divisible. A submodule $N$ of $M$ is $K$-high [9] in $M$, if it is maximal with the property of being disjoint from $K$. A submodule $N$ of $M$ is said to be essential in $M$ if $N \cap K = 0$ for every non-zero submodule $K$ of $M$, and $M$ is said to be the essential extension of $N$. According to [10], the submodules $H_k(M)$ with $k \geq 0$, form a neighborhood system of zero, and thus a topology known as $h$-topology arises. Closed modules are also closed with respect to this topology. Thus, the closure of $N \subseteq M$ is defined as

$$\bar{N} = \bigcap_{k=0}^{\infty} \left(N + H_k(M)\right).$$

Therefore, the submodule $N \subseteq M$ is closed with respect to $h$-topology if $\bar{N} = N$.

## 2. Main results

This section is concerned with finding those classes $\mathfrak{F}$ of QTAG-modules in which every $h$-pure submodule is an isotype submodule. For a QTAG-module $M$ and for an ordinal $\sigma$, $H_\sigma(M)$ is defined as $H_\sigma(M) = \bigcap_{\rho<\sigma} H_\rho(M)$ in [11], by using transfinite induction. Then

$$M^1 = \bigcap_{k=1}^{\infty} H_k(M) = H_\omega(M),$$

where $\omega$ is the first infinite ordinal. According to [12], a submodule $N$ of $M$ is said to be $\sigma$-pure if $H_\beta(M) \cap N = H_\beta(N)$ for all $\beta \leq \sigma$, and a submodule $N$ of $M$ is said to be isotype in $M$ if it is $\sigma$-pure for every ordinal $\sigma$. It is worthwhile to notice that some of the results related to these concepts have already been reported in [5]. For some crucial properties of $h$-pure submodules, we refer the interested reader to [14].

In order to develop the main results, we need to prove first some elementary but crucial lemmas.

**Lemma 2.1.** *Let $N$ be an $h$-pure submodule of a QTAG-module $M$. If $Soc^n(M)$ is a direct sum of an $h$-divisible module and a bounded module for some natural number $n$, then $N$ is $\sigma$-pure in $M$ for every ordinal $\sigma$.*

*Proof.* Clearly, $Soc^n(N)$ is $h$-pure in $Soc^n(M)$ for some $n$. By [4, Theorem 2.1], $Soc^n(N) = M_1 \oplus M_2$, where $M_1$ is $h$-divisible and $M_2$ is bounded. Now, $N = Soc^n(N) \oplus K$ for some $h$-pure submodule $K$ of $M$. Therefore,

$$H_\omega(N) = H_\omega\left(Soc^n(N)\right) \oplus H_\omega(K) = M_1 \oplus H_\omega(K)$$

and hence $H_\omega(N)$ is $h$-divisible. Consequently, $H_\beta(N) = H_\beta(M) \cap N$ for all $\beta \leq \sigma$, and we are done. □

**Lemma 2.2.** *Let $M$ be a QTAG-module and let $n$ be a natural number. If $Soc^n\left(H_{\omega+n}(M)\right)$ is not essential in $Soc^n\left(H_\omega(M)\right)$ and if either $Soc^n\left(H_{\omega+n+1}(M)\right)$ is nonzero or $H_{\omega+n+1}(M)$ is not closed module, then $M \notin \mathfrak{F}$.*

*Proof.* By hypotheses of the lemma, there is a nonzero uniform element $x \in Soc\left(H_\omega(M)\right)$ such that $xR \cap Soc^n\left(H_{\omega+n}(M)\right) = 0$ for some $n$. Take $y \in H_{\omega+n}(M)$ such that $y' \in Soc^n(M)$ where $d\left(yR/y'R\right) = 1$ and $e(y) = \infty$. Let

$$K = \langle Soc\left(H_{\omega+n}(M)\right), y', x+y \rangle$$

be any $h$-pure submodule of $M$ such that $xR \cap K = 0$. Now, if $x = z + t_1 y' + t_2(x+y)$ such that $d\left(yR/y'R\right) = 1$, where $t_1, t_2$ are integers and $z \in Soc\left(H_{\omega+n}(M)\right)$, then $(1-t_2)\,x = z + t_1 y' + t_2 y \in xR \cap H_{\omega+n}(M) = 0$, where $d\left(yR/y'R\right) = 1$. Hence, we have $(t_1+t_2)\,y' = 0$, a contradiction. This substantiates our claim.

Let $L$ be any $xR$-high submodule of $M$ containing $K$. Since $xR \subset H_\omega(M)$, $L$ is $h$-pure in $M$ (see [8]). Thus,

$$y' \in H_{\omega+n+1}(M) \cap L \backslash H_{\omega+n+1}(L),$$

where $d\,(yR/y'R) = 1$. Now, $y' = u'$ such that $d\,(yR/y'R) = 1$ and $d\,(uR/u'R) = 1$ for some $u \in H_{\omega+n}(L)$. Therefore, we have $y - u \in Soc\,(H_{\omega+n}(M)) \subset L, y \in L$ and $x \in L$, a contradiction. Consequently, we conclude that $M \notin \mathfrak{F}$. □

**Lemma 2.3.** *Let $M$ be a QTAG-module. Then $M \in \mathfrak{F}$ if and only if either $M$ is a direct sum of an $h$-divisible module and a bounded module or $M^1$ is elementary.*

*Proof.* If $M$ is a direct sum of an $h$-divisible module and a bounded module then $M \in \mathfrak{F}$ by Lemma 2.1. If $M^1$ is elementary and $N$ is an $h$-pure submodule of $M$ then $H_\omega(N) = N \cap H_\omega(M)$ and $H_{\omega+1}(N) = N \cap H_{\omega+1}(M) = 0$. Hence $M \in \mathfrak{F}$.

Conversely, we suppose that $M \in \mathfrak{F}$. Let $M^1 = X \oplus Y$, where $X$ is $h$-divisible and $Y$ is $h$-reduced. If both $X$ and $Y$ are nonzero then we write $Y = xR \oplus Z$, where $e(x) = n$ for some $n > 0$. Now, $H_n\left(M^1\right)$ is not essential in $M^1, H_{n+1}\left(M^1\right) \neq 0$. This is a contradiction because of Lemma 2.2. If $M^1$ is $h$-reduced but not bounded, then $M^1 = xR \oplus yR \oplus Z$, where $e(x) = n, e(y) = t$ and $t - n \geq 2$. Now, $H_n\left(M^1\right)$ is not essential in $M^1$, $H_{n+1}\left(M^1\right) \neq 0$ and again Lemma 2.2 implies a contradiction. Henceforth, $M^1$ is either $h$-divisible or bounded. Let $M^1$ be nonzero $h$-divisible; we write $M = M^1 \oplus K$ for some submodule $K$ of $M$. Now, if $K$ is not bounded then for any nonzero uniform element $x \in Soc\left(M^1\right)$, there is an $h$-pure submodule $L$ of $M$ such that $L \cap M^1 = xR$. Clearly, $L$ is not isotype in $M$, and hence $M$ is a direct sum of an $h$-divisible module and a bounded module. Let $M^1$ be bounded and suppose that $H_1\left(M^1\right) \neq 0$. If $K$ is any high submodule of $M$ then $K$ is not bounded. Finally, if $x$ is a nonzero uniform element of $Soc\left(H_1\left(M^1\right)\right)$, then there is an $h$-pure submodule $L$ of $M$ such that $L \cap M^1 = xR$. Thus, $L$ is not isotype in $M$ and hence $M^1$ is elementary. The proof of the lemma is now completed. □

**Lemma 2.4.** *Let $M$ be a closed QTAG-module. Then $M \in \mathfrak{F}$ if and only if $Soc^n(M) \in \mathfrak{F}$ for some $n$.*

*Proof.* The result follows from Lemmas 2.2 and 2.3. □

Now, we are ready to prove our first main result.

**Theorem 2.1.** *Let $M$ be a QTAG-module. The following statements are equivalent:*

**(i).** *Every $h$-pure submodule of $M$ is isotype in $M$.*

**(ii).** *For some $n \in \mathbb{N}$ either $Soc^n(M)$ is a direct sum of an $h$-divisible module and a bounded module, or $Soc^n(M)$ is unbounded such that $(Soc^n(M))^1$ is elementary and $H_\omega(M)$ is a closed module.*

*Proof.* $(ii) \Rightarrow (i)$. Let $N$ be an $h$-pure submodule of $M$. By Lemmas 2.3 and 2.4, $\bar{N}$ is isotype in $\bar{M}$, where $\bar{M}$ and $\bar{N}$ are the closures of $M$ and $N$, respectively. If $H_\omega(M)$ is closed and $\beta$ is an ordinal, with $\beta \geq \omega$, then

$$H_\beta(N) = H_\beta(\bar{N}) = \bar{N} \cap H_\beta(\bar{M}) = N \cap H_\beta(\bar{M}) = N \cap H_\beta(M).$$

If $Soc^n(M)$ is a direct sum of an $h$-divisible module and a bounded module then by Lemma 2.1, $H_\beta(N) = N \cap H_\beta(M)$ for every ordinal $\beta$. Thus, by definition, $N$ is isotype in $M$.

$(i) \Rightarrow (ii)$. Suppose that $M \in \mathfrak{F}$. By Lemma 2.3, for some $n \in \mathbb{N}$ either $Soc^n(M)$ is a direct sum of an $h$-divisible module and a bounded module or $(Soc^n(M))^1$ is elementary. If $(Soc^n(M))^1$ is a nonzero elementary module and $H_\omega(M)$ is not a closed module, then $H_1\left(Soc^n(M)\right)^1$ is not essential in $(Soc^n(M))^1$. Thus, Lemma 2.2 implies that $H_{\omega+2}(M)$ is not a closed module, which is a contradiction. Hence, it consequently follows that $H_\omega(M)$ is a closed module.

In order to complete the proof, it is sufficient to show that if $Soc^n(M)$ is unbounded, $(Soc^n(M))^1 = 0$ and $H_\omega(M)$ is not closed module, then $M \notin \mathfrak{F}$. Therefore, there is a linearly independent set $\{x_1, x_2, \ldots\} \in M$ such that $e\,(x_t) = t$. Let $y \in H_\omega(M)$ be an element of infinite order; there are elements $y_1, y_2, \ldots$, such that $d\,(y_t R/yR) = t-1$ for every $t \in \{1, 2, 3, \ldots\}$. Let $K = \langle y', x_1 + y_1, x_2 + y_2, \ldots\rangle$ be a submodule of $M$ such that $d\,(yR/y'R) = 1$. Suppose that $y \in K$ and it can be expressed as

$$y = r_0 y' + r_1\,(x_1 + y_1) + \cdots + r_s\,(x_s + y_s)\,,$$

where $d\,(yR/y'R) = 1$ and $r_0, \ldots, r_s$ are integers. Then

$$-\,(r_1 x_1 + \cdots + r_s x_s) = r_0 y' - y + r_1 y_1 + \cdots + r_s y_s,$$

where $d\left(yR/y'R\right)=1$. Thus, it follows that

$$-H_{s-1}\left(r_sx_sR\right)=H_{s-1}\left(\left(r_0y'-y+r_1y_1+\cdots+r_sy_s\right)R\right)\in Soc^n(M)\cap H_\omega(M)=0,$$

where $d\left(yR/y'R\right)=1$ and hence

$$-H_{s-2}\left(r_{s-1}x_{s-1}R\right)-H_{s-2}\left(r_sx_sR\right)=H_{s-2}\left(\left(r_0y'-y+\cdots+r_sy_s\right)R\right)\in Soc^n(M)\cap H_\omega(M)=0.$$

By continuing this process, we arrive at

$$\left(r_1x_1+\cdots+r_sx_s\right)\in Soc^n(M)\cap H_\omega(M)=0,$$

a contradiction. Let $L$ be a submodule of $M$ such that $K\subset L, y\notin L$. Since $L$ is $h$-pure in $M, H_t(M)\cap L\subset H_t(L)$ for some $t$. Let $u'\in L$ where $d\left(uR/u'R\right)=t+1$ for any uniform element $u\in M$. Consider $u'\notin L$ such that $d\left(uR/u'R\right)=t$. Then $y\in\langle u',L\rangle$ and $y=ku'+v$, where $d\left(uR/u'R\right)=t, v\in L$ and $k$ is an integer.

Furthermore, $ku'\in\langle y,L\rangle, u''\in L$, where $d\left(uR/u'R\right)=t$ and $d\left(u'R/u''R\right)=1$. Therefore, $u'\in\langle y,L\rangle$, where $d\left(uR/u'R\right)=t$ and thus $u'=sy+w$ for some $w\in L, s\geq 0$ and $d\left(uR/u'R\right)=t$. Hence, $u'-sH_t\left(y_{t+1}R\right)=w\in H_t(M)\cap L$, where $d\left(uR/u'R\right)=t$. By our hypothesis, we have $w=a'$, where $d\left(aR/a'R\right)=t$ for some $a\in L$. Now,

$$u'=sy'+a'=H_{t+1}\left(\left(a+sx_{t+1}+sy_{t+1}\right)R\right),$$

where $d\left(uR/u'R\right)=t+1, d\left(yR/y'R\right)=1$ and $d\left(aR/a'R\right)=t+1$. Hence, $u'\in H_{t+1}(L)$, where $d\left(uR/u'R\right)=t+1$. Finally, since $L$ is not isotype in $M$ provided that $d(yR/vR)=1$ for some $v\in H_\omega(L)$. Thus, we arrive at $y-v\in Soc^n(M)\cap H_\omega(M)=0$, a contradiction; hence, $y'\in L\cap H_{\omega+1}(M)\backslash H_{\omega+1}(L)$, where $d\left(yR/y'R\right)=1$. Consequently, we conclude that $M\notin\mathfrak{F}$. □

By utilizing Theorem 2.1, we now prove the next result.

**Theorem 2.2.** *Let $M$ be a QTAG-module. The following statements are equivalent:*

**(i).** *Every isotype submodule of $M$ is a direct summand of $M$.*

**(ii).** *Every $h$-pure submodule of $M$ is a direct summand of $M$.*

*Proof.* $(i)\Rightarrow(ii)$. Let $\bar{M}$ be the closure of $M$ and let $\bar{N}$ be the closure of the $h$-pure submodule $N$ in $M$. Since $\bar{N}$ is isotype in $M$, the closure $\bar{N}$ of $N$ is a direct summand of $M$. Choose $M=M_1\oplus M_2\oplus M_3$, where $M_1$ is $h$-reduced closed, $M_2$ is $h$-divisible and $M_3$ is a direct sum of uniserial modules. Now, suppose that $Soc^n\left(M_1\right)$ is bounded for some non-negative integer $n$. Then, $Soc^n\left(M_1\right)$ contains a proper basic submodule $B$ of $M$ (see [13]). Therefore, $B$ is isotype in $Soc^n\left(M_1\right)$ and hence in $M$. Consequently, $B$ is a direct summand of $Soc^n\left(M_1\right)$, say

$$Soc^n\left(M_1\right)=B\oplus M_4,$$

where $M_4$ is $h$-divisible; this is a contradiction. Thus, in view of Theorem 2.1, every $h$-pure submodule $N$ of $M$ is isotype in $M$ and hence a direct summand of $M$.

The implication $(ii)\Rightarrow(i)$ is obvious. □

The following result shows that Theorem 2.2 can slightly be extended.

**Theorem 2.3.** *Let $M$ be a QTAG-module. The following statements are equivalent:*

**(i).** *Every isotype submodule of $M$ is an absolute direct summand of $M$.*

**(ii).** *Every $h$-pure ($h$-neat) submodule of $M$ is an absolute direct summand of $M$.*

*Proof.* $(i)\Rightarrow(ii)$. Let $N$ be an isotype submodule of $M$. We observe that $N$ is an absolute direct summand of $M$. However, each isotype submodule $N$ of $M$ is a direct summand of $M$ and it follows that every direct summand of $M$ is an absolute direct summand of $M$, as desired.

The implication $(ii)\Rightarrow(i)$ is trivial. □

Next, we prove a result concerning $h$-neatness.

**Theorem 2.4.** *Let $M$ be a QTAG-module. The following statements are equivalent:*

**(i).** *Every $h$-neat submodule of $M$ is isotype in $M$.*

**(ii).** *Every $h$-neat submodule of $M$ is $h$-pure in $M$.*

*Proof.* The implication $(i) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$. Suppose that $N$ is an $h$-neat submodule of $M$. We know that $N$ is $h$-pure in $M$. Thus, by virtue of Theorem 2.1, $N$ is isotype in $M$, which allows us to infer that $M \in \mathfrak{F}$, as desired. □

Now, we give an example to show that there exists an $h$-neat submodule $N$ of $M$, which is not $h$-pure in $M$.

**Example 2.1.** *Let $M = xR \oplus yR$, where $x$ and $y$ are the uniform elements in $M$ such that $e(x) = 3$ and $e(y) = 1$. Then, we get an uniform element $z \in N$ such that $z = (x' + y)\, R$ and $d\,(xR/x'R) = 1$.*
*Now, for $n \geq 0, n\,(x' + y) = a'$ where $d\,(xR/x'R) = d\,(aR/a'R) = 1$ for some uniform element $a \in M$ such that $a = ux + vy$ and $u, v \in N$. Then*

$$nx' + ny = u'x + v'y = u'x,$$

*where $d\,(xR/x'R) = d\,(uR/u'R) = d\,(vR/v'R) = 1$ and $e(y) = 1$.*
*Notice that*

$$ny = (u' - nv)\, x \in xR \cap yR = 0.$$

*Moreover, by the same argument, for another $r \geq 0$, we have*

$$n\,(x' + y) = r\,(x' + y)\,,$$

*where $d\,(xR/x'R) = 1$. It is readily checked that $N$ is $h$-divisible in $M$, and thus, $N$ is $h$-neat submodule of $M$.*
*Since $N$ is a direct sum of uniserial modules such that $d\,(H_2(N)) = 1$, we get $x' = b'$, where $d\,(xR/x'R) = 2, d\,(bR/b'R) = 1$. Indeed, $b = x' + 0 \in N \cap H_2(M)$ and we see that $H_2(N) \neq N \cap H_2(M)$, which insures that $N$ is not $h$-pure in $M$.*

We end this section with the following result.

**Corollary 2.1.** *Let $M$ be a QTAG-module. Every $h$-pure submodule $N$ of $M$ is isotype in $M$ if and only if $M$ is elementary.*

*Proof.* This result can be proved by using the same idea as used in the proofs of Lemma 2.3 and Theorem 2.1. □

## 3. Open problems

In this section, we pose the following three problems related to the present study.

**Problem 3.1.** *Let $\mathfrak{F}$ be a class of QTAG-modules. If $N \subseteq M$ is $h$-neat and $M$ is a QTAG-module, does it follow that $M \in \mathfrak{F} \Leftrightarrow M/N \in \mathfrak{F}$?*

**Problem 3.2.** *Let $M$ be a direct sum of uniserial modules and assume that $k \geq 0$. What are the conditions under which any $h$-pure submodule between $M$ and $Soc^k(M)$ is uniserial?*

**Problem 3.3.** *Is the following statement true? If $M$ is a QTAG-module with an isotype submodule $N$ such that $(M/N)^1$ is a direct sum of uniserial modules, then $Soc(M/N)$ is the direct sum of an $h$-divisible module and a bounded module.*

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