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Research Article

New extensions of the Hermite-Hadamard inequality

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Abstract

Some new results related to generalized Hermite-Hadamard-type inequalities are established. For obtaining new inequalities, various approaches are utilized, including boundedness, convexity, and concavity. Considering special values of the parameters, it is demonstrated how the obtained inequalities reduce to the known ones.

Keywords: weighted integral operators; Hermite-Hadamard integral inequality; convex functions.

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1. Introduction

In recent decades, there has been a lot of research done on the concept of convexity and integral inequalities. The classic book [14] on inequalities — by Hardy, Littlewood, and Pólya — serves as the basis for research on this topic. After the appearance of the book [14], researchers obtained various variations of the inequalities concerning convexity and boundedness properties; these variations include Hermite-Hadamard-type inequalities [1–13, 15–17], Simpson-type inequalities [18–32] and Ostrowski-type inequalities [33–40, 43–45].

Definition 1.1. *A function $\phi : I \to \mathbb{R}$ is said to be convex if*

$$\phi\big(\lambda x + (1-\lambda)y\big) \leq \lambda\phi(x) + (1-\lambda)\phi(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$, where $I := [a,b]$. If the above inequality is reversed, then the function ϕ is called a concave function.

The idea of a convex function has been extended in various directions; these extensions include m-convex functions, n-convex functions, r-convex functions, h-convex functions, (h,m)-convex functions, s-convex functions (for example, see [29, 41, 42, 46–51], where a fairly complete panorama of the current development of these concepts is presented).

In this paper, some new results related to generalized Hermite-Hadamard-type inequalities are established. For obtaining new inequalities, various approaches are utilized, including boundedness, convexity, and concavity. Considering special values of the parameters, it is demonstrated how the obtained inequalities reduce to the known ones.

2. Main results

Theorem 2.1. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $\gamma'' \in L[a,b]$ where $a, b \in I^\circ$ with $a < b$. If $|\gamma''|$ is convex function on $[a,b]$, then*

$$\begin{aligned}|P(a,b,n,x,w,\gamma)| \quad &\leq \quad \frac{\left(\frac{x-a}{n+1}\right)^3}{2(b-a)} \int_0^1 |w(\delta)| \left[\frac{\delta}{n+1}|\gamma''(a)| + \frac{n+1-\delta}{n+1}|\gamma''(x)|\right] d\delta \\ &\quad + \frac{\left(\frac{b-x}{n+1}\right)^3}{2(b-a)} \int_0^1 |w(\delta)| \left[\frac{\delta}{n+1}|\gamma''(b)| + \frac{n+1-\delta}{n+1}|\gamma''(x)|\right] d\delta, \qquad (1)\end{aligned}$$

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where $P(a,b,n,x,w,\gamma)$ *is given by*

$$\begin{aligned}
P(a,b,n,x,w,\gamma) &= \frac{1}{2(b-a)}\Bigg\{\left[w(0)\gamma'(x)-w(1)\gamma'\left(\frac{a+nx}{n+1}\right)\right]\left(\frac{x-a}{n+1}\right)^2-\left[w'(1)\gamma\left(\frac{a+nx}{n+1}\right)-w'(0)\gamma(x)\right]\left(\frac{x-a}{n+1}\right)\\
&+\int_x^{\frac{a+nx}{n+1}} w''\left[\frac{(n+1)(u-x)}{a-x}\right]\gamma(u)\,du+\left[w(1)\gamma'\left(\frac{b+nx}{n+1}\right)-w(0)\gamma'(x)\right]\left(\frac{b-x}{n+1}\right)^2\\
&-\left[w'(1)\gamma\left(\frac{b+nx}{n+1}\right)-w'(0)\gamma(x)\right]\left(\frac{b-x}{n+1}\right)+\int_x^{\frac{b+nx}{n+1}} w''\left[\frac{(n+1)(u-x)}{b-x}\right]\gamma(u)\,du\Bigg\}. \qquad (2)
\end{aligned}$$

Proof. Integrating by parts gives

$$\begin{aligned}
&\frac{\left(\frac{x-a}{n+1}\right)^3}{2(b-a)}\int_0^1 w(\delta)\gamma''\left(\frac{\delta}{n+1}a+\frac{n+1-\delta}{n+1}x\right)d\delta+\frac{\left(\frac{b-x}{n+1}\right)^3}{2(b-a)}\int_0^1 w(\delta)\gamma''\left(\frac{\delta}{n+1}b+\frac{n+1-\delta}{n+1}x\right)d\delta\\
=&\frac{\left(\frac{x-a}{n+1}\right)^3}{2(b-a)}\left[\frac{w(\delta)\gamma'\left(\frac{\delta}{n+1}a+\frac{n+1-\delta}{n+1}x\right)}{\frac{a-x}{n+1}}\Bigg|_0^1-\int_0^1\frac{w'(\delta)\gamma'\left(\frac{\delta}{n+1}a+\frac{n+1-\delta}{n+1}x\right)}{\frac{a-x}{n+1}}d\delta\right]\\
&+\frac{\left(\frac{b-x}{n+1}\right)^3}{2(b-a)}\left[\frac{w(\delta)\gamma'\left(\frac{\delta}{n+1}b+\frac{n+1-\delta}{n+1}x\right)}{\frac{b-x}{n+1}}\Bigg|_0^1-\int_0^1\frac{w'(\delta)\gamma'\left(\frac{\delta}{n+1}b+\frac{n+1-\delta}{n+1}x\right)}{\frac{b-x}{n+1}}d\delta\right]\\
=&\frac{\left(\frac{x-a}{n+1}\right)^3}{2(b-a)}\left[\frac{w(1)\gamma'\left(\frac{a+nx}{n+1}\right)-w(0)\gamma'(x)}{\frac{a-x}{n+1}}-\int_0^1\frac{w'(\delta)\gamma'\left(\frac{\delta}{n+1}a+\frac{n+1-\delta}{n+1}x\right)}{\frac{a-x}{n+1}}d\delta\right]\\
&+\frac{\left(\frac{b-x}{n+1}\right)^3}{2(b-a)}\left[\frac{w(1)\gamma'\left(\frac{b+nx}{n+1}\right)-w(0)\gamma'(x)}{\frac{b-x}{n+1}}-\int_0^1\frac{w'(\delta)\gamma'\left(\frac{\delta}{n+1}b+\frac{n+1-\delta}{n+1}x\right)}{\frac{b-x}{n+1}}d\delta\right]\\
=&\frac{\left(\frac{x-a}{n+1}\right)^3}{2(b-a)}\left\{\frac{w(1)\gamma'\left(\frac{a+nx}{n+1}\right)-w(0)\gamma'(x)}{\frac{a-x}{n+1}}-\left[\frac{w'(\delta)\gamma\left(\frac{\delta}{n+1}a+\frac{n+1-\delta}{n+1}x\right)}{\left(\frac{a-x}{n+1}\right)^2}\Bigg|_0^1-\int_0^1\frac{w''(\delta)\gamma\left(\frac{\delta}{n+1}a+\frac{n+1-\delta}{n+1}x\right)}{\left(\frac{a-x}{n+1}\right)^2}d\delta\right]\right\}\\
&+\frac{\left(\frac{b-x}{n+1}\right)^3}{2(b-a)}\left\{\frac{w(1)\gamma'\left(\frac{b+nx}{n+1}\right)-w(0)\gamma'(x)}{\frac{b-x}{n+1}}-\left[\frac{w'(\delta)\gamma\left(\frac{\delta}{n+1}b+\frac{n+1-\delta}{n+1}x\right)}{\left(\frac{b-x}{n+1}\right)^2}\Bigg|_0^1-\int_0^1\frac{w''(\delta)\gamma\left(\frac{\delta}{n+1}b+\frac{n+1-\delta}{n+1}x\right)}{\left(\frac{b-x}{n+1}\right)^2}d\delta\right]\right\}\\
=&\frac{\left(\frac{x-a}{n+1}\right)^3}{2(b-a)}\left\{\frac{w(1)\gamma'\left(\frac{a+nx}{n+1}\right)-w(0)\gamma'(x)}{\frac{a-x}{n+1}}-\left[\frac{w'(1)\gamma\left(\frac{a+nx}{n+1}\right)-w'(0)\gamma(x)}{\left(\frac{a-x}{n+1}\right)^2}-\int_x^{\frac{a+nx}{n+1}}\frac{w''\left[\frac{(n+1)(u-x)}{a-x}\right]\gamma(u)\,du}{\left(\frac{a-x}{n+1}\right)^2\left(\frac{a-x}{n+1}\right)}\right]\right\}\\
&+\frac{\left(\frac{b-x}{n+1}\right)^3}{2(b-a)}\left\{\frac{w(1)\gamma'\left(\frac{b+nx}{n+1}\right)-w(0)\gamma'(x)}{\frac{b-x}{n+1}}-\left[\frac{w'(1)\gamma\left(\frac{b+nx}{n+1}\right)-w'(0)\gamma(x)}{\left(\frac{b-x}{n+1}\right)^2}-\int_x^{\frac{b+nx}{n+1}}\frac{w''\left[\frac{(n+1)(u-x)}{b-x}\right]\gamma(u)\,du}{\left(\frac{b-x}{n+1}\right)^2\left(\frac{b-x}{n+1}\right)}\right]\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
P(a,b,n,x,w,\gamma) =&\frac{1}{2(b-a)}\{\left[w(0)\gamma'(x)-w(1)\gamma'\left(\frac{a+nx}{n+1}\right)\right]\left(\frac{x-a}{n+1}\right)^2-\left[w'(1)\gamma\left(\frac{a+nx}{n+1}\right)-w'(0)\gamma(x)\right]\left(\frac{x-a}{n+1}\right)\\
&+\int_x^{\frac{a+nx}{n+1}} w''\left[\frac{(n+1)(u-x)}{a-x}\right]\gamma(u)\,du+\left[w(1)\gamma'\left(\frac{b+nx}{n+1}\right)-w(0)\gamma'(x)\right]\left(\frac{b-x}{n+1}\right)^2\\
&-\left[w'(1)\gamma\left(\frac{b+nx}{n+1}\right)-w'(0)\gamma(x)\right]\left(\frac{b-x}{n+1}\right)+\int_x^{\frac{b+nx}{n+1}} w''\left[\frac{(n+1)(u-x)}{b-x}\right]\gamma(u)\,du\}\\
=&\frac{\left(\frac{x-a}{n+1}\right)^3}{2(b-a)}\int_0^1 w(\delta)\gamma''\left(\frac{\delta}{n+1}a+\frac{n+1-\delta}{n+1}x\right)d\delta+\frac{\left(\frac{b-x}{n+1}\right)^3}{2(b-a)}\int_0^1 w(\delta)\gamma''\left(\frac{\delta}{n+1}b+\frac{n+1-\delta}{n+1}x\right)d\delta. \qquad (3)
\end{aligned}$$

Using the well-known triangular inequality of real numbers and the definition of convexity in (3), we get

$$\begin{aligned}
|P(a,b,n,x,w,\gamma)| &\leq \frac{\left(\frac{x-a}{n+1}\right)^3}{2(b-a)}\int_0^1 |w(\delta)|\left|\gamma''\left(\frac{\delta}{n+1}a+\frac{n+1-\delta}{n+1}x\right)\right|d\delta + \frac{\left(\frac{b-x}{n+1}\right)^3}{2(b-a)}\int_0^1 |w(\delta)|\left|\gamma''\left(\frac{\delta}{n+1}b+\frac{n+1-\delta}{n+1}x\right)\right|d\delta\\
&\leq \frac{\left(\frac{x-a}{n+1}\right)^3}{2(b-a)}\int_0^1 |w(\delta)|\left[\frac{\delta}{n+1}|\gamma''(a)|+\frac{n+1-\delta}{n+1}|\gamma''(x)|\right]d\delta\\
&\quad+\frac{\left(\frac{b-x}{n+1}\right)^3}{2(b-a)}\int_0^1 |w(\delta)|\left[\frac{\delta}{n+1}|\gamma''(b)|+\frac{n+1-\delta}{n+1}|\gamma''(x)|\right]d\delta.
\end{aligned}$$

□

Remark 2.1. The choices $n = 0$ and $w(\delta) = 1 - \delta^2$ in Theorem 2.1 yield Theorem 4 of [19].

Corollary 2.1. *With the assumptions made in the statement of Theorem 2.1, the following inequality holds:*

$$\begin{aligned}
|P(a,b,n,w,\gamma)| &\leq \frac{(b-a)^2}{16(n+1)^3}\int_0^1 |w(\delta)|\left[2\frac{n+1-\delta}{n+1}\left|\gamma''\left(\frac{a+b}{2}\right)\right|+\frac{\delta}{n+1}\Big(|\gamma''(a)|+|\gamma''(b)|\Big)\right]d\delta\\
&\leq \frac{(b-a)^2}{16(n+1)^3}\int_0^1 |w(\delta)|\Big[|\gamma''(a)|+|\gamma''(b)|\Big]d\delta. \qquad (4)
\end{aligned}$$

Proof. Taking $x = \frac{a+b}{2}$ in (1) we obtain the first inequality of (4) and then making use of the convexity of the function $|\gamma''|$, we arrive at the other inequality of (4). □

Theorem 2.2. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $\gamma'' \in L[a, b]$ and $|\gamma''|^q$ is convex on $[a, b]$ for some fixed $q > 1$ such that $p^{-1} + q^{-1} = 1$, then*

$$\begin{aligned}
|P(a,b,n,x,w,\gamma)| &\leq \left(\int_0^1 w^p(\delta)d\delta\right)^{\frac{1}{p}}\left[\frac{\left(\frac{x-a}{n+1}\right)^3}{2(b-a)}\left(\frac{|\gamma''(a)|^q}{2(n+1)}+\left(1-\frac{1}{2(n+1)}\right)|\gamma''(x)|^q\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\frac{\left(\frac{b-x}{n+1}\right)^3}{2(b-a)}\left(\frac{|\gamma''(b)|^q}{2(n+1)}+\left(1-\frac{1}{2(n+1)}\right)|\gamma''(x)|^q\right)^{\frac{1}{q}}\right]. \qquad (5)
\end{aligned}$$

Proof. Considering Equation (3) and then making use of the Hölder's inequality, we get

$$\begin{aligned}
|P(a,b,n,x,w,\gamma)| &\leq \frac{\left(\frac{x-a}{n+1}\right)^3}{2(b-a)}\left(\int_0^1 w^p(\delta)d\delta\right)^{\frac{1}{p}}\left(\int_0^1\left|\gamma''\left(\frac{\delta}{n+1}a+\frac{n+1-\delta}{n+1}x\right)\right|^q d\delta\right)^{\frac{1}{q}}\\
&\quad+\frac{\left(\frac{b-x}{n+1}\right)^3}{2(b-a)}\left(\int_0^1 w^p(\delta)d\delta\right)^{\frac{1}{p}}\left(\int_0^1\left|\gamma''\left(\frac{\delta}{n+1}b+\frac{n+1-\delta}{n+1}x\right)\right|^q d\delta\right)^{\frac{1}{q}}. \qquad (6)
\end{aligned}$$

Using the convexity of $|\gamma''|^q$, we get

$$\begin{aligned}
\int_0^1 |\gamma''\left(\frac{\delta}{n+1}a+\frac{n+1-\delta}{n+1}x\right)|^q d\delta &\leq \int_0^1\left(\frac{\delta}{n+1}|\gamma''(a)|^q+\frac{n+1-\delta}{n+1}|\gamma''(x)|^q\right)d\delta\\
&= \frac{|\gamma''(a)|^q}{2(n+1)}+\left(1-\frac{1}{2(n+1)}\right)|\gamma''(x)|^q.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_0^1 |\gamma''\left(\frac{\delta}{n+1}b+\frac{n+1-\delta}{n+1}x\right)|^q d\delta &\leq \int_0^1\left(\frac{\delta}{n+1}|\gamma''(b)|^q+\frac{n+1-\delta}{n+1}|\gamma''(x)|^q\right)d\delta\\
&= \frac{|\gamma''(b)|^q}{2(n+1)}+\left(1-\frac{1}{2(n+1)}\right)|\gamma''(x)|^q. \qquad (7)
\end{aligned}$$

The desired inequality follows from (6) and (7). □

Corollary 2.2. *Under the same conditions as given in the statement of Theorem 2.2, it holds that*

$$\begin{aligned}
&|P(a,b,n,w,\gamma)| \\
\leq\ & \frac{\left(\int_0^1 w^p(\delta)d\delta\right)^{\frac{1}{p}}}{\frac{16(n+1)^3}{(b-a)^2}} \left[\left(\frac{|\gamma''(a)|^q}{2(n+1)} + \left(1-\frac{1}{2(n+1)}\right)\left|\gamma''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\frac{|\gamma''(b)|^q}{2(n+1)} + \left(1-\frac{1}{2(n+1)}\right)\left|\gamma''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{\left(\int_0^1 w^p(\delta)d\delta\right)^{\frac{1}{p}}}{\frac{16(n+1)^3}{(b-a)^2}} \left[\left(1-\frac{1}{2(n+1)}\right)^{\frac{1}{q}} + (2(n+1))^{-\frac{1}{q}}\right]\Big(|\gamma''(a)| + |\gamma''(b)|\Big). \qquad (8)
\end{aligned}$$

Proof. The first inequality of (8) is established by putting $x = \frac{a+b}{2}$ in (5). The second inequality of (8) is derived by making use of the convexity of the function $|\gamma''|^q$. □

Theorem 2.3. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $\gamma'' \in L[a,b]$ and $|\gamma''|^q$ is concave function on $[a,b]$ for some fixed $q > 1$ and $p = \frac{q}{q-1}$, then*

$$|P(a,b,n,x,w,\gamma)| \leq \frac{\left(\int_0^1 w^p(\delta)d\delta\right)^{\frac{1}{p}}}{2(b-a)} \left[\left(\frac{x-a}{n+1}\right)^3 \left|\gamma''\left(\frac{x}{2}+\frac{a}{2(n+1)}\right)\right| + \left(\frac{b-x}{n+1}\right)^3 \left|\gamma''\left(\frac{x}{2}+\frac{b}{2(n+1)}\right)\right|\right]. \qquad (9)$$

Proof. Considering the Equation (3) and then making use of the Hölder's inequality for $q > 1$ and $p = \frac{q}{q-1}$, we get

$$\begin{aligned}
|P(a,b,n,x,w,\gamma)| \leq\ & \frac{\left(\frac{x-a}{n+1}\right)^3}{2(b-a)} \left(\int_0^1 w^p(\delta)d\delta\right)^{\frac{1}{p}} \left(\int_0^1 \left|\gamma''\left(\frac{\delta}{n+1}a + \frac{n+1-\delta}{n+1}x\right)\right|^q d\delta\right)^{\frac{1}{q}} \\
& + \frac{\left(\frac{b-x}{n+1}\right)^3}{2(b-a)} \left(\int_0^1 w^p(\delta)d\delta\right)^{\frac{1}{p}} \left(\int_0^1 \left|\gamma''\left(\frac{\delta}{n+1}b + \frac{n+1-\delta}{n+1}x\right)\right|^q d\delta\right)^{\frac{1}{q}}.
\end{aligned}$$

Since, the function $|\gamma''|^q$ is concave, by making use of Jensen's integral inequality, we get

$$\begin{aligned}
\int_0^1 \left|\gamma''\left(\frac{\delta}{n+1}a + \frac{n+1-\delta}{n+1}x\right)\right|^q d\delta \leq\ & \left|\gamma''\left[\int_0^1 \left(\frac{\delta}{n+1}a + \frac{n+1-\delta}{n+1}x\right)d\delta\right]\right|^q \\
=\ & \left|\gamma''\left(\frac{x}{2}+\frac{a}{2(n+1)}\right)\right|^q.
\end{aligned}$$

Similarly,

$$\int_0^1 \left|\gamma''\left(\frac{\delta}{n+1}b + \frac{n+1-\delta}{n+1}x\right)\right|^q d\delta \leq \left|\gamma''\left(\frac{x}{2}+\frac{b}{2(n+1)}\right)\right|^q.$$

Consequently, we get

$$|P(a,b,n,x,w,\gamma)| \leq \frac{\left(\int_0^1 w^p(\delta)d\delta\right)^{\frac{1}{p}}}{2(b-a)} \left[\left(\frac{x-a}{n+1}\right)^3 \left|\gamma''\left(\frac{x}{2}+\frac{a}{2(n+1)}\right)\right| + \left(\frac{b-x}{n+1}\right)^3 \left|\gamma''\left(\frac{x}{2}+\frac{b}{2(n+1)}\right)\right|\right].$$

□

Corollary 2.3. *Under the assumptions of Theorem 2.3, it holds that*

$$\begin{aligned}
|P(a,b,n,w,\gamma)| \leq\ & \frac{\left(\int_0^1 w^p(\delta)d\delta\right)^{\frac{1}{p}}}{2(b-a)} \left(\frac{b-a}{2(n+1)}\right)^3 \left[\left|\gamma''\left(\frac{(a+b)(n+1)+2a}{4(n+1)}\right)\right| + \left|\gamma''\left(\frac{(a+b)(n+1)+2b}{4(n+1)}\right)\right|\right] \\
\leq\ & \frac{\left(\int_0^1 w^p(\delta)d\delta\right)^{\frac{1}{p}}}{8(b-a)} \left(\frac{b-a}{n+1}\right)^3 \left|\gamma''\left(\frac{(a+b)(n+2)}{4(n+1)}\right)\right|. \qquad (10)
\end{aligned}$$

Proof. The first inequality of (10) is deduced by substituting $x = \frac{a+b}{2}$ in (9). The second inequality of (10) is established by making use of the concavity of the function $|\gamma''|^q$. □

Theorem 2.4. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $\gamma'' \in L[a,b]$ and $|\gamma''|^q$ is convex function on $[a,b]$ for some fixed $q > 1$, then*

$$\begin{aligned} |P(a,b,n,x,w,\gamma)| \leq\ & \frac{\left(\int_0^1 w(\delta)d\delta\right)^{1-\frac{1}{q}}}{2(b-a)}\left[\left(\frac{x-a}{n+1}\right)^3\left(\frac{|\gamma''(a)|^q}{n+1}\int_0^1 \delta w(\delta)d\delta + \frac{|\gamma''(x)|^q}{n+1}\int_0^1 (n+1-\delta)w(\delta)d\delta\right)^{\frac{1}{q}}\right. \\ & \left. + \left(\frac{b-x}{n+1}\right)^3\left(\frac{|\gamma''(b)|^q}{n+1}\int_0^1 \delta w(\delta)d\delta + \frac{|\gamma''(x)|^q}{n+1}\int_0^1 (n+1-\delta)w(\delta)d\delta\right)^{\frac{1}{q}}\right]. \end{aligned} \tag{11}$$

Proof. Considering Equation (3) again and then making use of the power-mean inequality for $q > 1$, we get

$$\begin{aligned} |P(a,b,n,x,w,\gamma)| \leq\ & \frac{\left(\frac{x-a}{n+1}\right)^3}{2(b-a)}\int_0^1 w(\delta)\left|\gamma''\left(\frac{\delta}{n+1}a + \frac{n+1-\delta}{n+1}x\right)\right|d\delta \\ & + \frac{\left(\frac{b-x}{n+1}\right)^3}{2(b-a)}\int_0^1 w(\delta)\left|\gamma''\left(\frac{\delta}{n+1}b + \frac{n+1-\delta}{n+1}x\right)\right|d\delta \\ \leq\ & \frac{\left(\frac{x-a}{n+1}\right)^3}{2(b-a)}\left(\int_0^1 w(\delta)d\delta\right)^{1-\frac{1}{q}}\left(\int_0^1 w(\delta)\left|\gamma''\left(\frac{\delta}{n+1}a + \frac{n+1-\delta}{n+1}x\right)\right|^q d\delta\right)^{\frac{1}{q}} \\ & + \frac{\left(\frac{b-x}{n+1}\right)^3}{2(b-a)}\left(\int_0^1 w(\delta)d\delta\right)^{1-\frac{1}{q}}\left(\int_0^1 w(\delta)\left|\gamma''\left(\frac{\delta}{n+1}b + \frac{n+1-\delta}{n+1}x\right)\right|^q d\delta\right)^{\frac{1}{q}}. \end{aligned} \tag{12}$$

By making use of the fact that the function $|\gamma''|^q$ is convex, we get

$$\begin{aligned} \int_0^1 w(\delta)\left|\gamma''\left(\frac{\delta}{n+1}a + \frac{n+1-\delta}{n+1}x\right)\right|^q d\delta \leq\ & \int_0^1 w(\delta)\left[\frac{\delta}{n+1}|\gamma''(a)|^q + \frac{n+1-\delta}{n+1}|\gamma''(x)|^q\right]d\delta \\ =\ & \frac{|\gamma''(a)|^q}{n+1}\int_0^1 \delta w(\delta)d\delta + \frac{|\gamma''(x)|^q}{n+1}\int_0^1 (n+1-\delta)\,w(\delta)d\delta. \end{aligned} \tag{13}$$

Similarly,

$$\int_0^1 w(\delta)\left|\gamma''\left(\frac{\delta}{n+1}b + \frac{n+1-\delta}{n+1}x\right)\right|^q d\delta \leq \frac{|\gamma''(b)|^q}{n+1}\int_0^1 \delta w(\delta)d\delta + \frac{|\gamma''(x)|^q}{n+1}\int_0^1 (n+1-\delta)\,w(\delta)d\delta. \tag{14}$$

Using (13) and (14) in (12), we get the inequality (11). □

Corollary 2.4. *Under the assumptions of Theorem 2.4, the following inequality holds:*

$$\begin{aligned} |P(a,b,n,w,\gamma)| \leq\ & \frac{\left(\int_0^1 w(\delta)d\delta\right)^{1-\frac{1}{q}}}{16}\frac{(b-a)^2}{(n+1)^3}\left[\left(\frac{|\gamma''(a)|^q}{n+1}\int_0^1 \delta w(\delta)d\delta + \frac{|\gamma''\left(\frac{a+b}{2}\right)|^q}{n+1}\int_0^1 (n+1-\delta)w(\delta)d\delta\right)^{\frac{1}{q}}\right. \\ & \left. + \left(\frac{|\gamma''(b)|^q}{n+1}\int_0^1 \delta w(\delta)d\delta + \frac{|\gamma''\left(\frac{a+b}{2}\right)|^q}{n+1}\int_0^1 (n+1-\delta)w(\delta)d\delta\right)^{\frac{1}{q}}\right] \\ \leq\ & \frac{\left(\int_0^1 w(\delta)d\delta\right)^{1-\frac{1}{q}}}{16}\frac{(b-a)^2}{(n+1)^{3+\frac{1}{q}}}\left[\left(\int_0^1 \delta w(\delta)d\delta\right)^{\frac{1}{q}} + \left(2^{q-1}\int_0^1 (n+1-\delta)w(\delta)d\delta\right)^{\frac{1}{q}}\right]\left[|\gamma''(a)| + |\gamma''(b)|\right]. \end{aligned} \tag{15}$$

Proof. The first inequality of (15) is established by putting $x = \frac{a+b}{2}$ in (11). The second inequality of (15) is deduced by making use of the convexity of the function $|\gamma''|^q$. □

Theorem 2.5. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $\gamma'' \in L[a,b]$ and if there exist constants $\zeta < \eta$ with $\eta > 0$ such that $-\infty < \zeta \leq \gamma'' \leq \eta < +\infty$, then*

$$\left|P\left(a,b,n,\frac{a+b}{2},w,\gamma\right)\right| \leq \frac{(b-a)^2}{8(n+1)^3}\eta\int_0^1 |w(\delta)|d\delta. \tag{16}$$

Proof. Considering Equation (3) and setting $x = \frac{a+b}{2}$, while taking the absolutely value, we obtain

$$\left| P\left(a, b, n, \frac{a+b}{2}, w, \gamma\right) \right| \leq \frac{(b-a)^2}{8(n+1)^3} \left| \int_0^1 w(\delta)\gamma''\left(\frac{\delta}{n+1}b + \frac{n+1-\delta}{n+1}x\right) d\delta \right|.$$

Adding and subtracting $\frac{\eta+\zeta}{2}$ to γ'', we obtain

$$\left| P\left(a, b, n, \frac{a+b}{2}, w, \gamma\right) \right| \leq \frac{(b-a)^2}{8(n+1)^3} \left| \int_0^1 w(\delta)\left(\gamma''\left(\frac{\delta}{n+1}b + \frac{n+1-\delta}{n+1}x\right) - \frac{\eta+\zeta}{2}\right) d\delta + \frac{\eta+\zeta}{2}\int_0^1 w(\delta)d\delta \right|.$$

Since

$$\left| \gamma''\left(\frac{\delta}{n+1}b + \frac{n+1-\delta}{n+1}x\right) - \frac{\eta+\zeta}{2} \right| \leq \frac{\eta-\zeta}{2},$$

which holds because of the boundedness of γ'', therefore we obtain

$$\left| P\left(a, b, n, \frac{a+b}{2}, w, \gamma\right) \right| \leq \frac{(b-a)^2}{8(n+1)^3} \left(\frac{\eta-\zeta}{2}\int_0^1 |w(\delta)|d\delta + \frac{\eta+\zeta}{2}\int_0^1 |w(\delta)|d\delta \right),$$

which (after simplification) gives the desired inequality. □

3. Applications

Proposition 3.1. *If* $0 < a < b$, $n \in \mathbb{Z}$, $|n(n-1)| \geq 3$ *and* $q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$|A(a^n, b^n) - L_n^n(a, b)| \leq \frac{\left(\int_0^1 w^p(\delta)d\delta\right)^{\frac{1}{p}}}{\frac{8(n+1)^3}{(b-a)^2}} \left[\left(1 - \frac{1}{2(n+1)}\right)^{\frac{1}{q}} + (2(n+1))^{-\frac{1}{q}} \right] A\left(a^{n-2}, b^{n-2}\right),$$

$$|A(a^n, b^n) - L_n^n(a, b)| \leq \frac{\left(\int_0^1 w(\delta)d\delta\right)^{1-\frac{1}{q}}}{\frac{8(n+1)^{3+\frac{1}{q}}}{(b-a)^2}} \left[\left(\int_0^1 \delta w(\delta)d\delta\right)^{\frac{1}{q}} + \left(2^{q-1}\int_0^1 (n+1-\delta)w(\delta)d\delta\right)^{\frac{1}{q}} \right] A\left(a^{n-2}, b^{n-2}\right).$$

Proof. Let $\gamma(x) = x^n$ with $x > 0$, $|n(n-1)| \geq 3$ and $n \in \mathbb{Z}$. Since the function γ satisfies all the conditions of Theorem 2.2, the desired inequalities follow from Corollaries 2.2 and 2.4. □

Proposition 3.2. *Let* $0 < a < b$ *and* $q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$|A(a^{-1}, b^{-1}) - L^{-1}(a, b)| \leq \frac{\left(\int_0^1 w^p(\delta)d\delta\right)^{\frac{1}{p}}}{\frac{4(n+1)^3}{(b-a)^2}} \left[\left(1 - \frac{1}{2(n+1)}\right)^{\frac{1}{q}} + (2(n+1))^{-\frac{1}{q}} \right] A\left(a^{-3}, b^{-3}\right),$$

$$|A(a^{-1}, b^{-1}) - L^{-1}(a, b)| \leq \frac{\left(\int_0^1 w(\delta)d\delta\right)^{1-\frac{1}{q}}}{\frac{4(n+1)^{3+\frac{1}{q}}}{(b-a)^2}} \left[\left(\int_0^1 \delta w(\delta)d\delta\right)^{\frac{1}{q}} + \left(2^{q-1}\int_0^1 (n+1-\delta)w(\delta)d\delta\right)^{\frac{1}{q}} \right] A\left(a^{-3}, b^{-3}\right).$$

Proof. Consider the function $\gamma(x) = \frac{1}{x}$ with $x > 0$. Since the function γ satisfies all the conditions of Theorem 2.2, the desired inequalities follow from Corollaries 2.2 and 2.4. □

4. Conclusion

Various inequalities have been obtained using the integral inequality given in Theorem 2.1. Variations concerning the convexity and boundedness of the function involved are investigated, which resulted in the new inequalities obtained in this paper. As noted in Remark 2.1, the settings $n = 0$ and $w(\delta) = 1 - \delta^2$ of specific parameters reduce to a known integral inequality derived by the authors of [19]. Applications of the obtained results to special means have also been given (in Section 3), which further verify the obtained results.

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