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Research Article

New bounds for the mean and the variance

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Abstract

In this article, new bounds for the mean and the variance of uniformly distributed discrete random variables are derived. It is shown that the new results, under certain conditions, are better than the bounds of Bhatia and Davies reported in [*Amer. Math. Monthly* **107** (2000) 353–357].

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1. Introduction

Let $x_1, x_2, \ldots, x_n$ be given positive real numbers. Let X be the discrete random variable uniformly distributed on $\{x_1, \ldots, x_n\}$. The *mean* of X is defined as

$$\mu(X) = \frac{x_1 + \ldots + x_n}{n},$$

while the *variance* of X is defined as

$$\sigma^2(X) = \frac{(x_1 - \mu(X))^2 + \ldots + (x_n - \mu(X))^2}{n}.$$

Let M and m be positive real numbers satisfying $0 < m \leq x_i \leq M$ for each $i = 1, 2, \ldots, n$. In the past, several bounds on the variance of X, in terms of M and m, were given. Popoviciu in [3] proved the following inequality

$$\sigma^2(X) \leq \frac{1}{4}(M - m)^2. \tag{1}$$

The complementary Von Szokefalvi Nagy inequality [2] asserts that

$$\sigma^2(X) \geq \frac{(M - m)^2}{2n}. \tag{2}$$

Popoviciu's inequality (1) was improved by Bhatia and Davis in [1]; they proved

$$\sigma^2(X) \leq (M - \mu(X))(\mu(X) - m). \tag{3}$$

A question about the relation between the variances of the variables X and X^2 was given on the 30th International Mathematical Competition for undergraduate students, Vojtěch Jarník, held in Czech Republic, 2022; this question asks to prove the following inequality

$$\sigma^2(X) \geq \left(\frac{m}{2M^2}\right)^2 \sigma^2(X^2). \tag{4}$$

In this paper, new bounds for the variances of X are derived, which under certain conditions are better than the bound (given in (3)) of Bhatia and Davis and than the bound obtained by Von Szokefalvi Nagy given in (2). Moreover, the inequality (4) is generalized. The paper is ended with two new bounds for the mean of X.

2. Better bounds for the variance of X

New bounds for the variance in terms of M, m, and $\mu(X)$

We start with an improvement of the well-known inequality (3) of Bhatia and Davis published in [1].

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Theorem 2.1. *Let $x_1, \ldots, x_n$ be given real numbers with $0 < m \leq x_i \leq M$ for each $i \in \{1, \ldots, n\}$ and let "a" numbers among $x_1, \ldots, x_n$ take values M, and b numbers take value m. Let X be the discrete random variable uniformly distributed on $\{x_1, x_2, \ldots, x_n\}$. Then*

$$\sigma^2(X) \leq (M - \mu(X))(\mu(X) - m) - \frac{(n-a-b)(M-m-1)}{n}.$$

Proof. For x_i which are different than M and m we get

$$M - m - 1 \leq (M - x_i)(x_i - m) = -x_i^2 + x_i(M+m) - Mm \tag{5}$$

with equality if and only if $x_i = m+1$ or $x_i = M-1$. After adding all inequalities in (5) we get

$$(n-a-b)(M-m-1) \ \leq \ -\sum_{m<x_i<M} x_i^2 - (n-a-b)Mm + (n\mu(X) - aM - bm)(M+m).$$

Thus, we get

$$\sum_{i=1}^{n} x_i^2 \leq aM^2 + bm^2 - (n-a-b)(M-m-1) - (n-a-b)Mm + (n\mu(X) - aM - bm)(M+m). \tag{6}$$

From

$$\sigma^2(X) = \frac{\sum_{i=1}^{n} x_i^2}{n} - \mu^2(X)$$

and from (6), we get

$$\sigma^2(X) \leq (M - \mu(X))(\mu(X) - m) - \frac{(n-a-b)(M-m-1)}{n},$$

where the equality holds if and only if $x_i \in \{m, m+1, M-1, M\}$. □

Theorem 2.2. *Let $x_1, \ldots, x_n$ be given real numbers with $0 < m \leq x_i \leq M$ for each $i \in \{1, \ldots, n\}$. Let X be the discrete random variable uniformly distributed on $\{x_1, x_2, \ldots, x_n\}$. Then*

$$\sigma^2(X) \leq \frac{(M^2+m^2)}{2Mmn(n-2)}(n\mu(X) - M - m)^2 + \frac{M^2+m^2}{n} - \mu^2(X).$$

Proof. We consider the expression

$$(Mx_i - mx_j)(Mx_j - mx_i) = (M^2+m^2)x_i x_j - Mm(x_i^2 + x_j^2). \tag{7}$$

Without loss of generality, we assume that $x_1 = M$ and $x_n = m$. The terms $Mx_i - mx_j$ and $Mx_j - mx_i$ are non-negative for all $i, j = 1, 2, \ldots, n$. By summing (7) over $i, j = 2, 3, \ldots, n-1$, we get

$$(M^2+m^2)(x_2 + \ldots + x_{n-1})^2 - Mm(2(n-2)(x_2^2 + \ldots + x_{n-1}^2)) \geq 0$$

that is,

$$(M^2+m^2)(n\mu(X) - M - m)^2 - Mm(2(n-2)(x_1^2 + \ldots + x_n^2 - M^2 - m^2)) \geq 0. \tag{8}$$

From (8), we get

$$\frac{x_1^2 + \ldots + x_n^2}{n} \leq \frac{(M^2+m^2)}{2Mmn(n-2)}(n\mu(X) - M - m)^2 + \frac{(M^2+m^2)}{n}.$$

Thus, we have

$$\sigma^2(X) = \frac{x_1^2 + \ldots + x_n^2}{n} - \mu^2(X) \leq \frac{(M^2+m^2)}{2Mmn(n-2)}(n\mu(X) - M - m)^2 + \frac{M^2+m^2}{n} - \mu^2(X).$$

□

Remark 2.1. *Unfortunately, we are not able to compare our bound with the bound obtained by Bhatia and Davis. However, by using Wolfram Alpha we found a lot of cases when our bound is better than the existing bounds.*

We close this subsection with a lower bound for the variance of X in terms of M, n, and $\mu(X)$.

Theorem 2.3. *Let $x_1, \ldots, x_n$ be given real numbers with $0 < m \leq x_i \leq M$ for each $i \in \{1, \ldots, n\}$. Let X be the discrete random variable uniformly distributed on $\{x_1, x_2, \ldots, x_n\}$. Then*

$$\sigma^2(X) \geq \frac{(M - \mu(X))^2}{n-1}. \tag{9}$$

Proof. Without loss of generality, we assume that $M = x_1$. We begin by proving the inequality:

$$x_1^2 + x_2^2 + \ldots + x_n^2 \geq \frac{(x_1 + x_2 + \ldots + x_n)^2}{n} + \frac{((n-1)x_1 - (x_2 + x_3 + \ldots + x_n))^2}{n(n-1)}. \tag{10}$$

The argument proceeds as follows. If we let $S = x_2 + x_3 + \ldots + x_n$, the inequality (10) is equivalent to

$$n(n-1)(x_1^2 + x_2^2 + \ldots + x_n^2) \geq (n-1)(x_1 + S)^2 + ((n-1)x_1 - S)^2$$

which in turn is equivalent to

$$(n-1)(x_1^2 + x_2^2 + \ldots + x_n^2) \geq S^2 + (n-1)x_1^2. \tag{11}$$

Dividing (11) by $n-1$ yields the equivalent inequality

$$x_2^2 + x_3^2 + \ldots + x_n^2 \geq \frac{(x_2 + x_3 + \ldots + x_n)^2}{n-1}$$

which holds because of the inequality between the quadratic and arithmetic means of the numbers $x_2, x_3, \ldots, x_n$. Consequently, (10) holds as well. Now, the inequality (10) is equivalent to

$$x_1^2 + \ldots + x_n^2 \geq n\mu^2(X) + \frac{n(M - \mu(X))^2}{n-1}.$$

Hence

$$\sigma^2(X) = \frac{x_1^2 + \ldots + x_n^2}{n} - \mu^2(X) \geq \frac{(M - \mu(X))^2}{n-1}.$$

□

Remark 2.2. *It is easy to show that if $\mu(X) \leq M - \sqrt{\frac{n-1}{2n}}(M - m)$ then the inequality (9) is better than the Von Szokefali Nagi inequality (2).*

A generalization of a competition problem

The Original Competition Problem. The 3rd problem (Category II) of the 30th International Mathematical Competition for undergraduate students, Vojtěch Jarník, held in Czech Republic, 2022, was the following:

Let $x_1, \ldots, x_n$ be given real numbers with $0 < m \leq x_i \leq M$ for each $i \in \{1, \ldots, n\}$. Let X be the discrete random variable uniformly distributed on $\{x_1, x_2, \ldots, x_n\}$. The mean μ and the variance σ^2 of X are defined as

$$\mu(X) = \frac{x_1 + \ldots + x_n}{n} \text{ and } \sigma^2(X) = \frac{(x_1 - \mu(X))^2 + \ldots + (x_n - \mu(X))^2}{n}.$$

By X^2 denote the discrete random variable uniformly distributed on $\{x_1^2, \ldots, x_n^2\}$. Prove that

$$\sigma^2(X) \geq \left(\frac{m}{2M^2}\right)^2 \sigma^2(X^2). \tag{12}$$

This problem was solved by 3 (out of 40) students, see [5]. In the next theorem, we generalize this problem and give a significantly shorter proof than that of the original problem.

Theorem 2.4. *Let $x_1, \ldots, x_n$ be given real numbers with $0 < m \leq x_i \leq M$ for each $i \in \{1, \ldots, n\}$. Let X be the discrete random variable uniformly distributed on $\{x_1, x_2, \ldots, x_n\}$. The mean μ and the variance σ^2 of X are defined as*

$$\mu(X) = \frac{x_1 + \ldots + x_n}{n} \text{ and } \sigma^2(X) = \frac{(x_1 - \mu(X))^2 + \ldots + (x_n - \mu(X))^2}{n}.$$

For $i \geq 1$, by X^i denote the discrete random variable uniformly distributed on $\{x_1^i, \ldots, x_n^i\}$. Then

$$\sigma^2(X^i) \geq \left(\frac{im^{i-1}}{jM^{j-1}}\right)^2 \sigma^2(X^j). \tag{13}$$

Proof. We have

$$\sigma^2(X^i) = \frac{(x_1^i - \mu(X^i))^2 + \ldots + (x_n^i - \mu(X^i))^2}{n} = \frac{(x_1^{2i} + \ldots + x_n^{2i}) - n\mu^2(X^i)}{n}$$
$$= \frac{n(x_1^{2i} + \ldots + x_n^{2i}) - (x_1^i + \ldots + x_n^i)^2}{n^2} = \frac{\sum_{1 \leq k < l \leq n}(x_k^i - x_l^i)^2}{n^2}.$$

For each $k \neq l$, it suffices to show that

$$(x_k^i - x_l^i)^2 \geq \left(\frac{im^{i-1}}{jM^{j-1}}\right)^2 (x_k^j - x_l^j)^2 \Leftrightarrow (x_k^{i-1} + x_k^{i-2}x_l + \ldots + x_l^{i-1}) \geq \left(\frac{im^{i-1}}{jM^{j-1}}\right)(x_k^{j-1} + x_k^{j-2}x_l + \ldots + x_l^{j-1}).$$

From $0 < m \leq x_i \leq M$, we get

$$\frac{x_k^{i-1} + x_k^{i-2}x_l + \ldots + x_l^{i-1}}{x_k^{j-1} + x_k^{j-2}x_l + \ldots + x_l^{j-1}} \geq \frac{im^{i-1}}{jM^{j-1}}.$$

□

Remark 2.3. *Setting* $i = 1$ *and* $j = 2$ *in* (13), *we get*

$$\sigma^2(X) \geq \left(\frac{1}{2M}\right)^2 \sigma^2(X^2). \tag{14}$$

Since $\frac{1}{2M} \geq \frac{m}{2M^2}$, *we observe that the inequality* (14) *is better than the inequality* (12).

Remark 2.4. *The inequality in* (14) *could be considered by combining the Popoviciu's and Szokefalvi Nagi inequality. In such a case, we need to show that*

$$\frac{(M-m)^2}{2n} \geq \left(\frac{1}{2M}\right)^2 \cdot \frac{1}{4}(M^2 - m^2)^2,$$

which obviously does not hold for $n \geq 8$.

A Solution of the Competition Problem. For the sake of completeness, we outline a solution for the original competition problem. First, we prove the following lemma:

Lemma 2.1. *If* x *and* y *are strictly positive real numbers, then*

$$\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \geq 2 + \frac{(x-y)^2}{2(x^2+y^2)}.$$

Proof. We prove the following equivalent inequality

$$\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \geq 2 + \frac{\left(\frac{x}{y}\right)^2 - 2\left(\frac{x}{y}\right) + 1}{2\left(\left(\frac{x}{y}\right)^2 + 1\right)}.$$

Let $t^2 = \frac{x}{y}, t > 0$. The required inequality is equivalent to the inequalities

$$t + \frac{1}{t} \geq 2 + \frac{t^4 - 2t^2 + 1}{2(t^4+1)} \Leftrightarrow 2t^6 - 5t^5 + 2t^4 + 2t^3 + 2t^2 - 5t + 2 \geq 0.$$

Now, we easily show $2t^6 - 5t^5 + 2t^4 + 2t^3 + 2t^2 - 5t + 2 = (t-1)^4(2t^2 + 3t + 2) \geq 0$. □

Let $a_i = \frac{x_i^2}{x_1^2 + \ldots + x_n^2}$ and $b_i = \frac{1}{n}$ for $i = 1, \ldots, n$. By applying Lemma 2.1 for $x = a_i$ and $y = b_i$, we obtain

$$\frac{x_i^2}{x_1^2 + \ldots + x_n^2} + \frac{1}{n} \geq \left(2 + \frac{(x_i^2 n - (x_1^2 + \ldots + x_n^2))^2}{2(x_i^4 n^2 + (x_1^2 + \ldots + x_n^2)^2)}\right) \frac{x_i}{\sqrt{(n(x_1^2 + \ldots + x_n^2)}}. \tag{15}$$

Now, if we sum up the obtained n inequalities in (15), we get

$$2 \geq \frac{2}{\sqrt{n(x_1^2 + \ldots + x_n^2)}} \sum_{i=1}^n x_i + \frac{m}{\sqrt{n(x_1^2 + \ldots + x_n^2)}} \cdot \frac{1}{2(M^4 + \mu^2(X^2))} \cdot \sum_{i=1}^n (x_i^2 - \frac{x_1^2 + \ldots + x_n^2}{n})^2 \Leftrightarrow$$

$$\sqrt{\frac{x_1^2 + \ldots + x_n^2}{n}} \geq \frac{\sum_{i=1}^n x_i}{n} + \frac{m \cdot \sigma^2(X^2)}{4(M^4 + \mu^2(X^2))} = \mu(X) + \frac{m \cdot \sigma^2(X^2)}{4(M^4 + \mu^2(X^2))} \Leftrightarrow$$

$$\sqrt{\mu(X^2)} \geq \mu(X) + \frac{m \cdot \sigma^2(X^2)}{4(M^4 + M^4)} = \mu(X) + \frac{m \cdot \sigma^2(X^2)}{8M^4}.$$

Finally, we get

$$\sigma^2(X) = (\sqrt{\mu(X^2)} - \mu(X))(\sqrt{\mu(X^2)} + \mu(X)) \geq \frac{m\sigma^2(X^2)}{8M^4} \cdot 2m = \left(\frac{m}{2M^2}\right)^2 \cdot \sigma^2(X^2).$$

3. New bounds for the mean

Schweitzer in [4] proved the inequality (16) given in the following theorem.

Theorem 3.1. *For $0 < m < M$, and $x_i \in [m, M]$ for $i \in \{1, \ldots, n\}$,*

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\right) \leq \frac{(m+M)^2}{4mM}. \tag{16}$$

As a consequence of Theorem 3.1, we obtain the next result.

Theorem 3.2. *Let $x_1, \ldots, x_n$ be given real numbers with $0 < m \leq x_i \leq M$ for each $i \in \{1, \ldots, n\}$. Let X be the discrete random variable uniformly distributed on $\{x_1, x_2, \ldots, x_n\}$. Then*

$$\mu(X)\mu\left(\frac{1}{X}\right) \leq \frac{(m+M)^2}{4mM}.$$

It is well-known that $\mu\left(\frac{1}{X}\right) \geq \frac{1}{\mu(X)}$, that is, $\mu(X)\left(\frac{1}{X}\right) \geq 1$, (this relation is based on the inequality between the arithmetic and the harmonic means for the numbers $x_1, \ldots, x_n$). In the next result, we improve this relation by using Lemma 2.1.

Theorem 3.3. *Let $x_1, \ldots, x_n$ be given real numbers with $0 < m \leq x_i \leq M$ for each $i \in \{1, \ldots, n\}$. Let X be the discrete random variable uniformly distributed on $\{x_1, x_2, \ldots, x_n\}$. Then*

$$\mu(X)\mu\left(\frac{1}{X}\right) \geq \left(1 + \frac{(M-m)^2}{2n(M^2+m^2)}\right)^2.$$

Proof. Without loss of generality, we assume $M = x_1 \geq x_2 \geq \ldots \geq x_n = m$. By using Cauchy-Schwarz inequality, we have

$$\begin{aligned} n^2\mu(X)\mu\left(\frac{1}{X}\right) &= (x_1 + \ldots + x_n)\left(\frac{1}{x_n} + \frac{1}{x_2} + \ldots + \frac{1}{x_1}\right) \\ &\geq \left(\sqrt{\frac{x_1}{x_n}} + \sqrt{\frac{x_2}{x_2}} + \ldots + \sqrt{\frac{x_n}{x_1}}\right)^2 \\ &= \left(\sqrt{\frac{x_1}{x_n}} + \sqrt{\frac{x_n}{x_1}} + n - 2\right)^2. \end{aligned}$$

Now, from Lemma 2.1, the required result follows. □

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