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Research Article

On the strength and domination number of graphs

Yukio Takahashi[1,*], Rikio Ichishima[2], Francesc A. Muntaner-Batle[3]

[1]*Department of Science and Engineering, Faculty of Electronics and Informatics, Kokushikan University, 4-28-1 Setagaya, Setagaya-ku, Tokyo 154-8515, Japan*
[2]*Department of Sport and Physical Education, Faculty of Physical Education, Kokushikan University, 7-3-1 Nagayama, Tama-shi, Tokyo 206-8515, Japan*
[3]*Graph Theory and Applications Research Group, School of Electrical Engineering and Computer Science, Faculty of Engineering and Built Environment, The University of Newcastle, NSW 2308, Australia*

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Abstract

A numbering f of a graph G of order n is a labeling that assigns distinct elements of the set $\{1, 2, \ldots, n\}$ to the vertices of G. The strength $\text{str}_f(G)$ of a numbering $f : V(G) \to \{1, 2, \ldots, n\}$ of G is defined by $\text{str}_f(G) = \max\{f(u) + f(v) \mid uv \in E(G)\}$, that is, $\text{str}_f(G)$ is the maximum edge label of G. The strength $\text{str}(G)$ of G is $\text{str}(G) = \min\{\text{str}_f(G) \mid f \text{ is a numbering of } G\}$. In this paper, we present sharp lower bounds for the strength of a graph in terms of its domination number as well as its (edge) covering and (edge) independence numbers. We also provide a necessary and sufficient condition for the strength of a graph to attain an earlier bound in terms of its subgraph structure. In addition, we establish a sharp lower bound for the domination number of a graph under certain conditions.

Keywords: strength; (edge) covering number; (edge) independence number; graph labeling; combinatorial optimization; domination number.

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1. Introduction

We refer to the book by Chartrand and Lesniak [5] for graph-theoretical notation and terminology not described in this paper. In particular, the *vertex set* of a graph G is denoted by $V(G)$, while the *edge set* of G is denoted by $E(G)$. The graph with n vertices and no edges is referred to as the *empty graph*.

We will use the notation $[a, b]$ for the interval of integers x such that $a \leq x \leq b$. For a graph G of order n, a *numbering* f of G is a labeling that assigns distinct elements of the set $[1, n]$ to the vertices of G, where each $uv \in E(G)$ is labeled $f(u) + f(v)$. The *strength* $\text{str}_f(G)$ *of a numbering* $f : V(G) \to [1, n]$ of G is defined by

$$\text{str}_f(G) = \max\{f(u) + f(v) \mid uv \in E(G)\},$$

that is, $\text{str}_f(G)$ is the maximum edge label of G and the *strength* $\text{str}(G)$ of a graph G itself is

$$\text{str}(G) = \min\{\text{str}_f(G) \mid f \text{ is a numbering of } G\}.$$

A numbering f of a graph G for which $\text{str}_f(G) = \text{str}(G)$ is called a *strength labeling* of G. Since empty graphs nK_1 do not have edges, this definition does not apply to such graphs. Consequently, we may define $\text{str}(nK_1) = +\infty$ for every positive integer n. This type of numberings was introduced in [11] as a generalization of the problem of finding whether a graph is super edge-magic or not (see [6] for the definition of a super edge-magic graph, and also consult either [1] or [7] for alternative and often more useful definitions of the same concept).

There are other related parameters that have been studied in the area of graph labelings. Excellent sources for more information on this topic are found in the extensive survey by Gallian [8], which also includes information on other kinds of graph labeling problems as well as their applications.

Several bounds for the strength of a graph have been found in terms of other parameters defined on graphs (see [9, 11, 15, 16]). Among others, the following result established in [11] that provides a lower bound for the strength of a graph G in terms of its order and minimum degree $\delta(G)$ is particularly useful.

Lemma 1.1. *For every graph G of order n with $\delta(G) \geq 1$,*

$$\text{str}(G) \geq n + \delta(G).$$

*Corresponding author (takayu@kokushikan.ac.jp).

It is worth mentioning that the lower bound given in Lemma 1.1 is sharp in the sense that there are infinitely many graphs G for which $\mathrm{str}\,(G) = |V\,(G)| + \delta\,(G)$ (see [9, 11, 12, 15] for a detailed list of such graphs and other sharp bounds).

For every nonempty graph G of order n, it is clear that $3 \leq \mathrm{str}\,(G) \leq 2n - 1$. In fact, it was shown in [13] that for every $k \in [1, n-1]$, there exists a graph G of order n satisfying $\delta\,(G) = k$ and $\mathrm{str}\,(G) = n + k$.

In the process of settling the problem (proposed in [11]) of finding sufficient conditions for a graph G of order n with $\delta\,(G) \geq 1$ to ensure that $\mathrm{str}\,(G) = n + \delta\,(G)$, an equivalent definition of the following class of graphs was defined in [14]. For integers $k \geq 2$, let F_k be the graph with $V\,(F_k) = \{v_i \,|i \in [1, k]\,\}$ and

$$E\,(F_k) = \{v_i v_j \,|i \in [1, \lfloor k/2 \rfloor] \text{ and } j \in [1+i, k+1-i]\,\}\,.$$

Let $\overline{G}$ denote the complement of a graph G. The following result found in [14] provides a necessary and sufficient condition for a graph G of order n to hold the inequality $\mathrm{str}\,(G) \leq 2n - k$, where $k \in [2, n-1]$.

Theorem 1.1. *Let G be a graph of order n. Then $\mathrm{str}\,(G) \leq 2n - k$ if and only if $\overline{G}$ contains F_k as a subgraph, where $k \in [2, n-1]$.*

Theorem 1.1 plays an important role in the study of the strength of graphs (see [17, 18] for instance).

2. Results

In this section, we present some results involving the domination number of a graph and a new sharp lower bound for the strength of a graph without isolated vertices. The following result provides a lower bound for the strength of a graph in terms of its domination number.

Lemma 2.1. *For every graph G of order n,*

$$\mathrm{str}\,(G) \geq 2n - 2\gamma\,(G) + 1.$$

Proof. Let G be a graph with $V\,(G) = \{v_i \mid i \in [1, n]\,\}$, and consider a strength labeling f of G. Since $1 \leq \gamma\,(G) \leq n$, it follows that the set

$$S = [n - \gamma\,(G)\,, n]$$

contains at least two integers. By the pigeonhole principle, at least two integers in S are assigned to two adjacent vertices, say $f\,(v_s)$ and $f\,(v_t)$, where $s, t \in [1, n]$. Now, assume, without loss of generality, that $f\,(v_s) > f\,(v_t)$. Then

$$\begin{aligned} \min\{f\,(v_s) + f\,(v_t) \mid s, t \in [1, n]\,\} &\geq (n - \gamma\,(G)) + (n - \gamma\,(G) + 1) \\ &= 2n - 2\gamma\,(G) + 1. \end{aligned}$$

Thus,

$$\mathrm{str}\,(G) = \mathrm{str}_f\,(G) \geq 2n - 2\gamma\,(G) + 1,$$

completing the proof. □

The bound given in Lemma 2.1 is sharp in the sense that there are infinitely many graphs G for which

$$\mathrm{str}\,(G) = 2\,|V\,(G)| - 2\gamma\,(G) + 1.$$

To see this, it suffices to consider the complete graph K_n of order n. It is straightforward to see that $\mathrm{str}\,(K_n) = 2n - 1$ and $\gamma\,(K_n) = 1$ $(n \geq 2)$. This implies that $\mathrm{str}\,(K_n) = 2n - 2\gamma\,(K_n) + 1$ $(n \geq 2)$. The following result provides a necessary and sufficient condition for a graph G of order n to hold for $\mathrm{str}\,(G) = 2n - 2\gamma\,(G) + 1$.

Theorem 2.1. *Let G be a graph of order n with $\gamma\,(G) = k$, where $k \in [2, \lceil n/2 \rceil]$. Then*

$$\mathrm{str}\,(G) = 2n - 2\gamma\,(G) + 1$$

if and only if $\overline{G}$ contains F_{2k-1} as a subgraph.

Proof. First, suppose that $\mathrm{str}\,(G) = 2n - 2k + 1$, where $\gamma\,(G) = k$ $(k \in [2, \lceil n/2 \rceil])$. Let $V\,(G) = \{v_i \mid i \in [1, n]\}$, and assume, without loss of generality, that there exists a strength labeling of G that assigns i to v_i $(i \in [1, n])$. Since $\mathrm{str}\,(G) = 2n-2k+1$, every two vertices v_i and v_j for which $i + j > 2n - 2k + 1$ are not adjacent in G. This means that every two vertices v_i and

v_j for which $i+j > 2n-2k+1$ are adjacent in $\overline{G}$. Let $v_i = w_{n+1-i}$ ($i \in [1,n]$) so that $V(\overline{G}) = \{w_i \mid i \in [1,n]\}$. Then if w_{n+1-i} and w_{n+1-j} are adjacent in $\overline{G}$, it follows that

$$\begin{aligned}(n+1-i)+(n+1-j) &= 2n+2-(i+j)\\ &< 2n+2-(2n-2k+1) = 2k+1.\end{aligned}$$

Thus, $\overline{G}$ contains F_{2k-1} as a subgraph.

Next, suppose that $\overline{G}$ contains F_{2k-1} as a subgraph, where $\gamma(G) = k$ ($k \in [2, \lceil n/2 \rceil]$). It follows from Theorem 1.1 that

$$\text{str}(G) \leq 2n-(2k-1) = 2n-2\gamma(G)+1.$$

It also follows from Lemma 2.1 that $\text{str}(G) \geq 2n-2\gamma(G)+1$ and therefore $\text{str}(G) = 2n-2\gamma(G)+1$. □

The following result found in [16] provides a necessary and sufficient condition for a graph G of order n to hold for $\text{str}(G) = 2n-2\beta(G)+1$, where $\beta(G)$ denotes the independence number of G.

Theorem 2.2. *Let G be a graph of order n with $\beta(G) = k$, where $k \in [2, \lceil n/2 \rceil]$. Then $\text{str}(G) = 2n-2\beta(G)+1$ if and only if $\overline{G}$ contains F_{2k-1} as a subgraph.*

The next result follows from Theorems 2.1 and 2.2, which shows a connection between the domination number and independence number.

Corollary 2.1. *Let G be a graph of order n with $\gamma(G) = k$, where $k \in [2, \lceil n/2 \rceil]$, and assume that $\overline{G}$ contains F_{2k-1} as a subgraph. Then $\gamma(G) = \beta(G)$.*

The following result is an immediate consequence of Lemma 2.1.

Corollary 2.2. *Let G be a graph of order n with $\text{str}(G) = n+\delta(G)$, where $\delta(G) \geq 1$. Then*

$$\gamma(G) \geq \lceil (n-\delta(G)+1)/2 \rceil.$$

There are infinitely many graphs attaining the bound given in Corollary 2.2. For instance, if $G = K_n$ ($n \geq 2$), then $\text{str}(G) = 2n-1$. Also, we have

$$|V(G)| = n \text{ and } \delta(G) = n-1.$$

This implies that $|V(G)| - \delta(G) = 1$. On the other hand, we have

$$\gamma(G) = 1 \text{ and } \lceil (|V(G)| - \delta(G)+1)/2 \rceil = 1.$$

It is known that the domination number of a graph without isolated vertices is bounded above by all of the covering and independence numbers (see [5, p. 307] for instance). Note that we denote $\alpha(G)$, $\alpha_1(G)$ and $\beta_1(G)$ to be the covering, edge covering and edge independence numbers of G, respectively.

Theorem 2.3. *If G is a graph without isolated vertices, then*

$$\gamma(G) \leq \min\{\alpha(G), \alpha_1(G), \beta(G), \beta_1(G)\}.$$

Theorem 2.3 together with Lemma 2.1 provides the lower bound, given in the following corollary, for the strength of a graph without isolated vertices.

Corollary 2.3. *For every graph G without isolated vertices,*

$$\text{str}(G) \geq 2n-2\min\{\alpha(G), \alpha_1(G), \beta(G), \beta_1(G)\}+1.$$

It is known from [11] that $\text{str}(C_{2n+1}) = 2n+3$ ($n \geq 1$). Also, note that

$$\alpha(C_{2n+1}) = \alpha_1(C_{2n+1}) = n+1 \text{ and } \beta(C_{2n+1}) = \beta_1(C_{2n+1}) = n\,(n \geq 1).$$

By means of these, it indicates that the bound given in Corollary 2.3 is sharp.

McCuaig and Shepherd [20] showed that if G is a connected graph of order n with $\delta(G) \geq 2$ and is not one of the seven exceptional graphs illustrated in Figure 1, then $\gamma(G) \leq 2n/5$. Reed [23] showed that if G is a graph of order n with $\delta(G) \geq 3$, then $\gamma(G) \leq 3n/8$. These results together with Lemma 2.1 provide us the lower bounds, given in the next two corollaries, for the strength of a graph.

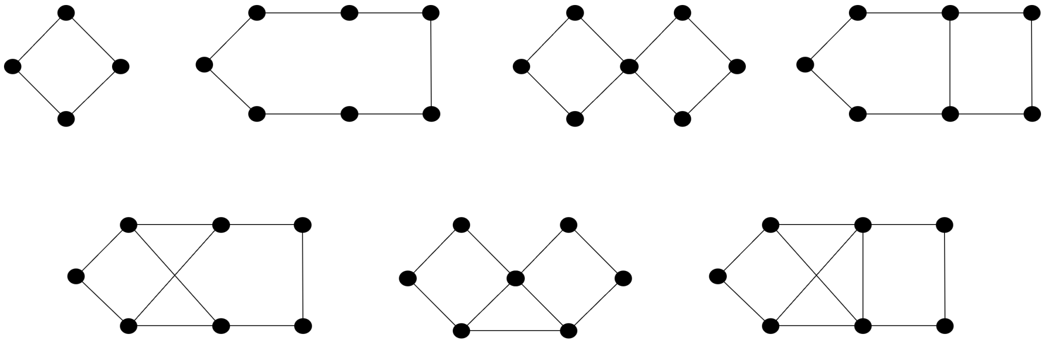

Figure 1: The seven exceptional graphs for Corollary 2.4.

Corollary 2.4. *If G is a connected graph of order n with $\delta(G) \geq 2$ and is none of the graphs of Figure 1, then*

$$\operatorname{str}(G) \geq 6n/5 + 1.$$

Corollary 2.5. *If G is a graph of order n with $\delta(G) \geq 3$, then*

$$\operatorname{str}(G) \geq 5n/4 + 1.$$

To close this section, we mention some general results providing upper bounds for the domination of a graph. Combining these results with Lemma 2.1, we can obtain new lower bounds for the strength of a graph.

The following upper bound for the domination number of a graph was independently proved by Alon and Spencer [2], Arnautov [3], Lovász [19], and Payan [21].

Theorem 2.4. *For any graph G of order n with $\delta(G) \geq 2$,*

$$\gamma(G) \leq \frac{1 + \ln(1 + \delta(G))}{1 + \delta(G)} n.$$

In [4] and [10], McCaro and Roditty provided the following upper bound for the domination number of a graph.

Theorem 2.5. *For any graph G of order n with $\delta(G) \geq 1$,*

$$\gamma(G) \leq \left(1 - \frac{\delta(G)}{(1 + \delta(G))^{1 + \frac{1}{\delta(G)}}}\right) n.$$

The preceding two upper bounds were recently improved by Rad [22] as the next two results indicate.

Theorem 2.6. *If G is a graph of order n with minimum degree $\delta \geq 2$ and maximum degree Δ, then for any integer $k \geq 1$,*

$$\gamma(G) \leq \frac{n}{1+\delta}\left[\ln(1+\delta) + 1 - (\delta - \ln(1+\delta)) \sum_{i=1}^{k} \left(\frac{\ln(1+\delta)}{1+\delta}\right)^{i(1+\Delta)}\right].$$

Theorem 2.7. *If G is a graph of order n with minimum degree $\delta \geq 2$ and maximum degree Δ, then for any integer $k \geq 1$,*

$$\gamma(G) \leq \left[1 - \frac{\delta}{(1+\delta)^{1+\frac{1}{\delta}}} - \frac{\delta}{(1+\delta)^{1+\frac{1}{\delta}}} \sum_{i=1}^{k} \left(1 - \frac{1}{(1+\delta)^{\frac{1}{\delta}}}\right)^{i(1+\Delta)}\right] n.$$

With the aid of the preceding four theorems and Lemma 2.1, it is possible to provide the lower bounds, given in the next four corollaries, for the strength of a graph.

Corollary 2.6. *For any graph G of order n with $\delta(G) \geq 2$,*

$$\operatorname{str}(G) \geq 2\left(1 - \frac{1 + \ln(1 + \delta(G))}{1 + \delta(G)}\right) n + 1.$$

Corollary 2.7. *For any graph G of order n with $\delta(G) \geq 1$,*

$$\operatorname{str}(G) \geq 2\left(\frac{\delta(G)}{(1 + \delta(G))^{1 + \frac{1}{\delta(G)}}}\right) n + 1.$$

Corollary 2.8. *If G is a graph of order n with minimum degree $\delta \geq 2$ and maximum degree Δ, then for any integer $k \geq 1$,*

$$\mathrm{str}\,(G) \geq 2\left[1 - \frac{1}{1+\delta}\left(\ln(1+\delta) + 1 - (\delta - \ln(1+\delta))\sum_{i=1}^{k}\left(\frac{\ln(1+\delta)}{1+\delta}\right)^{i(1+\Delta)}\right)\right]n + 1.$$

Corollary 2.9. *If G is a graph of order n with minimum degree $\delta \geq 2$ and maximum degree Δ, then for any integer $k \geq 1$,*

$$\mathrm{str}\,(G) \geq 2\left[\frac{\delta}{(1+\delta)^{1+\frac{1}{\delta}}} - \frac{\delta}{(1+\delta)^{1+\frac{1}{\delta}}}\sum_{i=1}^{k}\left(1 - \frac{1}{(1+\delta)^{\frac{1}{\delta}}}\right)^{i(1+\Delta)}\right]n + 1.$$

The preceding four corollaries (and Corollaries 2.4 and 2.5) provide just a few examples of various lower bounds for the strength of a graph that can be deduced from Lemma 2.1.

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