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*Research Article*

# Truncated Bresse-Timoshenko beam with fractional Laplacian damping

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**Abstract**

This article focuses on a Timoshenko beam model introduced by Elishakoff. This model is free of the second frequency spectrum and solves the paradox of equal wave speeds, related to Timoshenko's model. Damping created by a fractional Laplacian is considered, which includes internal damping, Kelvin-Voigt damping, and intermediate damping. Exponential stability is shown without requiring any relationship between the system coefficients.

**Keywords:** fractional Laplacian damping; Timoshenko system; well-posedness; semigroups.

**2020 Mathematics Subject Classification:** 35B35, 35B40, 35L45, 35Q74.

## 1. Introduction

In this paper, we investigate the global existence and decay properties of solutions for the system

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x + A^\theta\varphi_t = 0, \text{ in } ]0, L[\ \times\ ]0, \infty[, \tag{1}$$

$$-\rho_2\varphi_{ttx} - b\psi_{xx} + \kappa(\varphi_x + \psi) = 0, \text{ in } ]0, L[\ \times\ ]0, \infty[, \tag{2}$$

where all coefficients are positive; $\rho_1 = \rho S$ and $\rho_2 = \rho I$, where $S$ and $I$ are the cross-sectional area and the second moment of the cross-sectional area, respectively; $b = EI$ and $\kappa = kGS$, where $E$, $G$, and $k$ are Young's modulus, the modulus of rigidity, and the transverse shear factor, respectively; $A$ is a positive and self-adjoint operator with the compact inverse and $0 \leq \theta \leq 1$; $\varphi$ is the transverse displacement and $\psi$ is the rotation of the neutral axis due to bending. Also, we consider the initial data given by

$$\varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x),\ \ \psi(x,0) = \psi_0(x),\ x \in ]0, L[, \tag{3}$$

and Neumann-Dirichlet boundary conditions given by

$$\varphi_x(0,t) = \varphi_x(L,t) = \psi(0,t) = \psi(L,t) = 0,\ t \geq 0. \tag{4}$$

System (1)-(2), known as the truncated version of Bresse-Timoshenko's beam, was introduced by Elishakoff [9] and is free of the second spectrum, present in the pioneer model of Timoshenko [25] given by

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \tag{5}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) = 0. \tag{6}$$

About the pioneer Timoshenko system, we have a vast literature. See, for instance, [21, 26] with references in it. When system (5)-(6) is partially damped, the non-physical second frequency spectrum in Timoshenko's beam imposed that the exponential stability holds if and only if the wave speeds of the equations of the system are equal, that is,

$$\frac{\rho_1}{k} = \frac{\rho_2}{b}. \tag{7}$$

For a historical review of Timoshenko's theory, including essential phases of his life and recent arguments about the Timoshenko-Ehrenfest partnership, that describes both shear deformation and rotational bending effects on the beam, see [10, 11, 13]. In [3], the relation between the physical inconsistency known as the second spectrum of frequency and

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the exponential decay of truncated Bresse-Timoshenko beam equation with a damping mechanism just on angle rotation $\psi$ is analyzed. The authors also gave physical explanations of why the partially dissipative Timoshenko systems decay exponentially under condition (7), as previously proved by Soufyane [24]. The discovery of the second spectrum, that act in opposition to the dissipative properties of the system, is credited to Manevich and Kolakowski [16] and Nesterenko [18]. The spotlight is currently in systems free of the second frequency spectrum, as considered here. Apalara et al. [4] proved exponential stability for the Timoshenko system free of the second spectrum with just a thermal dissipation effect taking into account the Fourier law. In [3] was proved that the exponential decay with just one frictional damping on the rotation angle holds without the condition (7). A similar problem with delay was considered in [2] and showed exponential stability without any conditions on the coefficients of the system.

We introduce the one-dimensional Laplacian operator

$$A \quad = -\partial_{xx} : L^2(0,L) \to L^2(0,L).$$

$A^\theta$ is an intermediate fractional dissipative mechanism that includes the internal damping $u_t = A^\theta$ for $\theta = 0$ and the Kelvin-Voigt damping $-u_{txx} = A^\theta$ when $\theta = 1$. In addition, we solve the following problems

$$\begin{aligned} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x + \varphi_t = 0, \ \text{ in } \ ]0,L[ \ \times \ ]0,\infty[, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \kappa(\varphi_x + \psi) = 0, \ \text{ in } \ ]0,L[ \ \times \ ]0,\infty[, \end{aligned}$$

and

$$\begin{aligned} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x - \varphi_{txx} = 0, \ \text{ in } \ ]0,L[ \ \times \ ]0,\infty[, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \kappa(\varphi_x + \psi) = 0, \ \text{ in } \ ]0,L[ \ \times \ ]0,\infty[. \end{aligned}$$

The intermediate damping created by fractional Laplacian damping was previously considered by Medeiros and Milla [17]. Following the ideas as in [14], the authors proved the exponential decay of solutions for a wave equation of Kirchhoff type

$$u_{tt} - M(|A^{1/2}u|^2)Au + A^\theta u_t = f, \ 0 < \theta \le 1.$$

The intermediate damping $A^\theta$, $0 < \theta < 1$, is essential because the fractional derivative produces a more realistic physical situation than the integer derivative. In the equation of a membrane-like electrical network, the dissipative mechanism given by an intermediate damping acts between the electrical damping potential for $\theta = 0$ and the Laplacian of the electric potential for $\theta = 1$. Fractional powers were introduced by [6]. More details about fractional powers of linear operators can be found in Section 5 of [12]. Akagi et al. [1] provide a definition of the fractional Laplacian operator and the rigorous formulation of the Poisson problem

$$(-\Delta)^\theta u = f \ \text{ in } \ \Omega,$$

where $\Omega$ is an open and bounded set of $\mathbb{R}^n$, $0 < \theta < 1$ and $f$ is a function with suitable regularity.

Intermediate damping deals with the concepts of non-integer-order derivatives and can be related to fractional calculus. Fractional calculus's origin dates back to the seventeenth century. The concepts of non-integer-order derivatives are used in biology [22], medicine [23], geo-hydrology [5], and physics [15]. For a brief literature review on intermediate damping, we cite some recent works: Exponential stability for laminated beams with intermediate damping was considered in [8]. Polynomial decay for a system of two-coupled plate equations with intermediate damping was proved in [20]. Optimal decay rates for Kirchhoff plates with intermediate damping were studied in [7]. The asymptotic behavior of a linear plate equation with effects of rotational inertia and intermediate damping in the memory term was analyzed in [19].

To the best of our knowledge, the present study is the first contribution to the literature regarding truncated Bresse-Timoshenko beam with intermediate damping. The remainder of this paper is organized as follows. Section 2 deals with preliminaries. The existence of strong and weak solutions is given in Section 3 by using the Faedo-Galerkin method. In Section 4, the exponential stability is proved by the energy method by using suitable estimates for multipliers to construct a Lyapunov functional.

## 2. Preliminaries

The following notations are used in the rest of the paper:

$$\|\varphi\|_p = \|\varphi\|_{L^p(0,L)}, \quad \langle\varphi,\psi\rangle = \langle\varphi,\psi\rangle_{L^2(0,L)}, \quad \|\varphi\| = \|\varphi\|_{L^2(0,L)}.$$

The operator $A = -\partial_{xx}$ with domain $D(A) = H^2(0,L) \cap H_0^1(0,L)$ is positive and self-adjoint with compact inverse in the Hilbert space $L^2(0,L)$. The spectral theory allows us to define the powers $A^\theta$ for $\theta \in \mathbb{R}$. Therefore, for all $\theta > 0$, the operator $A^\theta$ is self-adjoint and positive on $L^2(0,L)$. Moreover, $D(A^\theta)$ is a Hilbert space endowed with inner product and norm defined by

$$\langle \varphi, \psi \rangle_{D(A^\theta)} = \langle A^\theta \varphi, A^\theta \psi \rangle, \qquad \|\varphi\|^2_{D(A^\theta)} = \langle A^\theta \varphi, A^\theta \varphi \rangle = \|A^\theta\|^2.$$

Furthermore, for $\theta_1 \geq \theta_2$, we have the following dense and continuous embedding $D(A^{\theta_1}) \hookrightarrow D(A^{\theta_2})$. Throughout this manuscript, we use $0 \leq \theta \leq 1$. Recall that

$$D(A^\theta) = H^2(0,L) \cap H_0^1(0,L) \subset D(A^{1/2}) = H_0^1(0,L) \subset D(A^{\theta/2}) \subset L^2(0,L),$$

with all inclusions dense and continuous.

Since $A$ is a self-adjoint positive operator with compact inverse, it is known by the spectral theory that the spectrum of this operator is constituted only by positive eigenvalues. We introduce the following Hilbert spaces $\mathcal{H}(0,L) \times H_0^1(0,L)$ and $\mathcal{H}(0,L) \times (H^2(0,L) \cap H_0^1(0,L))$, where

$$\mathcal{H}(0,L) = \{(\varphi, \psi) \in H_0^1(0,L) \times H_0^1(0,L) \ : \ \kappa\varphi_{xx} - A^\theta \psi \in L^2(0,L)\},$$

and the space

$$\mathcal{V} = \{(\varphi, \psi) \in H^2(0,L) \cap H_0^1(0,L) \times H^2(0,L) \cap H_0^1(0,L); \ \ \kappa\varphi_{xxx} - A^\theta \psi_x \in L^2(0,L)\}.$$

The norm in space $\mathcal{H}(0,L) \times H_0^1(0,L)$ is defined by:

$$\|(\varphi, \varphi_t, \psi)\|^2_{\mathcal{H}(0,L)\times H_0^1(0,L)} = \frac{\rho_1}{2} \parallel \varphi_t \parallel_2^2 + \frac{\kappa}{2} \parallel \varphi_x + \psi \parallel_2^2 + \frac{b}{2} \parallel \psi_x \parallel_2^2 + \frac{\rho_2}{2} \parallel \varphi_{tx} \parallel_2^2 + \frac{\rho_2}{2\kappa\rho_1} \parallel \kappa(\varphi_x + \psi)_x - A^\theta \varphi_t \parallel_2^2 .$$

Clearly there exists a constant $\kappa_0 > 0$ such that

$$\|\varphi_x\|^2 \leq \kappa_0 \left( b\|\psi_x\|^2 + \kappa \|\varphi_x + \psi\|^2 \right). \tag{8}$$

**Definition 2.1.** *We say that a strong solution of system (1)-(4) is a ternary of functions* $(\varphi, \varphi_t, \psi)$ *such that*

$$\begin{aligned} \rho_1 \varphi_{tt} + k(\varphi_x + \psi) + A^\theta \varphi_t = 0, \ \ &a.e. \ in \ \ ]0,L[ \ \times \ ]0,T[, \\ -\rho_2 \varphi_{ttx} - b\psi_{xx} + k(\varphi_x + \psi) = 0, \ \ &a.e. \ in \ \ ]0,L[ \ \times \ ]0,T[, \end{aligned}$$

*and*

$$(\varphi(0), \varphi_t(0), \psi(0)) = (\varphi_0, \varphi_1, \psi_0).$$

**Definition 2.2.** *We say that a weak solution of system (1)-(4) is a ternary of functions* $(\varphi, \varphi_t, \psi)$ *such that*

$$\begin{aligned} \rho_1 \varphi_{tt} - k(\varphi_x - \psi)_x + A^\theta \varphi_t = 0, \ \ &a.e. \ in \ \ ]0,L[ \ \times \ ]0,T[, \\ \rho_2(\varphi_{tt}, w_x) + b(\psi_x, w_x) + k(\varphi_x + \psi, w) = 0, \ \ &\forall \ w \in H_0^1(0,L), \end{aligned}$$

*in the sense* $\mathcal{D}'(0,T)$ *and* $(\varphi(0), \varphi_t(0), \psi(0)) = (\varphi_0, \varphi_1, \psi_0)$.

## 3. Well-posedness

This section studies the existence and uniqueness of weak and strong solutions of system (1)-(4).

### 3.1. Faedo-Galerkin method

#### 3.1.1. Approximate system

Consider in $H_*^1(0,L)$ and $H_0^1(0,L)$ the bases $\{w_j(x)\}_j$ and $\{\widetilde{w}_j(x)\}_j$, where $w_j(x) = \cos\left(\frac{j\pi}{L}x\right)$ and $\widetilde{w}_j(x) = \sin\left(\frac{j\pi}{L}x\right)$. Note that $\Delta w_j(x) = -\left(\frac{j\pi}{L}\right)^2 w_j(x)$ and $\Delta \widetilde{w}_j(x) = -\left(\frac{j\pi}{L}\right)^2 \widetilde{w}_j(x)$. Also, $\{w_j(x)\}_j$ and $\{\widetilde{w}_j(x)\}_j$ are orthogonal in $L^2(0,L)$. We define $W_m = [w_1(x), w_2(x), \cdots, w_m(x)]$ and $\widetilde{W}_m = [\widetilde{w}_1(x), \widetilde{w}_2(x), \cdots, \widetilde{w}_m(x)]$ $m$-dimensional subspaces, formed by the $m$-first base elements $\{w_j(x)\}_j$ and $\{\widetilde{w}_j(x)\}_j$, respectively. Thus, the approximate problem associated with (1)-(4) consists of finding functions of the form

$$(\varphi^m, \psi^m) = \left( \sum_{j=1}^m P_{mj}(t) w_j(x), \sum_{j=1}^m Q_{mj}(t) \widetilde{w}_j(x) \right) \in W_m \times \widetilde{W}_m,$$

where the coefficients $P_{mj}(t)$ and $Q_{mj}(t)$ are determined to satisfy the system of ordinary differential equations given by

$$\rho_1(\varphi^m_{tt}, w) - \kappa((\varphi^m_x + \psi^m)_x, w) + (A^\theta \varphi^m_t, w) = 0, \ \ \forall\, w \in W_m, \tag{9}$$

$$-\rho_2(\varphi^m_{ttx}, \widetilde{w}) - b(\psi^m_{xx}, \widetilde{w}) + \kappa(\varphi^m_x + \psi^m, \widetilde{w}) = 0, \ \ \forall\, \widetilde{w} \in \widetilde{W}_m, \tag{10}$$

$$\varphi^m(x,0) = \varphi^m_0(x), \ \varphi^m_t(x,0) = \varphi^m_1(x), \ \psi^m(x,0) = \psi^m_0(x). \tag{11}$$

By density argument, we have the following convergences

$$\begin{aligned} (\varphi^m_0(x), \varphi^m_1(x)) \ &\to \ (\varphi_0, \varphi_1) \ \text{ strong in } \mathcal{V}_1, \\ \psi^m_0(x) \ &\to \ \psi_0(x) \ \text{ strong in } H^2(0,L) \cap H^1_0(0,L). \end{aligned}$$

Using the theory of ordinary differential equations, the problem (9)-(10) has solutions $P_{mj}(t)$ and $Q_{mj}(t)$ defined over an interval $[0, t_m]$, where $0 < t_m < T$.

#### 3.1.2. Step 1: First a priori estimates

In Equations (9) and (10), substituting $w = \varphi^m_t$ and $\widetilde{w} = \psi^m_t$, respectively, and carrying out the calculations accordingly, we obtain

$$\frac{d}{dt}\left\{\frac{\rho_1}{2} \parallel \varphi^m_t \parallel^2_2 + \frac{\kappa}{2} \parallel \varphi^m_x + \psi^m \parallel^2_2 + \frac{b}{2} \parallel \psi^m_x \parallel^2_2\right\} + \parallel A^{\theta/2}\varphi^m_t \parallel^2_2 + \rho_2(\varphi^m_{tt}, \psi^m_{tx}) = 0. \tag{12}$$

Note that

$$(\varphi^m_{tt}, \psi^m_{tx}) = \frac{1}{\kappa}\left(\varphi^m_{tt}, \kappa(\varphi^m_x + \psi^m)_{tx} - A^\theta \varphi^m_{tt}\right) + \frac{1}{2}\frac{d}{dt} \parallel \varphi^m_{tx} \parallel^2_2 + \frac{1}{\kappa} \parallel A^{\theta/2}\varphi^m_{tt} \parallel^2_2 . \tag{13}$$

Using (13) in (12), we arrive at

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{\rho_1}{2}\varphi^m_t \parallel^2_2 + \frac{\kappa}{2} \parallel \varphi^m_x + \psi^m \parallel^2_2 + \frac{b}{2} \parallel \psi^m_x \parallel^2_2 + \frac{\rho_2}{2} \parallel \varphi^m_{tx} \parallel^2_2\right\} \\ &+ \ \frac{\rho_2}{\kappa}\left(\varphi^m_{tt}, \kappa(\varphi^m_x + \psi^m)_{tx} - A^\theta \varphi^m_{tt}\right) + \parallel A^{\theta/2}\varphi^m_t \parallel^2_2 + \frac{\rho_2}{\kappa} \parallel A^{\theta/2}\varphi^m_{tt} \parallel^2_2 = 0. \end{aligned} \tag{14}$$

Now, making $w = \kappa(\varphi^m_x + \psi^m)_{tx} - A^\theta \varphi^m_{tt}$ on (9), we get

$$\left(\varphi^m_{tt}, \kappa(\varphi^m_x + \psi^m)_{tx} - A^\theta \varphi^m_{tt}\right) = \frac{1}{2\rho_1}\frac{d}{dt} \parallel \kappa(\varphi^m_x + \psi^m)_x - A^\theta \varphi^m_t \parallel^2_2 . \tag{15}$$

Using (15) in (14), we arrive at

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{\rho_1}{2} \parallel \varphi^m_t \parallel^2_2 + \frac{\kappa}{2} \parallel \varphi^m_x + \psi^m \parallel^2_2 + \frac{b}{2} \parallel \psi^m_x \parallel^2_2 + \frac{\rho_2}{2\kappa\rho_1} \parallel \kappa(\varphi^m_x + \psi^m)_x - A^\theta \varphi^m_t \parallel^2_2 + \frac{\rho_2}{2} \parallel \varphi^m_{tx} \parallel^2_2\right\} \\ &+ \parallel A^{\theta/2}\varphi^m_t \parallel^2_2 + \frac{\rho_2}{\kappa} \parallel A^{\theta/2}\varphi^m_{tt} \parallel^2_2 = 0. \end{aligned}$$

Integrating from $0$ to $t$ we obtain a positive constant $C_1$, such that

$$\begin{aligned} &\frac{\rho_1}{2} \parallel \varphi^m_t \parallel^2_2 + \frac{\kappa}{2} \parallel \varphi^m_x + \psi^m \parallel^2_2 + \frac{b}{2} \parallel \psi^m_x \parallel^2_2 + \frac{\rho_2}{2\kappa\rho_1} \parallel \kappa(\varphi^m_x + \psi^m)_x - A^\theta \varphi^m_t \parallel^2_2 + \frac{\rho_2}{2} \parallel \varphi^m_{tx} \parallel^2_2 \\ &+ \int_0^t \left\{\parallel A^{\theta/2}\varphi^m_t \parallel^2_2 + \frac{\rho_2}{\kappa} \parallel A^{\theta/2}\varphi^m_{tt} \parallel^2_2\right\} dt \le C_1. \end{aligned}$$

From the previous estimate, we have

$$\begin{aligned} \varphi^m_x + \psi^m &\text{ is limited in } L^\infty\big(]0,T[; L^2(0,L)\big), \\ \varphi^m_t &\text{ is limited in } L^\infty\big(]0,T[; H^1_0(0,L)\big), \\ \varphi^m_t &\text{ is limited in } L^2\big(]0,T[; H^\theta_0(0,L)\big), \\ \psi^m &\text{ is limited in } L^\infty\big(]0,T[; H^1_0(0,L)\big), \\ \kappa(\varphi^m_x + \psi^m)_x - A^\theta \varphi^m_t &\text{ is limited in } L^\infty\big(]0,T[; L^2(0,L)\big), \\ \varphi^m_{tt} &\text{ is limited in } L^2\big(]0,T[; H^\theta_0(0,L)\big). \end{aligned}$$

Furthermore, using (8) we conclude that

$$\varphi^m \text{ is limited in } L^\infty\big(]0,T[; H^1_0(0,L)\big).$$

### 3.1.3. Step 2: Second a priori estimates

Substituting $w = -\varphi_{txx}$ and $\widetilde{w} = -\psi_{txx}$ in Equations (9) and (10), respectively, and analogously to the first estimate, we obtain

$$\frac{d}{dt}\left\{\frac{\rho_1}{2} \parallel \varphi_{tx}^m \parallel_2^2 + \frac{\kappa}{2} \parallel \varphi_{xx}^m + \psi_x^m \parallel_2^2 + \frac{b}{2} \parallel \psi_{xx}^m \parallel_2^2 + \frac{\rho_2}{2\kappa\rho_1} \parallel \kappa(\varphi_x^m + \psi^m)_{xx} - A^\theta \varphi_{tx}^m \parallel_2^2 + \frac{\rho_2}{2} \parallel \varphi_{txx}^m \parallel_2^2\right\}$$
$$+ \parallel A^\theta \varphi_t^m \parallel_2^2 + \frac{\rho_2}{\kappa} \parallel A^\theta \varphi_{tt}^m \parallel_2^2 = 0.$$

Integrating from $0$ to $t$, we get

$$\frac{\rho_1}{2} \parallel \varphi_{tx}^m \parallel_2^2 + \frac{\kappa}{2} \parallel \varphi_{xx}^m + \psi_x^m \parallel_2^2 + \frac{b}{2} \parallel \psi_{xx}^m \parallel_2^2 + \frac{\rho_2}{2\kappa\rho_1} \parallel \kappa(\varphi_x^m + \psi^m)_{xx} - A^\theta \varphi_{tx}^m \parallel_2^2 + \frac{\rho_2}{2} \parallel \varphi_{txx}^m \parallel_2^2$$
$$+ \int_0^t \left\{\parallel A^\theta \varphi_t^m \parallel_2^2 + \frac{\rho_2}{\kappa} \parallel A^\theta \varphi_{tt}^m \parallel_2^2\right\} dt \le C_2.$$

where $C_2$ is a constant that is independent of $m$ and $t$. From the above estimate, we deduce

$$\begin{aligned}
\varphi_{txx}^m &\text{ is limited in } L^\infty\big(]0,T[; L^2(0,L)\big),\\
\psi_{xx}^m &\text{ is limited in } L^\infty\big(]0,T[; L^2(0,L)\big),\\
\kappa(\varphi_x^m + \psi^m)_{xx} - A^\theta \varphi_{tx}^m &\text{ is limited in } L^\infty\big(]0,T[; L^2(0,L)\big),\\
\varphi_t^m &\text{ is limited in } L^2\big(]0,T[; H_0^{2\theta}(0,L)\big),\\
\varphi_{tt}^m &\text{ is limited in } L^2\big(]0,T[; H_0^{2\theta}(0,L)\big).
\end{aligned}$$

Furthermore, there are constants $c_2$ and $c_3$ such that

$$\begin{aligned}
\parallel \varphi_{xx}^m \parallel_2^2 &\le c_2(\parallel \kappa(\varphi_x^m + \psi^m)_x \parallel_2^2,\\
\parallel \kappa\varphi_{xxx}^m - A^\theta \varphi_{tx}^m \parallel_2^2 &\le c_3(\parallel \kappa(\varphi_x^m + \psi^m)_{xx} - A^\theta \varphi_{tx}^m \parallel_2^2 + \parallel \psi_{xx}^m \parallel_2^2),
\end{aligned}$$

so,

$$\begin{aligned}
\varphi_{xx}^m &\text{ is limited in } L^\infty\big(]0,T[; L^2(0,L)\big),\\
\kappa\varphi_{xxx}^m - A^\theta \varphi_{tx}^m &\text{ is limited in } L^\infty\big(]0,T[; L^2(0,L)\big).
\end{aligned}$$

From a priori estimates, we have

$$\begin{aligned}
(\varphi^m, \varphi_t^m) &\text{ is limited in } L^\infty\big(]0,T[; \mathcal{V}\big),\\
\varphi_t^m &\text{ is limited in } L^2\big(]0,T[; H_0^{2\theta}(0,L)\big),\\
\psi^m &\text{ is limited in } L^\infty\big(]0,T[; H^2(0,L) \cap H_0^1(0,L)\big),\\
\varphi_{tt}^m &\text{ is limited in } L^2\big(]0,T[; H_0^{2\theta}(0,L)\big).
\end{aligned}$$

### 3.1.4. Step 3: Passage to limits

By the Banach-Alouglu-Bourbaki corollary, we can extract a subsequence of $(\varphi^m)$ and $(\psi^m)$ that we still denote by $(\varphi^m)$ and $(\psi^m)$ such that

$$\begin{aligned}
(\varphi^m, \varphi_t^m) &\rightharpoonup (\varphi, \varphi_t) \text{ weak star in } L^\infty\big(]0,T[; \mathcal{V}\big),\\
\varphi_t^m &\rightharpoonup \varphi_t \text{ weak in } L^2\big(]0,T[; H^2(0,L) \cap H_0^1(0,L)\big),\\
\psi^m &\rightharpoonup \psi \text{ weak star in } L^\infty\big(]0,T[; H^2(0,L) \cap H_0^1(0,L)\big),\\
\varphi_{tt}^m &\rightharpoonup \varphi_{tt} \text{ weak in } L^2\big(]0,T[; H^2(0,L) \cap H_0^1(0,L)\big).
\end{aligned}$$

From the Du-Bois-Raymond lemma, it follows that

$$\begin{aligned}
\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x - \mu\varphi_{txx} = 0, &\text{ a.e. in } ]0,L[ \times ]0,T[,\\
-\rho_2\varphi_{ttx} - b\psi_{xx} + \kappa(\varphi_x + \psi) = 0, &\text{ a.e. in } ]0,L[ \times ]0,T[.
\end{aligned}$$

### 3.1.5. Step 4: Continuous dependence and uniqueness

Let $\{\varphi,\varphi_t,\psi\}$ and $\{\widetilde{\varphi},\widetilde{\varphi_t},\widetilde{\psi}\}$ be strong system solutions of (1)-(4) corresponding to the initial conditions $\{\varphi_0,\varphi_1,\psi_0\}$ and $\{\widetilde{\varphi}_0,\widetilde{\varphi}_1,\widetilde{\psi}_0\}$, respectively. Under these conditions, the elements of the set $\{y,y_t,z\}=\{\varphi-\widetilde{\varphi},\varphi_t-\widetilde{\varphi_t},\psi-\widetilde{\psi}\}$ satisfy the following equations

$$\rho_1 y_{tt}-\kappa(y_x+z)_x+A^\theta y_t=0, \tag{16}$$
$$-\rho_2 y_{ttx}-bz_{xx}+\kappa(y_x+z)=0, \tag{17}$$

with the initial conditions $\{y(x,0),y_t(x,0),z(x,0)\}=\{\varphi_0-\widetilde{\varphi}_0,\varphi_1-\widetilde{\varphi}_1,\psi_0-\widetilde{\psi}_0\}$, where

$$\begin{aligned}(y,y_t)&\in L^\infty\big(]0,T[\,;\mathcal{V}\big),\\ z&\in L^\infty\big(]0,T[\,;H^2(0,L)\cap H_0^1(0,L)\big).\end{aligned}$$

Multiplying Equations (16), (17), by $y_t$, $z_t$, respectively, and integrating on $(0,L)$, we obtain

$$\frac{d}{dt}\left\{\|\, y_t\,\|_2^2+\|\, y_x+z\,\|_2^2+\|\, z_x\,\|_2^2+\|\, y_{tx}\,\|_2^2+\|\,\kappa(y_x+z)_x+\mu y_{txx}\,\|_2^2\right\}\le 0. \tag{18}$$

Integrating (18) on $(0,t)$, we get

$$\begin{aligned}&\|y_t\,\|_2^2+\|\, y_x+z\,\|_2^2+\|\, z_x\,\|_2^2+\|\, y_{tx}\,\|_2^2+\|\,\kappa(y_x+z)_x+\mu y_{txx}\,\|_2^2\\ &\quad\le\|\, y_1\,\|_2^2+\|\, y_{0,x}+z_0\,\|_2^2+\|\, z_{0,x}\,\|_2^2+\|\, y_{1,x}\,\|_2^2+\|\,\kappa(y_{0,x}+z_0)_x+\mu y_{1,xx}\,\|_2^2\,.\end{aligned} \tag{19}$$

Inequality (19) directly leads to the continued dependence on the initial data for a strong solution. Furthermore, we have the uniqueness of the strong solution to the problem (1)-(4).

## 3.2. Strong and weak solutions

**Theorem 3.2.1.** *If $(\varphi_0,\varphi_1,\psi_0)\in\mathcal{V}\times H^2(0,L)\cap H_0^1(0,L)$, then the system (1)-(4) is well-posed for the strong solution. Moreover,*

$$\begin{aligned}(\varphi,\varphi_t)\;&\in\;L^\infty\big(]0,T[;\mathcal{V}\big),\\ \psi\;&\in\;L^\infty\big(]0,T[;H^2(0,L)\cap H_0^1(0,L)\big),\\ \varphi_t\;&\in\;L^2\big(]0,T[;H_0^{2\theta}(0,L)\big),\\ \varphi_{tt}\;&\in\;L^2\big(]0,T[;H_0^{2\theta}(0,L)\big).\end{aligned}$$

*Proof.* The proof is a consequence of the Faedo-Galerkin method. □

**Theorem 3.2.2.** *If $(\varphi_0,\varphi_1,\psi_0)\in\mathcal{H}\times H_0^1(0,L)$, then the system (1)-(4) is well-posed for the weak solution such that*

$$(\varphi,\varphi_t)\in L^\infty\big(]0,T[\,;\,\mathcal{H}\big),\quad \psi\in L^\infty\big(]0,T[\,;H_0^1(0,L)\big),\quad \varphi_{tt}\in L^2\big(]0,T[\,;H_0^1(0,L)\big).$$

*Proof.* The existence of a weak solution will be proved by approximating a sequence of strong solutions found in Theorem 3.2.1.

**Existence**:

Given $(\varphi_0,\varphi_1,\psi_0)\in\mathcal{H}\times H_0^1(0,L)$, there are sequences $(\varphi_0^m,\varphi_1^m)$ and $(\psi_0^m)$ into $\mathcal{V}$ and $H^2(0,L)\cap H_0^1(0,L)$, respectively, such that

$$(\varphi_0^m,\varphi_1^m)\to(\varphi_0,\varphi_1)\text{ strong in }\mathcal{H}, \tag{20}$$
$$\psi_0^m\to\psi_0\text{ strong in }H_0^1(0,L). \tag{21}$$

For each $m$, Theorem 3.2.1 guarantees the existence of a unique strong solution $\{\varphi^m,\varphi_t^m,\psi^m\}$, such that

$$\begin{aligned}(\varphi^m,\varphi_t^m)\;&\in\;L^\infty\big(]0,\;T[\,;\mathcal{V}\big),\\ \psi^m\;&\in\;L^\infty\big(]0,T[\,;H^2(0,L)\cap H_0^1(0,L)\big),\end{aligned}$$

where

$$\rho_1\varphi_{tt}^m-\kappa(\varphi_x^m+\psi^m)-\mu\varphi_{txx}^m=0,\ \text{ a.e. in }]0,L[\times[0,T], \tag{22}$$

$$-\rho_2\varphi^m_{ttx} - b\psi^m_{xx} + \kappa(\varphi^m_x + \psi^m) = 0, \text{ a.e. in } ]0,L[\times[0,T], \tag{23}$$

and

$$(\varphi^m(0), \varphi^m_t(0), \psi^m(0)) = (\varphi^m_0, \varphi^m_1, \psi^m_0).$$

Multiplying Equations (22) and (23) by $\varphi^m_t$ and $\psi^m_t$, respectively, and then integrating over $(0,L)$, we get

$$\frac{d}{dt}\left\{\frac{\rho_1}{2}\parallel\varphi^m_t\parallel^2_2 + \frac{\kappa}{2}\parallel\varphi^m_x+\psi^m\parallel^2_2 + \frac{b}{2}\parallel\psi^m_x\parallel^2_2\right\} + \parallel A^{\theta/2}\varphi^m_t\parallel^2_2 + \rho_2(\varphi^m_{tt},\psi^m_{tx}) = 0.$$

Using again the following identity

$$(\varphi^m_{tt},\psi^m_{tx}) = \frac{1}{\kappa}\left(\varphi^m_{tt}, \kappa(\varphi^m_x+\psi^m)_{tx} - A^\theta\varphi^m_{ttxx}\right) + \frac{1}{2}\frac{d}{dt}\parallel\varphi^m_{tx}\parallel^2_2 + \frac{1}{\kappa}\parallel A^{\theta/2}\varphi^m_{tt}\parallel^2_2,$$

and multiplying (22) by $\kappa(\varphi^m_x+\psi^m)_{tx} + \mu\varphi^m_{ttxx}$, we get

$$\frac{d}{dt}\left\{\frac{\rho_1}{2}\parallel\varphi^m_t\parallel^2_2 + \frac{\kappa}{2}\parallel\varphi^m_x+\psi^m\parallel^2_2 + \frac{b}{2}\parallel\psi^m_x\parallel^2_2 + \frac{\rho_2}{2}\parallel\varphi^m_{tx}\parallel^2_2 + \frac{\rho_2}{2\kappa\rho_1}\parallel\kappa(\varphi^m_x+\psi^m)_x - A^\theta\varphi^m_t\parallel^2_2\right\}$$
$$+\parallel A^{\theta/2}\varphi^m_t\parallel^2_2 + \frac{\rho_2}{\kappa}\parallel A^{\theta/2}\varphi^m_{tt}\parallel^2_2 = 0.$$

Integrating from $0$ to $t$ and using the convergences (20) and (21), we get

$$\frac{\rho_1}{2}\parallel\varphi^m_t\parallel^2_2 + \frac{\kappa}{2}\parallel\varphi^m_x+\psi^m\parallel^2_2 + \frac{b}{2}\parallel\psi^m_x\parallel^2_2 + \frac{\rho_2}{2}\parallel\varphi^m_{tx}\parallel^2_2 + \frac{\rho_2}{2\kappa\rho_1}\parallel\kappa(\varphi^m_x+\psi^m)_x - A^\theta\varphi^m_t\parallel^2_2$$
$$+\int_0^t\left\{\parallel A^{\theta/2}\varphi^m_t\parallel^2_2 + \frac{\rho_2}{\kappa}\parallel A^{\theta/2}\varphi^m_{tt}\parallel^2_2\right\}dt \le C_1,$$

where $C_1$ is a constant that is independent of $m$ and $t$. Thus,

$$\begin{aligned}
\varphi^m_x + \psi^m &\text{ is limited in } L^\infty\big(]0,T[;L^2(0,L)\big),\\
\varphi^m_t &\text{ is limited in } L^\infty\big(]0,T[\,;H^1_*(0,L)\big),\\
\varphi^m_t &\text{ is limited in } L^2\big(]0,T[\,;H^\theta_0(0,L)\big),\\
\psi^m &\text{ is limited in } L^\infty\big(]0,T[\,;H^1_0(0,L)\big),\\
\kappa(\varphi^m_x+\psi^m)_x - A^\theta\varphi^m_t &\text{ is limited in } L^\infty\big(]0,T[\,;L^2(0,L)\big),\\
\varphi^m_{tt} &\text{ is limited in } L^2\big(]0,T[\,;H^\theta_0(0,L)\big).
\end{aligned}$$

Furthermore, there is a constant $c_1$, such that

$$\parallel\varphi^m_x\parallel^2_2 \le c_1(\parallel\varphi^m_x+\psi^m\parallel^2_2 + \parallel\psi^m\parallel^2_2), \text{ and } \kappa\varphi^m_{xx} - A^\theta\varphi^m_t = \kappa(\varphi^m_x+\psi^m)_x + \mu\varphi^m_{txx} - \kappa\psi^m_x,$$

so

$$\varphi^m \text{ is limited in } L^\infty\big(]0,T[;H^1_*(0,L)\big), \text{ and } \kappa\varphi^m_{xx} - A^\theta\varphi^m_t \text{is limited in } L^\infty\big(]0,T[;L^2(0,L)\big).$$

By the Banach-Alouglu-Bourbaki corollary, we can extract a subsequence of $(\varphi^m)$ and $(\psi^m)$ that we will still denote by $(\varphi^m)$ and $(\psi^m)$ such that

$$\begin{aligned}
\varphi^m \rightharpoonup \varphi &\text{ weak star in } L^\infty\big(]0,T[\,;H^1_0(0,L)\big),\\
\varphi^m_t \rightharpoonup \varphi_t &\text{ weak star in } L^\infty\big(]0,T[\,;H^1_0(0,L)\big),\\
\psi^m \rightharpoonup \psi &\text{ weak star in } L^\infty\big(]0,T[\,;H^1_0(0,L)\big),\\
\varphi^m_t \rightharpoonup \varphi_t &\text{ weak in } L^2\big(]0,T[\,;H^\theta_0(0,L)\big),\\
\varphi^m_{tt} \rightharpoonup \varphi_{tt} &\text{ weak in } L^2\big(]0,T[\,;H^\theta_0(0,L)\big),\\
\kappa\varphi^m_{xx} - A^\theta\varphi^m_t \rightharpoonup \kappa\varphi_{xx} - A^\theta\varphi_t &\text{ weak star in } L^\infty\big(]0,\ T[\,;L^2(0,L)\big),
\end{aligned}$$

resulting that

$$\rho_1\varphi_{tt} - \kappa(\varphi_x+\psi)_x + \mu\varphi_{txx} = 0, \text{ a.e. in } ]0,L[\ \times\ ]0,T[,$$
$$\rho_2(\varphi_{tt},w_x) + b(\psi_x,w_x) + \kappa(\varphi_x+\psi,w) = 0,\ \forall\, w\in H^1_0(0,L) \text{ in } \mathcal{D}'(0,T).$$

**Uniqueness**:

Let $(\varphi,\varphi_t,\psi)$ and $(\widetilde{\varphi},\widetilde{\varphi_t},\widetilde{\psi})$ be weak solutions of the system (1)-(4) corresponding to the initial data $(\varphi_0,\varphi_1,\psi_0)$. Under these conditions, we have that $(y,y_t,z)=(\varphi-\widetilde{\varphi},\varphi_t-\widetilde{\varphi_t},\psi-\widetilde{\psi})$ satisfy the following equations

$$
\begin{aligned}
&\rho_1 y_{tt}-\kappa(y_x+z)_x+A^\theta y_t=0, \quad \text{a.e. in } ]0,L[\ \times\ ]0,T[, \\
&\rho_2(y_{tt},w_x)+b(z_x,w_x)+\kappa(y_x+z,w)=0,\ \forall\, w\in H_0^1(0,L),
\end{aligned}
\tag{24}
$$

with the initial data $\{y(x,0),y_t(x,0),z(x,0)\}=\{0,0,0\}$, where

$$(y,y_t)\in L^\infty\big(]0,T[;\mathcal{H}\big), \text{ and } z\in L^\infty\big(]0,T[;H_0^1(0,L)\big).$$

Note that as we do not have the space of $z_t$, the duality $(y_{tt},z_{tx})$ does not make sense. Consequently, we define the following functionals

$$\sigma^1(t)=\begin{cases} -\displaystyle\int_t^s y(r)dr, & 0<t<s,\\ 0, & s\le t<T,\end{cases} \quad \text{and} \quad \sigma^2(t)=\begin{cases} -\displaystyle\int_t^s z(r)dr, & 0<t<s.\\ 0, & s\le t<T.\end{cases}$$

So, $\sigma^2\in L^\infty\big(]0,T[;H_0^1(0,L)\big)$ and thus the duality $(y_{tt},\sigma^2_x)$. Also, we have

$$\rho_1\int_0^s(y_{tt},\sigma^1)dt+\kappa\int_0^s(y_x+z,\sigma^1_x)dt+\int_0^s(A^{\theta/2}y_t,A^{\theta/2}\sigma^1)dt = 0, \tag{25}$$

$$\rho_2\int_0^s(y_{tt},\sigma^2_x)dt+b\int_0^s(z_x,\sigma^2_x)dt+\kappa\int_0^s(y_x+z,\sigma^2)dt = 0. \tag{26}$$

Adding Equations (25) and (26), we get

$$\rho_1\int_0^s(y_{tt},\sigma^1)dt+\kappa\int_0^s(y_x+z,\sigma^1_x+\sigma^2)dt+b\int_0^s(z_x,\sigma^2_x)dt+\rho_2\int_0^s(y_{tt},\sigma^2_x)dt+\int_0^s(A^{\theta/2}y_t,A^{\theta/2}\sigma^1)dt=0. \tag{27}$$

Noticing that $\sigma^i(t)=\sigma^i_1(t)-\sigma^i_1(s)$ for $i=1,2$, and $\sigma^1_t(t)=y(t)$, $\sigma^2_t(t)=z(t)$ for $t\in(0,s)$, we obtain

$$\int_0^s(y_{tt},\sigma^1)dt=-\frac{1}{2}\int_0^s\frac{d}{dt}\parallel y_t\parallel_2^2 dt=-\frac{1}{2}\parallel y(s)\parallel_2^2, \tag{28}$$

$$\int_0^s(y_x+z,\sigma^1_x+\sigma^2)dt=-\frac{1}{2}\parallel\sigma^1_x(0)+\sigma^2(0)\parallel_2^2, \tag{29}$$

$$\int_0^s(z_x,\sigma^2_x)dt=\int_0^s\frac{d}{dt}\parallel\sigma^2_x\parallel_2^2 dt=-\frac{1}{2}\parallel\sigma^2_x(0)\parallel_2^2, \tag{30}$$

$$\int_0^s(y_{tt},\sigma^2_x)dt=-\frac{\rho_1}{2\kappa}\parallel y_t(s)\parallel_2^2-\frac{1}{2}\parallel y_x(s)\parallel_2^2-\frac{\mu}{\kappa}\int_0^s\parallel y_t\parallel_2^2 dt, \tag{31}$$

and

$$\int_0^s(A^{\theta/2}y_t,A^{\theta/2}\sigma^1)dt=-\int_0^s\parallel A^{\theta/2}y(t)\parallel_2^2 dt. \tag{32}$$

Using (28), (29), (30), (31) and (32) in (27), we get

$$
\begin{aligned}
\frac{\rho_1}{2}\parallel y(s)\parallel_2^2&+\frac{\kappa}{2}\parallel\sigma^1_x(0)+\sigma^2(0)\parallel_2^2+\frac{b}{2}\parallel\sigma^2_x(0)\parallel_2^2+\frac{\rho_1\rho_2}{2\kappa}\parallel y_t(s)\parallel_2^2+\frac{\rho_2}{2}\parallel y_x(s)\parallel_2^2\\
&+\frac{\mu\rho_2}{\kappa}\int_0^s\parallel y_t\parallel_2^2 dt+\int_0^s\parallel A^{\theta/2}y(t)\parallel_2^2 dt=0.
\end{aligned}
$$

So, $y=0$ and $y_t=0$ lead to $\varphi=\widetilde{\varphi}$ and $\varphi_t=\widetilde{\varphi}_t$. Also, from (24), we have $z_x=0$. Since $z\in H_0^1(0,L)$, we have $z=0$. Therefore, $\psi=\widetilde{\psi}$ and thus we conclude that the solution is unique.

**Continuous dependence**:

The continuous dependence on the initial data for weak solutions follows directly from (19) which gives us the continuous dependence for strong solutions by density arguments. □

## 4. Asymptotic behavior

In this section, we show that the energy of the system decays exponentially regardless of the relationship

$$\frac{\rho_1}{k} = \frac{\rho_2}{b}.$$

### 4.1. Technical lemmas

We define for all $t \geq 0$, the energy functional of the system (1)-(4) by

$$E(t) \overset{\text{def}}{=} \frac{1}{2}\int_0^L \left[\rho_1|\varphi_t|^2 + \rho_2|\varphi_{tx}|^2 + b|\psi_x|^2 + \kappa|\varphi_x + \psi|^2 + \frac{\rho_2}{\kappa\rho_1}\left|\kappa(\varphi_x + \psi)_x - A^\theta\varphi_t\right|^2\right] dx.$$

The following two technical lemmas can be proved by straightforward calculations.

**Lemma 4.1.1.** *The functional* $E(t)$ *of the system* (1)-(4) *satisfies the inequality*

$$\frac{d}{dt}E(t) \leq -\frac{1}{c}\int_0^L \left(|\varphi_t|^2 + \frac{\rho_2}{\kappa}|\varphi_{tt}|^2\right) dx.$$

**Lemma 4.1.2.** *Let* $(\varphi, \psi_t, \psi)$ *be a system solution of* (1)-(4)*. The functional*

$$\mathcal{F}(t) = \int_0^L \left[\rho_1\varphi_t\varphi + \frac{1}{2}|A^{\theta/2}\varphi|^2 + \frac{\rho_2}{2\kappa}|A^{\theta/2}\varphi_t|^2 + \varphi_{tx}\varphi_t\right] dx$$

*satisfies*

$$\frac{d}{dt}\mathcal{F}(t) = -\kappa\int_0^L |\varphi_x + \psi|^2\, dx - b\int_0^L |\psi_x|^2 dx - \frac{\rho_2}{\kappa\rho_1}\int_0^L \left|\kappa(\varphi_x + \psi)_x - A^\theta\varphi_t\right|^2 dx + \rho_1\int_0^L |\varphi_t|^2 dx + \rho_2\int_0^L |\varphi_{tx}|^2 dx.$$

Now, we define the Lyapunov functional $\mathcal{L}(t) = \mathcal{F}(t) + NE(t)$, where $N$ is a positive constant that will be fixed later.

**Lemma 4.1.3.** *There are positive constants* $c_1$ *and* $c_2$ *such that* $c_1E(t) \leq \mathcal{L}(t) \leq c_2E(t)$.

*Proof.* From the definition of $\mathcal{L}(t)$ we have

$$|\mathcal{L}(t) - NE(t)| = |\mathcal{F}(t)|.$$

Using Young's inequality and (8) we obtain a constant $\eta > 0$ such that

$$|\mathcal{F}(t)| \leq \eta\int_0^L \left[\rho_1|\varphi_t|^2 + \rho_2\varphi_{tx} + b|\psi_x|^2 + \kappa|\varphi_x + \psi|^2\right] dx \leq \eta E(t).$$

Thus, $(N - \eta)E(t) \leq \mathcal{L}(t) \leq (N + \eta)E(t)$. We conclude the proof by taking $N > \eta$. □

**Lemma 4.1.4.** *The following inequality is true*

$$\frac{d}{dt}\mathcal{L}(t) \leq -\beta E(t). \tag{33}$$

*Proof.* It follows from Lemma 4.1.1 and Lemma 4.1.2 that

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) &= \frac{d}{dt}\mathcal{F}(t) + N\frac{d}{dt}E(t) \\
&\leq -\kappa\int_0^L |\varphi_x + \psi|^2\, dx - b\int_0^L |\psi_x|^2 dx - \frac{\rho_2}{\kappa\rho_1}\int_0^L \left|\kappa(\varphi_x + \psi)_x - A^\theta\varphi_t\right|^2 dx \\
&\quad + \rho_1\int_0^L |\varphi_t|^2 dx + \rho_2\int_0^L |\varphi_{tx}|^2 dx - \frac{N}{c}\int_0^L \varphi_{tx}^2 dx - N\frac{\rho_2}{c\kappa}\int_0^L \varphi_{ttx}^2 dx \\
&\leq -\kappa\int_0^L |\varphi_x + \psi|^2\, dx - b\int_0^L |\psi_x|^2 dx - \frac{\rho_2}{\kappa\rho_1}\int_0^L \left|\kappa(\varphi_x + \psi)_x - A^\theta\varphi_t\right|^2 dx \\
&\quad - \rho_1\int_0^L |\varphi_t|^2 dx + 2\rho_1\int_0^L |\varphi_t|^2 dx + \rho_2\int_0^L |\varphi_{tx}|^2 dx - \frac{N}{c}\int_0^L |\varphi_{tx}|^2 dx.
\end{aligned}$$

Using Poincarè inequality we obtain

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) &\leq -\kappa\int_0^L |\varphi_x + \psi|^2\, dx - b\int_0^L |\psi_x|^2 dx - \frac{\rho_2}{\kappa\rho_1}\int_0^L \left|\kappa(\varphi_x + \psi)_x - A^\theta\varphi_t\right|^2 dx \\
&\quad - \rho_1\int_0^L |\varphi_t|^2 dx - \left[\frac{N}{c} - (2\rho_1 c + \rho_2)\right]\int_0^L |\varphi_{tx}|^2 dx.
\end{aligned}$$

Taking $N > 2c\,(\rho_1 c + \rho_2)$, we assure that (33) holds. □

## 4.2. Exponential decay

**Theorem 4.2.1.** *There are two positive constants $M$ and $\omega$ that do not depend on the initial conditions and do not depend on any relationship between their coefficients such that*

$$E(t) \leq ME(0)e^{-\omega t}\,;\ \ \forall\ \ t > 0.$$

*Proof.* Using Lemma 4.1.3 and Lemma 4.1.4 we obtain

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{\beta}{c_2}\mathcal{L}(t),$$

which implies that

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\frac{\beta}{c_2}t}.$$

Using again Lemma 4.1.3 we obtain

$$E(t) \leq ME(0)e^{-\omega t},\ \ \forall\, t > 0,\ \text{ where }\ \omega = \frac{\beta}{c_2}\ \text{ and }\ M = \frac{c_2}{c_1}.$$

□

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