**Contributions to Mathematics**
www.shahindp.com/locate/cm

*Contrib. Math.* **8** (2023) 30–37
DOI: 10.47443/cm.2023.036

*Research Article*

# Applications of Radon's inequalities to generalized topological descriptors

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(Received: 26 June 2023. Received in revised form: 7 August 2023. Accepted: 25 August 2023. Published online: 29 August 2023.)



**Abstract**

Given a graph $G$, many of its topological descriptors have the additive form $D_p(G) = \sum_i c_i^p$, where the $c_i$s are positive parameters associated with $G$, and $p$ is an arbitrary real number. Sometimes these expressions are generalizations of descriptors with the simpler form $D(G) = \sum_i c_i$. It is shown how Radon's inequality and its refinements can be used to find a variety of bounds among members of these families of generalized descriptors. The particular case of sums of powers of normalized Laplacian eigenvalues is thoroughly discussed.

**Keywords:** general first Zagreb index; Laplace eigenvalues; general sum-connectivity index.

**2020 Mathematics Subject Classification:** 05C09, 05C92.

## 1. Introduction

Let $G = (V, E)$ be a simple, connected, undirected graph where $V = \{v_1, ..., v_n\}$ is the set of vertices and $E$ is the set of edges. We denote by $d_i$ the degree of the vertex $v_i$ and assume that $\Delta = d_1 \geq d_2 \geq ... \geq d_n = \delta$. It is well known that $\sum_{i=1}^n d_i = 2|E|$. Let $A(G)$ be the adjacency matrix of $G$ and $D(G)$ be the diagonal matrix of vertex degrees of $G$. The matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix of $G$, with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$, while the matrix $\mathcal{L}(G) = D(G)^{-1/2} L(G) D(G)^{-1/2}$ is known as the normalized Laplacian matrix, with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_{n-1} > \lambda_n = 0$. The $A_\alpha$ matrix of $G$ was defined in [11] as $A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G)$, for $0 \leq \alpha \leq 1$. If we denote its eigenvalues by $\gamma_1(\alpha) \geq \gamma_2(\alpha) \geq \cdots \geq \gamma_n(\alpha)$ and we restrict ourselves to $\frac{1}{2} < \alpha < 1$, we can guarantee that all these eigenvalues are non-negative. For more details on graph theory, we refer the reader to [5].

In this article, we are interested in topological descriptors of a graph $G$ with the form

$$D_p(G) = \sum_{i=1}^{N} c_i^p, \tag{1}$$

where the $c_i$s are some positive parameters associated with $G$ and $p$ is an arbitrary real number. Sometimes these descriptors arise as generalizations of other descriptors which were originally thought of as particular cases of $p$. Examples of these, without attempting to be exhaustive, are: the general first Zagreb index

$$M_1^p(G) = \sum_{i=1}^{n} d_i^p,$$

which generalizes the first Zagreb index, obtained when $p = 2$; also, the general Randić index (see [6]) as

$$R_p(G) = \sum_{ij \in E} (d_i d_j)^p;$$

and also, the general sum-connectivity index

$$H_p(G) = \sum_{ij \in E} (d_i + d_j)^p,$$

introduced in [17], which can be seen as another way to generalize $M_1^2(G)$, since $H_1(G) = M_1^2(G)$. We direct the reader to the survey [4] where these three and other general indices are discussed.

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Another example is the general atom-bond connectivity (ABC), considered in [15], as

$$ABC_p(G) = \sum_{ij \in E} \left( \frac{d_i + d_j - 2}{d_i d_j} \right)^p,$$

which is a generalization of the original ABC index, where $p = \frac{1}{2}$, introduced in [3].

To the best of our knowledge, no *general* Kirchhoff index has been defined, but we could do so with the formula

$$K_p(G) = \sum_{i<j} R_{ij}^p, \tag{2}$$

where $R_{ij}$ is the effective resistance, computed with Ohm's laws, between vertices $v_i$ and $v_j$. In addition to the original Kirchhoff index $K(G)$, obtained when $p = 1$ in (2), we will be working in this article with the multiplicative degree-Kirchhoff index:

$$K^*(G) = \sum_{i<j} d_i d_j R_{ij}.$$

Sometimes the descriptors that we look at, do not generalize former descriptors, but still have the form (1); for example:

$$s_p(G) = \sum_{i=1}^{n-1} \mu_i^p,$$

where the $\mu_i$s are the non-zero Laplacian eigenvalues of $G$, and

$$s_p^*(G) = \sum_{i=1}^{n-1} \lambda_i^p,$$

where the $\lambda_i$s are the non-zero normalized Laplacian eigenvalues of $G$. These latter descriptors were introduced in [16]. Also, worth mentioning is the following recently defined descriptor (see [8]):

$$s_p^\alpha(G) = \sum_{i=1}^{n} \gamma_i(\alpha)^p,$$

where the $\gamma_i(\alpha)$s are the eigenvalues of $A_\alpha(G)$ and $\frac{1}{2} < \alpha < 1$.

We see that the index of the summation in (1) can run in one of the sets $\{1, 2, \dots, n\}$, or $\{1, 2, \dots, n-1\}$, or the set of edges $E$, or all the pairs of indices $i, j$ such that $i < j$, conveniently ordered. Thus, $N$ can be $n$, $n-1$, $|E|$ or $\binom{n}{2}$, and the context will make clear which case is being considered. We exhibit a variety of relations among members of the above-mentioned families of descriptors by applying Radon's inequality to $D_p(G)$ and then we focus on some particular cases.

## 2. Radon's inequalities

We begin with the main tool of this article, Radon's inequalities, found in [13]:

**Lemma 2.1.** *If $a_1, a_2, \dots, a_N$ and $b_1, b_2, \dots, b_N$ are positive real numbers and $p \geq 1$ or $p \leq 0$, then*

$$\sum_{i=1}^{N} \frac{a_i^p}{b_i^{p-1}} \geq \frac{\left(\sum_{i=1}^{N} a_i\right)^p}{\left(\sum_{i=1}^{N} b_i\right)^{p-1}}. \tag{3}$$

*The opposite inequality holds whenever $0 \leq p \leq 1$. The equality is attained in case $p = 0$ or $p = 1$, or if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_N}{b_N}$.*

Now, we can prove the following result.

**Theorem 2.1.** *For any descriptor $D_p(G)$ of the form given in (1), $\alpha$ and $\beta$ arbitrary real numbers, and $p \geq 1$ or $p \leq 0$, the inequality*

$$D_{p\alpha-(p-1)\beta}(G) \geq \frac{(D_\alpha(G))^p}{(D_\beta(G))^{p-1}} \tag{4}$$

*holds; the opposite inequality holds whenever $0 \leq p \leq 1$. The equality in (4) is attained in case $p = 0$ or $p = 1$ or if $c_1^{\alpha-\beta} = c_2^{\alpha-\beta} = \cdots = c_N^{\alpha-\beta}$.*

*Proof.* Take $a_i = c_i^\alpha$ and $b_i = c_i^\beta$ in (3). We get that

$$D_{\alpha p-\beta(p-1)}(G) = \sum_i \frac{(c_i^\alpha)^p}{(c_i^\beta)^{(p-1)}} \geq \frac{(\sum_i c_i^\alpha)^p}{\left(\sum_i c_i^\beta\right)^{p-1}} = \frac{(D_\alpha(G))^p}{(D_\beta(G))^{p-1}}.$$

The statement about the equality follows because in this case $\dfrac{a_i}{b_i} = c_i^{\alpha-\beta}$. □

The next is an almost trivial observation that helps us identify extremal graphs involving $D_p$-type descriptors.

**Corollary 2.1.** *For all the descriptors of the type $D_p$ defined above, where the $c_i$s are degrees, effective resistances, Laplacian or normalized Laplacian eigenvalues, the complete graph satisfies the equality in (4). For those descriptors where the summation runs over all edges, and the $c_i$s can be described in terms of a function of sums and products of degrees over the edges, the star graph $S_n$ satisfies the equality in (4).*

*Proof.* For the complete graph $K_n$ with $n$ vertices, all degrees, all effective resistances and all non-zero Laplacian and normalized Laplacian eigenvalues are equal. Therefore, for all descriptors using these parameters, the values $c_i^{\alpha-\beta}$ in Theorem 2.1 are all equal, and the equality in (4) is attained for $K_n$. For the star graph $S_n$, sums and products of degrees are constants: $d_i + d_j = n$ and $d_i d_j = n-1$ for all edges $ij \in E$, and a similar argument applies. □

We illustrate Theorem 2.1 and Corollary 2.1 with three simple applications. Consider first $M_1^p(G) = \sum_{v\in V} d_v^p$, then $M_1^0(G) = n$ and $M_1^1(G) = 2|E|$ and therefore, applying (4) with $\alpha = 1$ and $\beta = 0$, we get

$$M_1^p(G) \geq \frac{(2|E|)^p}{n^{p-1}}, \tag{5}$$

for $p \geq 1$ or $p \leq 0$, and with the opposite inequality in case $0 \leq p \leq 1$. The equality in (5) is attained by the complete graph $K_n$, in view of Corollary 2.1. In fact, it is easy to see that the equality in (5) is attained by any $d$-regular graph, for which the bound becomes $nd^p$. In the recent article [10], the following bound was obtained for $M_1^p(G)$ in terms of the largest and smallest degrees, $\Delta$ and $\delta$, respectively, when $\Delta(\delta+1)$ is even:

$$M_1^p(G) \geq \Delta\delta^p + \Delta^p. \tag{6}$$

We notice that the bound (6) is worse than (5) in case the graph $G$ is $d$-regular, with $d < n-1$, because it only attains the value $(d+1)d^p < nd^p$.

For another simple application, consider the general sum-connectivity index

$$H_p(G) = \sum_{xy\in E} (d_x + d_y)^p.$$

Note that $H_0(G) = |E|$ and $H_1(G) = M_1^2(G)$. Again, taking $\alpha = 1$ and $\beta = 0$ in (4) we have

$$H_p(G) \geq \frac{\left(M_1^2(G)\right)^p}{|E|^{p-1}}, \tag{7}$$

for $p \geq 1$ or $p \leq 0$, and with the opposite inequality in case $0 \leq p \leq 1$. The equality in (7) is attained by the star graph $S_n$ in view of Corollary 2.1. In fact, the equality in (7) is attained by any graph for which $d_i + d_j$ is a constant value over all $ij \in E$. This result can be found as Proposition 1 in [17].

For another application of (4), consider $s_p^\alpha(G)$ with $\frac{1}{2} < \alpha < 1$. It is well-known that $s_0^\alpha(G) = n$ and $s_1^\alpha(G) = 2\alpha|E|$. Then, with the same choices of $\alpha$ and $\beta$ as before in (4), for the $s_p^\alpha(G)$ descriptor, we obtain

$$s_p^\alpha(G) \geq \frac{[s_1^\alpha(G)]^p}{[s_0^\alpha(G)]^{p-1}} = \frac{(2\alpha|E|)^p}{n^{p-1}}, \tag{8}$$

in case $p \leq 0$ or $p \geq 1$, with the opposite inequality if $0 \leq p \leq 1$. This was shown in Theorem 4.1 of [8].

The power of Theorem 2.1 and Corollary 2.1 resides both in the variety of indices to which they apply, and also in the flexibility for the choices of $\alpha$ and $\beta$. As a rule, the more particular cases of the values $D_p(G)$ that are known in closed form, the more significant bounds that we can get. We illustrate this idea in the next section for the case of the index $s_p^*(G)$, the sum of the $p$ powers of the normalized Laplacian eigenvalues.

## 3. Sums of powers of normalized Laplacian eigenvalues

As mentioned in [2], the following particular cases of the descriptor $s_p^*(G)$ are known:

$$s_0^*(G) = n - 1, \tag{9}$$

$$s_1^*(G) = n, \tag{10}$$

$$s_2^*(G) = n + 2R_{-1}(G), \tag{11}$$

where $R_{-1}(G)$ is the generalized Randić index with $\alpha = -1$, and

$$s_{-1}^*(G) = \frac{1}{2|E|} K^*(G).$$

From (4) we get

$$s_{p\alpha-(p-1)\beta}^*(G) \geq \frac{(s_\alpha^*(G))^p}{(s_\beta^*(G))^{p-1}} \tag{12}$$

for $p \geq 1$ or $p \leq 0$. If $0 \leq p \leq 1$, the opposite inequality in (12) is valid. Choosing $\alpha = 1$ and $\beta = 0$ in (12), we obtain:

$$s_p^*(G) \geq \frac{n^p}{(n-1)^{p-1}} \tag{13}$$

for $p \leq 0$ or $p \geq 1$; with the opposite inequality if $0 \leq p \leq 1$. We remark that by Corollary 2.1 and (13),

$$s_p^*(K_n) = \frac{n^p}{(n-1)^{p-1}}$$

and thus $s_p^*(G)$ attains its minimum for $G = K_n$, when $p \leq 0$ or $p \geq 1$, and its maximum for the same $G = K_n$ when $0 \leq p \leq 1$. In particular, when $p = 2$ then from (13) we get

$$R_{-1}(G) \geq \frac{n}{2(n-1)},$$

and when $p = -1$ we get

$$K^*(G) \geq 2|E| \frac{(n-1)^2}{n};$$

the last two bounds for the indices $R_{-1}(G)$ and $K^*(G)$ are well known in the literature (see Theorem 3.2 in [7] and Corollary 4 in [12]), both are attained by $K_n$.

In (12), choosing $\alpha = 2, \beta = 0$, and using $q$ instead of $p$, we get:

$$s_{2q}^*(G) \geq \frac{(n + 2R_{-1}(G))^q}{(n-1)^{q-1}}.$$

Changing variables, $p = 2q$, we obtain

$$s_p^*(G) \geq \sqrt{\frac{(n + 2R_{-1}(G))^p}{(n-1)^{p-2}}} \tag{14}$$

for $p \leq 0$ or $p \geq 2$. With the opposite inequality if $0 \leq p \leq 2$.

In (12), choosing $\alpha = 2, \beta = 1$, and using $q$ instead of $p$, we obtain

$$s_{q+1}^*(G) \geq \frac{(n + 2R_{-1}(G))^q}{n^{q-1}}.$$

Changing variables, $p = q + 1$, we obtain

$$s_p^*(G) \geq \frac{(n + 2R_{-1}(G))^{p-1}}{n^{p-2}} \tag{15}$$

for $p \leq 1$ or $p \geq 2$. For $1 \leq p \leq 2$ the opposite inequality holds.

All inequalities (13), (14) and (15) become equalities for the complete graph $K_n$. Putting together all these bounds, and selecting the best of them, we obtain the next theorem.

**Theorem 3.1.** *For any graph $G$ we have the lower bounds*

$$s_p^*(G) \geq \frac{n^p}{(n-1)^{\eta-1}} \quad for \quad p \leq 0 \quad and \quad 1 \leq p \leq 2,$$

$$s_p^*(G) \geq \frac{(n+2R_{-1}(G))^{p-1}}{n^{p-2}} \quad for \quad 0 \leq p \leq 1 \quad and \quad p \geq 2,$$

*and the upper bounds*

$$s_p^*(G) \leq \frac{n^p}{(n-1)^{p-1}} \quad for \quad 0 \leq p \leq 1,$$

$$s_p^*(G) \leq \frac{(n+2R_{-1}(G))^{p-1}}{n^{p-2}} \quad for \quad 1 \leq p \leq 2.$$

*The equalities are attained by the complete graph $K_n$.*

*Proof.* From the fact that

$$R_{-1}(G) \geq \frac{n}{2(n-1)},$$

it can be verified that

$$\frac{n^p}{(n-1)^{p-1}} \leq \sqrt{\frac{(n+2R_{-1}(G))^p}{(n-1)^{p-2}}} \tag{16}$$

for $p \geq 0$, with the opposite inequality when $p \leq 0$. Likewise,

$$\sqrt{\frac{(n+2R_{-1}(G))^p}{(n-1)^{p-2}}} \leq \frac{(n+2R_{-1}(G))^{p-1}}{n^{p-2}}, \tag{17}$$

for $p \geq 2$, with the opposite inequality when $p \leq 2$.

For lower bounds we obtain the following: in the interval $p \leq 0$, all three bounds (13), (14) and (15) apply, but (13) is the best by (16) and (17); likewise, in the interval $p \geq 2$, all three bounds hold but the best is (15); finally, in the interval $0 \leq p \leq 1$, only (15) applies.

In the case of the upper bounds, the opposites of (13) and (14) apply in the interval $0 \leq p \leq 1$, but the opposite of (13) is the best, and in the interval $1 \leq p \leq 2$, both the opposites of (14) and (15) apply, but the opposite of (15) is the best. □

In [2], it was shown with majorization methods that

$$s_p^*(G) \geq W^p + \frac{(n-W)^p}{(n-2)^{p-1}}, \tag{18}$$

where

$$W = 1 + \sqrt{\frac{2R_{-1}(G)}{n(n-1)}}$$

if $p < 0$ or $p > 1$, with the opposite inequality holding when $0 < p < 1$. The equality is attained when $G = K_n$. We remark, for the sake of comparison, that Theorem 3.1 gives lower bounds for any real value of $p$, while in the inequality (18), $p$ is restricted to be less than $0$ or greater than $1$. Likewise, we provide upper bounds for $0 \leq p \leq 2$, whereas according to [2] the opposite in (18) holds when $0 \leq p \leq 1$. Also, in the interval $p \geq 2$, our lower bound performs better: For example, in the case of the $n$-cycle $C_n$, for which $R_{-1}(C_n) = \frac{n}{4}$, our lower bound is roughly equal to $1.5n$ when $n$ is large; whereas, when $n$ grows, $W$ approaches to $1$ and so the bound in (18) is roughly equal to $n$.

In [1], one can find the following bound when $G$ is a non-bipartite graph:

$$s_p^*(G) \geq 2^p + \left(1 - \frac{2R_{-1}}{n}\right)^p + \frac{(n-3+\frac{2R_{-1}}{n})^p}{(n-2)^{p-1}}, \tag{19}$$

for $p \leq 0$ or $p \geq 1$, with the opposite inequality holding if $0 \leq p \leq 1$.

The previous example of $C_n$ (now with $n$ odd, to make it non-bipartite) also shows that our bound performs asymptotically better for fixed $p \geq 2$, because (19) yields a lower bound roughly equal to $n$, while ours is roughly equal to $1.5n$.

## 4. Refinements

Radon's inequalities have undergone some refinements through the years, improving the inequalities in certain cases. For instance, we quote from [9] the following result:

**Lemma 4.1.** *For $p \geq 1$, $N \geq 2$, $a_i \geq 0$, $b_i > 0$ we have*

$$\sum_{i=1}^{N} \frac{a_i^p}{b_i^{p-1}} \geq \frac{\left(\sum_{i=1}^{N} a_i\right)^p}{\left(\sum_{i=1}^{N} b_i\right)^{p-1}} + \max_{1\leq i<j\leq N} \left( \frac{a_i^p}{b_i^{p-1}} + \frac{a_j^p}{b_j^{p-1}} - \frac{(a_i+a_j)^p}{(b_i+b_j)^{p-1}} \right).$$

With the help of Lemma 4.1, we can prove the following refinement of Theorem 2.1, using the same ideas, that will be the source of several lower and upper bounds shown after it:

**Theorem 4.1.** *For any descriptor defined as in (1) and $p \geq 1$, it holds that*

$$D_{\alpha p-\beta(p-1)}(G) \geq \frac{D_\alpha(G)^p}{D_\beta(G)^{p-1}} + \max_{1\leq i<j\leq N} \left( c_i^{\alpha p-\beta(p-1)} + c_j^{\alpha p-\beta(p-1)} - \frac{(c_i^\alpha + c_j^\alpha)^p}{(c_i^\beta + c_j^\beta)^{p-1}} \right). \tag{20}$$

The particular case $\alpha = 1$ and $\beta = 0$, treated in the previous section, when applied to (20) yields the following result:

**Corollary 4.1.** *For any descriptor defined as in (1) and $p \geq 1$, it holds that*

$$D_p(G) \geq \frac{D_1(G)^p}{D_0(G)^{p-1}} + \max_{1\leq i<j\leq N} \left( c_i^p + c_j^p - \frac{(c_i+c_j)^p}{2^{p-1}} \right). \tag{21}$$

If we take $D_p(G)$ to be $M_1^p(G)$ in (21), then we obtain

$$M_1^p(G) \geq \frac{(2|E|)^p}{n^{p-1}} + \Delta^p + \delta^p - \frac{(\Delta+\delta)^p}{2^{p-1}},$$

where the equality is attained by all $d$-regular graphs and by all $n$-vertex unicyclic graphs $U_n$ consisting of a cycle with a linear graph of any length between 1 and $n-3$ attached to any of the vertices of the cycle. In this case, it holds that $M_1^p(U_n) = 2^p n + 3^p + 1 - 2^{p+1}$, which is also the value of the lower bound.

Applying Corollary 4.1 now to $s_p^\alpha(G)$, we obtain

$$s_p^\alpha(G) \geq \frac{(2\alpha|E|)^p}{n^{p-1}} + \gamma_1(\alpha)^p + \gamma_n(\alpha)^p - \frac{(\gamma_1(\alpha)+\gamma_n(\alpha))^p}{2^{p-1}}, \tag{22}$$

for $p \geq 1$, which improves (8). Applying the Corollary 4.1 yet again, this time to $s_p^*(G)$, we obtain

$$s_p^*(G) \geq \frac{n^p}{(n-1)^{p-1}} + \lambda_1^p + \lambda_{n-1}^p - \frac{(\lambda_1+\lambda_{n-1})^p}{2^{p-1}}, \tag{23}$$

for $p \geq 1$, which improves (14). The particular case $p = 2$ yields

$$R_{-1}(G) \geq \frac{n}{2(n-1)} + \frac{(\lambda_1 - \lambda_{n-1})^2}{4}. \tag{24}$$

This bound improves (3).

If we consider $D_p(G)$ to be $s_p(G) = \sum_{i=1}^{n-1} \mu^p$, it is well known (see [2]) that $s_0(G) = n-1$, $s_1(G) = 2|E|$, $s_{-1}(G) = nK(G)$, and $s_{1/2}(G) = LEL(G)$, where $LEL(G)$ is usually called the Laplacian energy-like descriptor. Taking $\alpha = 0$, $\beta = 1$ and $p = 2$ in (20) we obtain

$$K(G) \geq n\left[\frac{(n-1)^2}{2|E|} + \max_{1\leq i<j\leq n-1} \left( \frac{1}{\mu_i} + \frac{1}{\mu_j} - \frac{4}{\mu_i + m_j} \right)\right],$$

or

$$K(G) \geq n\left[\frac{(n-1)^2}{2|E|} + \frac{(\mu_1 - \mu_{n-1})^2}{\mu_1\mu_{n-1}(\mu_1+\mu_{n-1})}\right]. \tag{25}$$

Also, taking $\alpha = \frac{1}{2}$, $\beta = 0$ and $p = 2$ in (20) for the descriptor $s_p(G)$, we get

$$LEL(G) \leq \sqrt{(n-1)\left[2|E| - \frac{(\mu_1^{1/2} - \mu_{n-1}^{1/2})}{2}\right]}. \tag{26}$$

The bounds (22), (23), (24), (25) and (26) could be expressed in terms of other parameters of the graphs in question, using bounds for the largest and smallest $A_\alpha$ eigenvalues and the largest and smallest Laplacian and normalized Laplacian eigenvalues, of which many can be found in the literature, but we will not pursue here that matter.

When it comes to upper bounds, there are also refinements to Radon's inequality. Quoting [14], we have the next lemma.

**Lemma 4.2.** *For $p \geq 1$, $n \geq 2$, $a_i \geq 0$, $b_i > 0$ the following inequality holds*

$$\sum_{i=1}^{N} \frac{a_i^p}{b_i^{p-1}} \leq \frac{\left(\sum_{i=1}^{N} a_i\right)^p}{\left(\sum_{i=1}^{N} b_i\right)^{p-1}} + \left[M^p + m^p - \frac{(M+m)^p}{2^{p-1}}\right] \sum_{i=1}^{N} b_i,$$

*where $m \leq \frac{a_i}{b_i} \leq M$.*

Lemma 4.2 translates immediately into the next result.

**Theorem 4.2.** *For any descriptor defined as in (1) and $p \geq 1$, it holds that*

$$D_{\alpha p - \beta(p-1)}(G) \leq \frac{D_\alpha(G)^p}{D_\beta(G)^{p-1}} + \left[M^p + m^p - \frac{(M+m)^p}{2^{p-1}}\right] D_\beta(G), \tag{27}$$

*where $m \leq c_i^{\alpha-\beta} \leq M$.*

The particular case of (27), when $\alpha = 1$ and $\beta = 0$, yields the compact formula

$$D_p(G) \leq \frac{D_1(G)^p}{D_0(G)^{p-1}} + \left[M^p + m^p - \frac{(M+m)^p}{2^{p-1}}\right] D_0(G), \tag{28}$$

where $m \leq c_i \leq M$.

In the case of $M_1^p(G)$, (28) implies

$$M_1^p(G) \leq \frac{(2|E|)^p}{n^{p-1}} + \left[\Delta^p + \delta^p - \frac{(\Delta+\delta)^p}{2^{p-1}}\right] n,$$

for all $p \geq 1$, where the equality holds for all $d$-regular graphs.

In the case of $H_p(G)$, the general sum-connectivity index, (28) yields

$$H_p(G) \leq \frac{\left[M_1^2(G)\right]^p}{|E|^{p-1}} + \left[M^p + m^p - \frac{(M+m)^p}{2^{p-1}}\right] |E|, \tag{29}$$

for $p \geq 1$, and where $M = \max_{ij \in E}(d_i + d_j)$ and $m = \min_{ij \in E}(d_i + d_j)$. The equality in (29) is attained by all graphs in which $d_i + d_j$ is a constant value for every $ij \in E$.

In the case of $s_p^\alpha(G)$, by (28) we get

$$s_p^\alpha(G) \leq \frac{(2\alpha|E|)^p}{n^{p-1}} + \left[\gamma_1(\alpha)^p + \gamma_n(\alpha)^p - \frac{(\gamma_1(\alpha)+\gamma_n(\alpha))^p}{2^{p-1}}\right] n,$$

for $p \geq 1$.

If we take $D_p(G) = s_p^*(G)$, $\alpha = 2$ and $\beta = 1$, and use $q$ instead of $p$ in (27) we obtain

$$s_{q+1}^* \leq \frac{(n + 2R_{-1}(G))^q}{n^{q-1}} + \left[M^q + m^q - \frac{(M+m)^q}{2^{q-1}}\right] n \tag{30}$$

for $q \geq 1$. Performing the necessary changes in (30), we arrive at the following inequality

$$s_p^*(G) \leq \frac{(n + 2R_{-1}(G))^{p-1}}{n^{p-2}} + \left[\lambda_1^{p-1} + \lambda_{n-1}^{p-1} - \frac{(\lambda_1 + \lambda_{n-1})^{p-1}}{2^{p-2}}\right] n,$$

this adds an upper bound, which is attained by $K_n$, to those in Theorem 3.1, for $p \geq 2$.

## 5. Final remarks

We have shown that Radon's inequalities can be applied to large families of topological descriptors in order to find numerous upper and lower bounds, some known in the literature and many others, which are new. Using refinements of Radon's inequalities, we have improved lower bounds and produced new upper bounds, typically involving additional parameters.

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