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Research Article

Some new lower bounds on the algebraic connectivity of graphs

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Abstract

The second-smallest eigenvalue of the Laplacian matrix of a graph G is called the algebraic connectivity of G, which is one of the most-studied parameters in spectral graph theory and network science. In this paper, we obtain some new lower bounds of the algebraic connectivity by rank-one perturbation matrix and compare them with known results.

Keywords: algebraic connectivity; spanning trees; first Zagreb index.

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1. Introduction

Let G be a simple graph with the vertex set $V(G)$ and edge set $E(G)$. The maximum degree, the minimum degree, the diameter and the edge connectivity of a graph G are denoted by Δ, δ, d and κ', respectively. Other undefined notations and terminologies can be found in [6].

The Laplacian matrix of G, denoted by $L(G)$, is given by $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of the vertex degrees of G and $A(G)$ is the adjacency matrix. The eigenvalues of the matrix $L(G)$ are known as Laplacian eigenvalues of G. Let $\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n$ be the Laplacian eigenvalues of a graph G of order n arranged in a non-increasing way. The Laplacian eigenvalue μ_{n-1} is often referred to as the algebraic connectivity of G (see [9]) and is denoted as $a(G)$. The algebraic connectivity is sometimes referred to as the Fiedler value, see for example [5]. It is a well-known fact that 0 is always a Laplacian eigenvalue with e, the all-ones vector, as an associated eigenvector; whose multiplicity corresponds to the number of connected components of G.

Algebraic connectivity has been a hot topic in spectral graph theory and network science in recent decades; hundreds of research articles have been published on this topic, for example, see [1–4, 8, 15, 16, 20, 25, 28]. However, the results on lower bounds of the algebraic connectivity are still relatively little known. For a graph G with n vertices and m edges, Fiedler [9] obtained the following lower bounds on the algebraic connectivity $a(G)$:

$$a(G) \geq 2\delta - n + 2, \tag{1}$$

$$a(G) \geq 2\kappa'\left(1 - \cos\frac{\pi}{n}\right), \tag{2}$$

$$a(G) \geq 2\kappa'\left(\cos\frac{\pi}{n} - \cos\frac{2\pi}{n}\right) - 2\cos\frac{\pi}{n}\left(1 - \cos\frac{\pi}{n}\right)\Delta. \tag{3}$$

In 1991, Mohar [23] found the following lower bound involving diameter on $a(G)$:

$$a(G) \geq \frac{4}{nd}. \tag{4}$$

In 2007, Lu et al. [19] presented another lower bound on $a(G)$ which involves diameter:

$$a(G) \geq \frac{2n}{2 + n(n-1)d - 2md}. \tag{5}$$

In this paper, we obtain some new lower bounds of the algebraic connectivity in terms of the maximum degree, the first Zagreb index, and the numbers of vertices, edges, and spanning trees. The obtained results enrich the existing research on the lower bounds of the algebraic connectivity of graphs.

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The rest of the paper is organized as follows. In Section 2, we give some preliminaries that are needed in the subsequent sections. The main results of this paper are given in Section 3. Section 4 gives three examples, which illustrate that the obtained bounds are better than some of the existing bounds in certain cases.

2. Preliminaries

The sum of resistance distances between all pairs of vertices of a graph G is known as the Kirchhoff index [14] of G and is denoted by $Kf(G)$. The following useful formula for calculating the Kirchhoff index of G is due to Gutman and Mohar [12]:

$$Kf(G) = \sum_{i=1}^{n-1} \frac{1}{\mu_i}.$$

This index is used often to measure how well connected a given network is [10, 14]. Lipman et al. [18] introduced the biharmonic distance $d_B^2(u,v)$ between two vertices u and v of a graph G:

$$d_B^2(u,v) = L_{uu}^{2+} + L_{vv}^{2+} - 2L_{uv}^{2+},$$

where L_{uv}^{2+} is the (u,v)-entry of the matrix obtained from the square of the Moore-Penrose inverse of $L(G)$. In [18], it was noted that the biharmonic distance has some advantages over the resistance distance and geodesic distance in computer graphics, geometric processing, shape analysis, etc. Based on the concept of the biharmonic distance, Yi et al. [27] introduced the biharmonic index; this index for a graph G is defined as

$$Bh(G) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} d_B^2(u,v) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i^2(G)}.$$

Lemma 2.1 (see [7]). *Let M be an arbitrary $n \times n$ matrix with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $\mathbf{u}_k$ be an eigenvector of M associated with the eigenvalue λ_k, and let $\boldsymbol{w}$ be an arbitrary n-dimensional vector. Then the matrix $M + \mathbf{u}_k \boldsymbol{w}^T$ has the following eigenvalues: $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k + \mathbf{u}_k^T \boldsymbol{w}, \lambda_{k+1}, \ldots, \lambda_n$.*

Lemma 2.2 (see [21]). *Let M be a symmetric positive definite matrix. Then the minimum eigenvalue of M satisfies*

$$\lambda_{\min}(M) \geq \left(\frac{n-1}{tr(M)}\right)^{n-1} \det(M),$$

where $tr(M)$ is trace of the matrix M.

Lemma 2.3 (see [11]). *Let G be a connected graph with n vertices and at least one edge. Then*

$$\mu_1(G) \geq \Delta + 1.$$

with equality if and only if $\Delta = n-1$.

Lemma 2.4 (see [9]). *If G is not a complete graph K_n with n vertices, then $\mu_{n-1}(G) \leq \delta$.*

Lemma 2.5 (see [17]). *Let M be an $n \times n$ complex matrix and let*

$$l_{k+1} = |\det M| \left(\frac{n-1}{||M||_F^2 - l_k^2}\right)^{\frac{n-1}{2}}, \quad k = 1, 2, 3, \ldots,$$

with

$$l_1 = |\det M| \left(\frac{n-1}{||M||_F^2}\right)^{\frac{n-1}{2}}.$$

Then the smallest singular value of M satisfies

$$\sigma_{\min} \geq \lim_{k \to \infty} l_k.$$

Lemma 2.6 (see [29]). *Let G be a bipartite graph with at least one edge. Then*

$$\mu_1 \geq \frac{Z_1}{m},$$

where Z_1 is called the first Zagreb index [13], which is equal to the sum of squares of the degrees of the vertices in G.

Lemma 2.7 (see [26]). *Let M be an $n \times n$ real symmetric positive definite matrix. Then the minimum eigenvalue of M satisfies*

$$\begin{aligned}
\lambda_{\min}(M) &\geq \frac{2tr(M^{-1})}{[tr(M^{-1})]^2 + tr(M^{-2})}, \\
\lambda_{\min}(M) &\geq \frac{1}{tr(M^{-1})}\left\{\frac{3}{2} - \frac{1}{2}\frac{tr(M^{-2})}{[tr(M^{-1})]^2}\right\}, \\
\lambda_{\min}(M) &\geq \frac{1}{tr(M^{-1})} \cdot n\left\{1 + \sqrt{(n-1)\left[n \cdot \frac{tr(M^{-2})}{[tr(M^{-1})]^2} - 1\right]}\right\}^{-1}.
\end{aligned}$$

Lemma 2.8 (see [22]). *Let G be a connected graph with n vertices, and let $\overline{G}$ be the complement of G. Then*

$$\mu_i(\overline{G}) = n - \mu_{n-i}(G) \quad \textit{for} \quad i = 1, 2, \ldots, n-1.$$

Lemma 2.9 (see [24, 30]). *Let G be a graph with the vertex-degree sequence $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$. Then*

$$\mu_1(G) \leq \delta + \frac{1}{2} + \sqrt{\sum_{i=1}^{n} d_i(d_i - \delta) + \left(\delta - \frac{1}{2}\right)^2},$$

and equality holds if and only if G is a regular graph with at least one bipartite component, or G is the disjoint union of a star graph and (possibly) K_2's.

3. Main results

Theorem 3.1. *Let G be a connected graph with n vertices, m edges, τ spanning trees and maximum degree Δ. Then*

$$a(G) \geq \begin{cases} \left(\dfrac{n-1}{2m+\Delta+1}\right)^{n-1}(\Delta+1)n\tau, & \textit{if } \ \Delta \leq \dfrac{2m}{n-2} - 1; \\[2ex] \left(\dfrac{n-2}{2m}\right)^{n-2} n\tau, & \textit{if } \ \Delta > \dfrac{2m}{n-2} - 1. \end{cases} \tag{6}$$

Proof. We consider the matrix $M = L(G) + \xi J$, where $\mu_{n-1} \leq n\xi \leq \mu_1$ and J is the all-ones matrix. Then

$$m_{ij} = \begin{cases} d_i + \xi, & \text{if } i = j; \\ -1 + \xi, & \text{if } v_iv_j \in E(G); \\ \xi, & \text{if } v_iv_j \notin E(G). \end{cases}$$

Thus $tr(M) = \sum_{i=1}^{n}(d_i + \xi) = 2m + n\xi$. Note that $M = L(G) + \xi J = L(G) + \xi ee^T$. By Lemma 2.1, the eigenvalues of M are $\mu_1, \mu_2, \ldots, \mu_{n-1}, n\xi$. Since $\mu_{n-1} \leq n\xi \leq \mu_1$, the matrix M is a symmetric positive definite matrix. By the matrix-tree theorem (see [23]), we have $\prod_{i=1}^{n-1}\mu_i = n\tau$. By Lemma 2.2, we have

$$\begin{aligned}
a(G) &= \lambda_{\min}(M) \\
&\geq \left(\frac{n-1}{tr(M)}\right)^{n-1} \det(M) \\
&= \left(\frac{n-1}{2m+n\xi}\right)^{n-1} n\xi \cdot n\tau \\
&= \left(\frac{n-1}{2m+n\xi}\right)^{n-1} n^2\xi\tau.
\end{aligned}$$

Let $f(x) = \left(\frac{n-1}{2m+nx}\right)^{n-1} n^2x\tau$. A simple calculation yields

$$f'(x) = n^2\tau\left(\frac{n-1}{2m+n\xi}\right)^{n-1}\left(\frac{-n^2x + 2nx + 2m}{2m+nx}\right).$$

Thus, $f(x)$ is increasing for $x \leq \frac{2m}{n(n-2)}$ and decreasing for $x \geq \frac{2m}{n(n-2)}$.

If $G \neq K_n$, then by Lemmas 2.3 and 2.4, we consider

$$I = \left[\frac{\delta}{n}, \frac{\Delta+1}{n}\right] \quad \left(\text{notice that } \frac{\delta}{n} < \frac{2m}{n(n-2)}\right).$$

It is immediate that $f(\frac{\Delta+1}{n}) \geq f(x)$ for $\frac{\delta}{n} \leq x \leq \frac{\Delta+1}{n} \leq \frac{2m}{n(n-2)}$, or $f(\frac{2m}{n(n-2)}) \geq f(x)$ for $\frac{\delta}{n} \leq x \leq \frac{2m}{n(n-2)} \leq \frac{\Delta+1}{n}$. Thus we have

$$a(G) \geq \begin{cases} \left(\dfrac{n-1}{2m+\Delta+1}\right)^{n-1} (\Delta+1)n\tau, & \text{if } \Delta \leq \dfrac{2m}{n-2} - 1; \\ \left(\dfrac{n-2}{2m}\right)^{n-2} n\tau, & \text{if } \Delta > \dfrac{2m}{n-2} - 1. \end{cases}$$

If $G = K_n$, then

$$a(G) = n > \left(\frac{n-1}{n}\right)^{n-1} n$$

for $\Delta = n-1 < \frac{2m}{n-2} - 1$.

By combining the above arguments, we arrive at the required result. □

Theorem 3.2. *Let G be a connected graph with n vertices, m edges, τ spanning trees and the first Zagreb index Z_1. Then*

$$a(G) \geq \lim_{k\to\infty} l_k,$$

where

$$l_{k+1} = (n^2\xi\tau)\left(\frac{n-1}{2m+Z_1+n^2\xi^2-l_k^2}\right)^{\frac{n-1}{2}}, \quad k = 1,2,3,\ldots,$$

with

$$l_1 = (n^2\xi\tau)\left(\frac{n-1}{2m+Z_1+n^2\xi^2}\right)^{\frac{n-1}{2}},$$

and $\mu_{n-1} \leq n\xi \leq \mu_1$.

Proof. Let $M = L(G) + \xi J$. By the proof of Theorem 3.1, we have

$$\begin{aligned} ||M||_F^2 &= 2m + Z_1 + n^2\xi^2, \\ \det M &= n^2\xi\tau. \end{aligned}$$

By Lemma 2.5, we have

$$a(G) = \sigma_{\min} \geq \lim_{k\to\infty} l_k,$$

where

$$l_{k+1} = (n^2\xi\tau)\left(\frac{n-1}{2m+Z_1+n^2\xi^2-l_k^2}\right)^{\frac{n-1}{2}}, \quad k = 1,2,3,\ldots,$$

with

$$l_1 = (n^2\xi\tau)\left(\frac{n-1}{2m+Z_1+n^2\xi^2}\right)^{\frac{n-1}{2}}.$$

□

Corollary 3.1. *Let G be a connected r-regular graph with n vertices and τ spanning trees. Then*

$$a(G) \geq (nr\tau)\left(\frac{n-1}{nr+nr^2+r^2-n^2r^2\tau^2\left(\frac{n-1}{nr+nr^2+r^2}\right)^{n-1}}\right)^{\frac{n-1}{2}}. \tag{7}$$

Proof. By taking $\xi = \frac{r}{n}$ and $k = 2$ in Theorem 3.2, we get the required result. □

Corollary 3.2. *Let G be a connected bipartite graph with n vertices, m edges, τ spanning trees and the first Zagreb index Z_1. Then*

$$a(G) \geq \left(\frac{nZ_1\tau}{m}\right)\left(\frac{(n-1)m^2}{2m^3+m^2Z_1+Z_1^2-n^2Z_1^2\tau^2\left(\frac{(n-1)m^2}{2m^3+m^2Z_1+Z_1^2}\right)^{n-1}}\right)^{\frac{n-1}{2}}. \tag{8}$$

Proof. We get the result by taking $\xi = \frac{Z_1}{nm}$ and $k = 2$ in Theorem 3.2 and using Lemma 2.6. □

The next result also trivially holds.

Corollary 3.3. *Let T be a tree with n vertices and the first Zagreb index Z_1. Then*

$$a(T) \geq \left(\frac{nZ_1}{n-1}\right)\left(\frac{(n-1)^3}{2(n-1)^3+(n-1)^2Z_1+Z_1^2-n^2Z_1^2\left(\frac{(n-1)^3}{2(n-1)^3+(n-1)^2Z_1+Z_1^2}\right)^{n-1}}\right)^{\frac{n-1}{2}}.$$

Theorem 3.3. *Let G be a connected graph with n vertices, m edges and τ spanning trees. If there is α such that $a(G) \leq \alpha$, then*

$$a(G) \geq \left(\frac{n-1}{2m+\alpha}\right)^{n-1} n\alpha\tau.$$

Proof. We consider the matrix $M = L(G) + \boldsymbol{e}\boldsymbol{w}^T$, where $\boldsymbol{w}^T = \frac{\alpha}{2}(1,1,0,\dots,0)^T$. Then $tr(M) = \alpha + \sum_{i=1}^n d_i = 2m + \alpha$. By Lemma 2.1, the eigenvalues of M are $\mu_1, \mu_2, \dots, \mu_{n-1}, \alpha$. Since $a(G) \leq \alpha$, the matrix M is a symmetric-positive definite matrix. By the matrix-tree theorem (see [23]), we have $\prod_{i=1}^{n-1} \mu_i = n\tau$. By Lemma 2.2, we have

$$a(G) = \lambda_{\min}(M) \geq \left(\frac{n-1}{tr(M)}\right)^{n-1}\det(M) = \left(\frac{n-1}{2m+\alpha}\right)^{n-1}\alpha\cdot n\tau = \left(\frac{n-1}{2m+\alpha}\right)^{n-1} n\alpha\tau.$$

□

Corollary 3.4. *Let $G \neq K_n$ be a connected graph with n vertices, m edges and τ spanning trees. Then*

$$a(G) \geq \left(\frac{n-1}{2m+\delta}\right)^{n-1} n\delta\tau. \tag{9}$$

Proof. From Lemma 2.4 and Theorem 3.3, the result follows. □

Theorem 3.4. *Let G be a connected graph with n vertices. If there is α such that $a(G) \leq \alpha$, then*

$$\begin{aligned}
a(G) &\geq \frac{2\left(\frac{1}{n}Kf(G)+\frac{1}{\alpha}\right)}{\left(\frac{1}{n}Kf(G)+\frac{1}{\alpha}\right)^2+\frac{1}{n}Bh(G)+\frac{1}{\alpha^2}},\\
a(G) &\geq \frac{1}{\frac{1}{n}Kf(G)+\frac{1}{\alpha}}\left\{\frac{3}{2}-\frac{1}{2}\frac{\frac{1}{n}Bh(G)+\frac{1}{\alpha^2}}{\left(\frac{1}{n}Kf(G)+\frac{1}{\alpha}\right)^2}\right\},\\
a(G) &\geq \frac{n^2}{Kf(G)+\frac{n}{\alpha}}\left\{1+\sqrt{(n-1)\left[\frac{Bh(G)+\frac{n}{\alpha^2}}{\left(\frac{1}{n}Kf(G)+\frac{1}{\alpha}\right)^2}-1\right]}\right\}^{-1}.
\end{aligned}$$

Proof. Let $M = L(G) + \boldsymbol{e}\boldsymbol{w}^T$, where $\boldsymbol{w}^T = \frac{\alpha}{2}(1,1,0,\dots,0)^T$. By the proof of Theorem 3.3, we have

$$\begin{aligned}
tr(M^{-1}) &= \frac{1}{n}Kf(G)+\frac{1}{\alpha},\\
tr(M^{-2}) &= \frac{1}{n}Bh(G)+\frac{1}{\alpha^2}.
\end{aligned}$$

By using Lemma 2.7, we get the required result. □

Corollary 3.5. *Let $G \neq K_n$ be a connected graph with n vertices and the minimum degree δ. Then*

$$\begin{aligned}
a(G) &\geq \frac{2\left(\frac{1}{n}Kf(G)+\frac{1}{\delta}\right)}{\left(\frac{1}{n}Kf(G)+\frac{1}{\delta}\right)^2+\frac{1}{n}Bh(G)+\frac{1}{\delta^2}},\\
a(G) &\geq \frac{1}{\frac{1}{n}Kf(G)+\frac{1}{\delta}}\left\{\frac{3}{2}-\frac{1}{2}\frac{\frac{1}{n}Bh(G)+\frac{1}{\delta^2}}{\left(\frac{1}{n}Kf(G)+\frac{1}{\delta}\right)^2}\right\},\\
a(G) &\geq \frac{n^2}{Kf(G)+\frac{n}{\delta}}\left\{1+\sqrt{(n-1)\left[\frac{Bh(G)+\frac{n}{\delta^2}}{\left(\frac{1}{n}Kf(G)+\frac{1}{\delta}\right)^2}-1\right]}\right\}^{-1}.
\end{aligned}$$

Proof. From Lemma 2.4 and Theorem 3.4, the result follows. □

Problem 3.1. *Find the graphs satisfying the equality in Theorems 3.1, 3.2, 3.3 and 3.4.*

Theorem 3.5. *Let G be a connected graph with n vertices, m edges, maximum degree Δ and the first Zagreb index Z_1. Then*

$$a(G) \geq \Delta + \frac{1}{2} - \frac{1}{2}\sqrt{4Z_1 + (2n-2\Delta-3)^2 + 4(n-1)(n\Delta-2m) - 8m\Delta} \tag{10}$$

with equality if and only if $\overline{G}$ is a regular graph with at least one bipartite component, or $\overline{G}$ is the disjoint union of a star graph and (possibly) K_2's.

Proof. Let $d_1(G) \geq d_2(G) \geq \cdots \geq d_n(G)$ be the vertex-degree sequence of G. Denote by $m(\overline{G})$ the number of edges in $\overline{G}$. Then

$$m(\overline{G}) = \frac{n(n-1)}{2} - m, \quad d_i(\overline{G}) = n - 1 - d_{n-i+1}(G).$$

By Lemmas 2.8 and 2.9, we have

$$\begin{aligned}
a(G) &= n - \mu_1(\overline{G}) \\
&\geq n - \left[\delta(\overline{G}) + \frac{1}{2} + \sqrt{\sum_{i=1}^{n} d_i(\overline{G})(d_i(\overline{G}) - \delta(\overline{G})) + \left(\delta(\overline{G}) - \frac{1}{2}\right)^2}\,\right] \\
&= n - (n-1-\Delta) - \frac{1}{2} - \sqrt{\sum_{i=1}^{n}(n-1-d_{n-i+1}(G))(\Delta - d_{n-i+1}(G)) + \left(n - \Delta - \frac{3}{2}\right)^2} \\
&= \Delta + \frac{1}{2} - \sqrt{n(n-1)\Delta - (n-1+\Delta)\sum_{i=1}^{n} d_{n-i+1}(G) + \sum_{i=1}^{n} d_{n-i+1}^2(G) + \left(n - \Delta - \frac{3}{2}\right)^2} \\
&= \Delta + \frac{1}{2} - \sqrt{n(n-1)\Delta - 2m(n-1+\Delta) + Z_1 + \left(n - \Delta - \frac{3}{2}\right)^2} \\
&= \Delta + \frac{1}{2} - \frac{1}{2}\sqrt{4Z_1 + (2n-2\Delta-3)^2 + 4(n-1)(n\Delta-2m) - 8m\Delta}
\end{aligned}$$

with equality if and only if $\overline{G}$ is a regular graph with at least one bipartite component, or $\overline{G}$ is the disjoint union of a star graph and (possibly) K_2's. □

Corollary 3.6. *Let $G \neq K_n$ be a connected r-regular graph with n vertices. Then $a(G) \geq 2r - n + 2$.*

4. Examples

In this section, we give some examples of the bounds on the algebraic connectivity given in (1)–(10). Based on our calculations, we believe that these bounds are incomparable.

Example 4.1. Let $G = K_9 + e$ be the graph obtained by attaching a pendant vertex to exactly one of the vertices of the complete graph K_9. By (6) and (10), we have $a(G) \geq 0.8924$ and $a(G) = 1$. However, by (1)–(5), we get $a(G) \geq -10$, $a(G) \geq 0.0979$, $a(G) \geq -0.5538$, $a(G) \geq 0.2$, $a(G) \geq 0.5882$, respectively. In fact, the exact value is $a(G) = 1$.

Example 4.2. Let G be the Petersen graph. Applying (6), (7) and (9), we have $a(G) \geq 0.5114$, $a(G) \geq 0.3773$ and $a(G) \geq 0.5009$, respectively. However, using (1)–(5), we get $a(G) \geq -2$, $a(G) \geq 0.2937$, $a(G) \geq 0.5729$, $a(G) \geq 0.2$ and $a(G) \geq 0.1639$, respectively. On the other hand, the exact value is $a(G) = 2$.

Example 4.3. Let G be the complete bipartite graph $K_{5,5}$. By (6), (8), (9) and (10), we have $a(G) \geq 1.6765$, $a(G) \geq 1.5404$, $a(G) \geq 1.6430$ and $a(G) \geq 2$, respectively. However, by (1)–(5), we get $a(G) \geq 2$, $a(G) \geq 0.4894$, $a(G) \geq 0.9549$, $a(G) \geq 0.2$ and $a(G) \geq 0.2439$, respectively. Also, note that the exact value is $a(G) = 5$.

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