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Research Article

Formulae concerning multiple harmonic-like numbers

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Abstract

By means of the generating function method as well as Stirling and Lah inversion, several summation formulae involving generalized harmonic-like numbers and other combinatorial numbers named after Stirling, Lah, Hal and Fubini are derived.

Keywords: multiple harmonic-like numbers; Stirling numbers; Lah numbers; Hal numbers; Fubini numbers.

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1. Introduction and motivation

It is well-known that the classical harmonic numbers are defined by

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{for} \quad n \in \mathbb{N},$$

where $\mathbb{N}$ is the set of positive integers. The generating function of H_n is given by

$$\sum_{n=1}^{\infty} H_n x^n = \frac{-\ln(1-x)}{1-x}.$$

Harmonic numbers have wide applications in number theory, combinatorics, and computer science. Properties as well as identities about them have already been explored extensively. In addition, many researchers also have studied other harmonic-like numbers defined in various ways [1–5, 9–11], and obtained a number of interesting results. For instance, Cheon and El-Mikkawy [1, 2] studied the following multiple harmonic-like numbers, which reduce, when $\ell = 1$, to the ordinary harmonic numbers:

$$H_n(\ell) = \sum_{1 \le k_1 + k_2 + \cdots + k_\ell \le n} \frac{1}{k_1 k_2 \cdots k_\ell},$$

and obtained its generating function, given as follows:

$$\sum_{n=1}^{\infty} H_n(\ell) x^n = \frac{\{-\ln(1-x)\}^\ell}{1-x}. \tag{1}$$

Assume that $[x^n]g(x)$ stands for the coefficient of x^n in the formal power series $g(x)$. Then, we get the following relation between $H_n(\ell)$ and Stirling numbers of the first kind:

$$\begin{aligned}
\sum_{m=\ell}^{n} \frac{(-1)^{m-\ell}}{m!} s(m,\ell) &= \sum_{m=\ell}^{n} (-1)^{m-\ell} [x^m] \frac{\ln^\ell(1+x)}{\ell!} \\
&= \frac{1}{\ell!} \sum_{m=\ell}^{n} (-1)^m [x^m] \frac{\{-\ln(1+x)\}^\ell}{1+x} \times (1+x) \\
&= \frac{1}{\ell!} \sum_{m=\ell}^{n} (-1)^m \sum_{k=0}^{m} [x^{m-k}] \frac{\{-\ln(1+x)\}^\ell}{1+x} \times [x^k](1+x) \\
&= \frac{1}{\ell!} \sum_{m=\ell}^{n} \{H_m(\ell) - H_{m-1}(\ell)\} = \frac{1}{\ell!} H_n(\ell),
\end{aligned}$$

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where $s(n,k)$ denotes the Stirling numbers of the first kind generated by

$$\frac{\ln^k(1+x)}{k!} = \sum_{n=k}^{\infty} s(n,k)\frac{x^n}{n!}. \tag{2}$$

Instead, by extracting the coefficient of x^n in the generating function (1), the above relation can also be obtained [8].

By means of Riordan arrays, Cheon and El-Mikkawy [2] proved the following summation formulae about the multiple harmonic-like numbers $H_n(\ell)$:

$$\sum_{\ell=1}^{n} \frac{(\pm 1)^\ell}{\ell!} H_n(\ell) = \begin{cases} n, & \text{“}+\text{”;} \\ -1, & \text{“}-\text{”,} \end{cases} \quad \text{and} \quad \sum_{\ell=1}^{n} B_\ell \frac{(-1)^\ell}{\ell!} H_n(\ell) = H_{n+1} - 1,$$

where B_n denotes the Bernoulli numbers. By examining the structure of these summation formulae, Guo and Chu [8] established, by making use of the following scheme [7, 8], seven classes of summation formulae involving the multiple harmonic-like numbers and other combinatorial numbers named after Bernoulli, Euler, Bell, Genocchi and Stirling numbers.

Suppose that the double-indexed sequence $\{\mathcal{D}(n,k)\}_{n\geq k}$, subject to $\mathcal{D}(n,k) = 0$ when $n < k$, and the sequence $\{\lambda_k\}$ have generating functions

$$\sum_{n=k}^{\infty} \mathcal{D}(n,k)x^n = h(x)g^k(x) \quad \text{and} \quad \sum_{k=\varepsilon}^{\infty} \lambda_k x^k = f(x),$$

respectively, where ε is a non-negative integer. Then, we can evaluate the sum

$$\begin{aligned} \sum_{k=\varepsilon}^{n} \lambda_k \mathcal{D}(n,k) &= \sum_{k=\varepsilon}^{n} \lambda_k [x^n] h(x) g^k(x) \\ &= [x^n] h(x) \sum_{k=\varepsilon}^{\infty} \lambda_k g^k(x) = [x^n] h(x) f(g(x)). \end{aligned} \tag{3}$$

In particular, when $h(x) \equiv 1$, we have

$$\sum_{k=\varepsilon}^{n} \lambda_k \mathcal{D}(n,k) = [x^n] f(g(x)).$$

In addition to the harmonic numbers H_n and $H_n(\ell)$ mentioned above, in this paper, we also investigate their alternating forms defined by

$$\mathcal{H}_n = \sum_{k=1}^{n} \frac{(-1)^k}{k} \quad \text{and} \quad \mathcal{H}_n(\ell) = \sum_{1\leq k_1+k_2+\cdots+k_\ell\leq n} \frac{(-1)^{k_1+k_2+\cdots+k_\ell}}{k_1 k_2 \cdots k_\ell},$$

with generating functions

$$\sum_{n=1}^{\infty} \mathcal{H}_n x^n = \frac{-\ln(1+x)}{1-x} \quad \text{and} \quad \sum_{n=1}^{\infty} \mathcal{H}_n(\ell) x^n = \frac{\{-\ln(1+x)\}^\ell}{1-x}.$$

In the next section, by making use of the scheme mentioned above we establish some summation formulae involving the numbers $H_n(\ell)$ as well as Lah, Hal, and Fubini numbers. In Section 3, we provide several identities involving alternating harmonic-like numbers $\mathcal{H}_n$, $\mathcal{H}_n(\ell)$, and other combinatorial numbers.

2. Identities involving $H_n(\ell)$

Formulae concerning $H_n(\ell)$, Lah, and Hal numbers

The Lah numbers $L(n,k)$ was discovered by Ivo Lah in 1955. These numbers are coefficients expressing rising factorials in terms of falling factorials [6, p.156]:

$$(-x)_n = (-1)^n \langle x \rangle_n = \sum_{k=0}^{n} L(n,k)(x)_k,$$

where

$$\langle x \rangle_0 = 1 \quad \text{and} \quad \langle x \rangle_n = \prod_{k=1}^{n} (x+k-1), \quad \text{for} \quad n \in \mathbb{N}.$$

The unsigned Lah numbers $|L(n,k)|$ count the number of ways to partition a set of n elements into k nonempty linearly ordered queues. Lah numbers $L(n,k)$ are generated by

$$\sum_{n=k}^{\infty} L(n,k)\frac{x^n}{n!} = \frac{1}{k!}\left(-\frac{x}{1+x}\right)^k,$$

and have the explicit formula known as

$$L(n,k) = (-1)^n \frac{n!}{k!}\binom{n-1}{k-1}.$$

The Hal numbers $H(n,k)$ are generated similarly by

$$\sum_{n=k}^{\infty} H(n,k)\frac{x^n}{n!} = \frac{1}{k!}\left(\frac{x}{x-1}\right)^k.$$

By the generating functions of $L(n,k)$ and $H(n,k)$, we immediately obtain their relation

$$L(n,k) = (-1)^{n+k} H(n,k). \tag{4}$$

Theorem 2.1 (Identities involving $H_n(\ell)$, $L(n,k)$, and $H(n,k)$)**.**

$$\sum_{k=\ell}^{n} (-1)^k \frac{k!}{\ell!} L(n,k) H_k(\ell) = s(n-1,\ell-1) + s(n-1,\ell), \tag{5}$$

$$\sum_{k=\ell}^{n} \frac{k!}{\ell!} H(n,k) H_k(\ell) = (-1)^n \big\{ s(n-1,\ell-1) + s(n-1,\ell) \big\}. \tag{6}$$

Proof. First, we prove the formula (5). Letting $\mathcal{D}(n,k) = L(n,k)$ and $\lambda_k = H_k(\ell)$. Then, from the generating functions of $H_n(\ell)$ and $L(n,k)$, we have

$$\begin{aligned}\sum_{k=\ell}^{n} (-1)^k \frac{k!}{\ell!} L(n,k) H_k(\ell) &= \sum_{k=\ell}^{n} (-1)^k \frac{k!}{\ell!} H_k(\ell)[x^n]\frac{n!}{k!}\Big(-\frac{x}{1+x}\Big)^k \\ &= n![x^n]\left(\frac{\ln^{\ell}(1+x)}{\ell!} + \frac{x\ln^{\ell}(1+x)}{\ell!}\right).\end{aligned}$$

By the generating function of Stirling numbers of the first kind (2), we evaluate the coefficient

$$[x^n]\left(\frac{\ln^{\ell}(1+x)}{\ell!} + \frac{x\ln^{\ell}(1+x)}{\ell!}\right) = \frac{s(n,\ell)}{n!} + \frac{s(n-1,\ell)}{(n-1)!}.$$

Keeping in mind the recurrence relation

$$s(n+1,k) = s(n,k-1) - ns(n,k),$$

we get the desired result.

The formula (6) follows analogously, or from (4) directly. □

By means of the Lah inversion

$$f(n) = \sum_{k=0}^{n} L(n,k) g(k) \Longleftrightarrow g(n) = \sum_{k=0}^{n} L(n,k) f(k),$$

we get, from Theorem 2.1, the counterpart two formulae, given in the next theorem.

Theorem 2.2 (Inversion formulae of (5) and (6))**.**

$$\sum_{k=\ell}^{n} L(n,k)\big\{ s(k-1,\ell-1) + s(k-1,\ell) \big\} = (-1)^n \frac{n!}{\ell!} H_n(\ell);$$

$$\sum_{k=\ell}^{n} (-1)^k H(n,k)\big\{ s(k-1,\ell-1) + s(k-1,\ell) \big\} = \frac{n!}{\ell!} H_n(\ell).$$

Formulae concerning $H_n(\ell)$ and Fubini numbers

The Fubini numbers (or the ordered Bell numbers) $\mathcal{F}_n$, counting the number of weak orderings of a set with n elements [6, p.228], are defined by

$$\mathcal{F}_n = \sum_{k=0}^{n} k!S(n,k),$$

where $S(n,k)$ denotes the Stirling numbers of the second kind with the generating function

$$\sum_{n=k}^{\infty} S(n,k)\frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}.$$

Fubini numbers can be generated by the exponential generating function

$$\sum_{n=0}^{\infty} \mathcal{F}_n \frac{x^n}{n!} = \frac{1}{2-e^x},$$

and satisfy the recurrence relation

$$\mathcal{F}_0 = 1 \text{ and } \mathcal{F}_n = \sum_{j=1}^{n} \binom{n}{j} \mathcal{F}_{n-j}.$$

Theorem 2.3 (Identities involving $H_n(\ell)$ and $\mathcal{F}_n$)**.**

$$\sum_{\ell=1}^{n} \frac{\mathcal{F}_\ell}{\ell!} H_n(\ell) = 2^n - 1, \tag{7}$$

$$\sum_{\ell=1}^{n} \frac{(-1)^\ell \mathcal{F}_\ell}{\ell!} H_n(\ell) = \begin{cases} 0, & n \equiv_2 0; \\ -1, & n \equiv_2 1, \end{cases} \tag{8}$$

where $m \equiv_k n$ stands for "m is congruent to n modulo k".

Proof. Choosing $\mathcal{D}(n,k) = H_n(k)$ and $\lambda_k = \frac{\mathcal{F}_k}{k!}$ in Scheme (3), and according to the generating functions of $H_n(\ell)$ and $\mathcal{F}_n$, we get

$$\begin{aligned}
\sum_{\ell=1}^{n} \frac{\mathcal{F}_\ell}{\ell!} H_n(\ell) &= \sum_{\ell=1}^{n} \frac{\mathcal{F}_\ell}{\ell!} [x^n] \frac{\{-\ln(1-x)\}^\ell}{1-x} = [x^n] \frac{1}{1-x} \sum_{\ell=1}^{\infty} \frac{\mathcal{F}_\ell}{\ell!} \{-\ln(1-x)\}^\ell \\
&= [x^n] \frac{1}{1-x} \left\{ \frac{1}{2-e^{-\ln(1-x)}} - \mathcal{F}_0 \right\} = [x^n] \left\{ \frac{1}{1-2x} - \frac{1}{1-x} \right\} \\
&= 2^n - 1.
\end{aligned}$$

Similarly, for the formula (8), we have

$$\sum_{\ell=1}^{n} \frac{(-1)^\ell \mathcal{F}_\ell}{\ell!} H_n(\ell) = [x^n] \left\{ \frac{1}{1-x^2} - \frac{1}{1-x} \right\},$$

and the proof follows by extracting the coefficients. □

3. Identities involving $\mathcal{H}_n$ and $\mathcal{H}_n(\ell)$

In this section, we examine the alternating harmonic-like numbers $\mathcal{H}_n$ and $\mathcal{H}_n(\ell)$.

Theorem 3.1 (Identity involving $\mathcal{H}_n$, $S(n,k)$, and $\mathcal{F}_n$)**.**

$$\sum_{k=1}^{n} k!\mathcal{H}_k S(n,k) = -n\mathcal{F}_{n-1}. \tag{9}$$

Proof. From the generating functions of $\mathcal{H}_n$, $S(n,k)$, and $\mathcal{F}_n$, by setting $\mathcal{D}(n,k) = S(n,k)$ and $\lambda_k = \mathcal{H}_k$, we get

$$\begin{aligned}
\sum_{k=1}^{n} k!\mathcal{H}_k S(n,k) &= \sum_{k=1}^{n} k!\mathcal{H}_k [x^n] \frac{n!(e^x-1)^k}{k!} = n![x^n] \sum_{k=1}^{\infty} \mathcal{H}_k (e^x-1)^k \\
&= n![x^n] \frac{-x}{2-e^x} = -n![x^{n-1}] \frac{1}{2-e^x} = -n\mathcal{F}_{n-1}.
\end{aligned}$$

□

By means of the Stirling inversion

$$f(n)=\sum_{k=0}^{n}s(n,k)g(k)\Longleftrightarrow g(n)=\sum_{k=0}^{n}S(n,k)f(k),$$

we have the summation formula given in the following theorem:

Theorem 3.2 (Inversion formula of (9)).

$$\sum_{k=1}^{n}ks(n,k)\mathcal{F}_{k-1}=-n!\mathcal{H}_n.$$

Similar to the relation between $H_n(\ell)$ and Stirling numbers of the first kind mentioned in Section 1, we deduce another relation between $\mathcal{H}_n(\ell)$ and $s(n,k)$.

Theorem 3.3 (Identity involving $\mathcal{H}_n(\ell)$ and $s(n,k)$).

$$\sum_{m=\ell}^{n}\frac{s(m,\ell)}{m!}=(-1)^{\ell}\frac{1}{\ell!}\mathcal{H}_n(\ell). \tag{10}$$

Proof. By the generating functions of $\mathcal{H}_n(\ell)$ and $s(n,k)$, we have

$$\begin{aligned}\sum_{m=\ell}^{n}\frac{s(m,\ell)}{m!}&=\sum_{m=\ell}^{n}[x^m]\frac{\ln^{\ell}(1+x)}{\ell!}=(-1)^{\ell}\frac{1}{\ell!}\sum_{m=\ell}^{n}[x^m]\frac{\{-\ln(1+x)\}^{\ell}}{1-x}\times(1-x)\\&=(-1)^{\ell}\frac{1}{\ell!}\sum_{m=\ell}^{n}\sum_{k=0}^{m}[x^{m-k}]\frac{\{-\ln(1+x)\}^{\ell}}{1-x}\times[x^k](1-x)\\&=(-1)^{\ell}\frac{1}{\ell!}\sum_{m=\ell}^{n}\{\mathcal{H}_m(\ell)-\mathcal{H}_{m-1}(\ell)\}=(-1)^{\ell}\frac{1}{\ell!}\mathcal{H}_n(\ell).\end{aligned}$$

□

In the next theorem, by using the scheme (3), we establish two transformation formulae involving the numbers $\mathcal{H}_n(\ell)$, $L(n,k)$, $H(n,k)$, and $s(n,k)$.

Theorem 3.4 (Identities involving $\mathcal{H}_n(\ell)$, $L(n,k)$, $H(n,k)$, and $s(n,k)$).

$$\frac{1}{\ell!}\sum_{k=\ell}^{n}k!L(n,k)\mathcal{H}_k(\ell)=2^n n!\sum_{k=\ell}^{n-1}\frac{(-1)^{n-k}}{2^{k+1}k!}s(k,\ell)+s(n,\ell), \tag{11}$$

$$\frac{1}{\ell!}\sum_{k=\ell}^{n}(-1)^k k!H(n,k)\mathcal{H}_k(\ell)=2^n n!\sum_{k=\ell}^{n-1}\frac{(-1)^{k}}{2^{k+1}k!}s(k,\ell)+(-1)^n s(n,\ell). \tag{12}$$

Proof. First, we prove the identity (11). From the generating functions of $L(n,k)$ and $\mathcal{H}_n(\ell)$, we verify that

$$\begin{aligned}\frac{1}{\ell!}\sum_{k=\ell}^{n}k!L(n,k)\mathcal{H}_k(\ell)&=\frac{1}{\ell!}\sum_{k=\ell}^{n}k!\mathcal{H}_k(\ell)[x^n]\frac{n!}{k!}\Big(-\frac{x}{x+1}\Big)^k\\&=\frac{n!}{\ell!}[x^n]\sum_{k=\ell}^{\infty}\mathcal{H}_k(\ell)\Big(-\frac{x}{x+1}\Big)^k=\frac{n!}{\ell!}[x^n]\frac{\{-\ln(1-\frac{x}{x+1})\}^{\ell}}{1+\frac{x}{x+1}}\\&=n![x^n]\frac{1+x}{1+2x}\frac{\ln^{\ell}(1+x)}{\ell!}=n!\sum_{k=\ell}^{n}[x^k]\frac{\ln^{\ell}(1+x)}{\ell!}\Big([x^{n-k}]\frac{1+x}{1+2x}\Big).\end{aligned}$$

Note that

$$[x^n]\frac{1+x}{1+2x}=\begin{cases}1, & n=0;\\(-1)^n2^{n-1}. & n\geq 1.\end{cases}$$

Thus, we have

$$\sum_{k=\ell}^{n}[x^k]\frac{\ln^\ell(1+x)}{\ell!}\left([x^{n-k}]\frac{1+x}{1+2x}\right)=\sum_{k=\ell}^{n-1}\frac{(-1)^{n-k}}{k!}s(k,\ell)2^{n-k-1}+s(n,\ell)$$
$$=2^n\sum_{k=\ell}^{n-1}\frac{(-1)^{n-k}}{2^{k+1}k!}s(k,\ell)+s(n,\ell).$$

By the same method as used in proving (11), or from (4), we prove the identity (12). □

Finally, we prove two summation formulae involving numbers $\mathcal{H}_n(\ell)$ and $\mathcal{F}_n$.

Theorem 3.5 (Identities involving $\mathcal{H}_n(\ell)$ and $\mathcal{F}_n$**).**

$$\sum_{\ell=1}^{n}\frac{(\pm1)^\ell}{\ell!}\mathcal{F}_\ell\mathcal{H}_n(\ell)=\begin{cases}\dfrac{(-2)^n-1}{3}, & \text{“}+\text{”}\\[2mm] n, & \text{“}-\text{”}.\end{cases} \tag{13}$$

Proof. Using the generating functions of $\mathcal{F}_n$ and $\mathcal{H}_n(\ell)$, we evaluate the sum

$$\sum_{\ell=1}^{n}\frac{(\pm1)^\ell}{\ell!}\mathcal{F}_\ell\mathcal{H}_n(\ell)=\sum_{\ell=1}^{n}\frac{1}{\ell!}\mathcal{F}_\ell[x^n]\frac{\left\{\mp\ln(1+x)\right\}^\ell}{1-x}$$
$$=\begin{cases}-[x^{n-1}]\dfrac{1}{1-x}\times\dfrac{1}{1+2x}, & \text{“}+\text{”};\\[3mm] [x^n]\left\{\dfrac{1}{(1-x)^2}-\dfrac{1}{1-x}\right\}, & \text{“}-\text{”}.\end{cases}$$

The proof follows extracting the coefficients

$$-\left[x^{n-1}\right]\frac{1}{1-x}\times\frac{1}{1+2x}=\frac{(-2)^n-1}{3}\quad\text{and}\quad[x^n]\left\{\frac{1}{(1-x)^2}-\frac{1}{1-x}\right\}=n.$$

□

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