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Research Article

On Gaussian Leonardo numbers

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Abstract

The Gaussian Leonardo sequence is a new sequence defined in this study. Some identities for this new sequence are given. Some relations among the Gaussian Fibonacci numbers, Gaussian Lucas numbers, and Gaussian Leonardo numbers are also proven. Moreover, a matrix representation of the Gaussian Leonardo numbers is obtained.

Keywords: Fibonacci numbers; Leonardo numbers; Gaussian Fibonacci numbers; Gaussian Lucas numbers; Gaussian Leonardo numbers.

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1. Introduction

Sequences of integers have an important role in various fields, including computer science, physics, cryptology, and coding theory. The Fibonacci and Lucas sequences are among the most famous sequences of integers. A Gaussian integer is a complex number whose both imaginary and real parts are integers. The Gaussian integers have been investigated by many researchers. Gaussian integers were first considered by Gauss in [6]. Horadam introduced the complex Fibonacci numbers in [7]. Jordan introduced Gaussian Fibonacci and Gaussian Lucas numbers in [8] and he extended some well-known relations about Fibonacci sequences to Gaussian Fibonacci numbers. Pethe and Horadam investigated generalized Gaussian Fibonacci numbers in [9]. Berzsenyi extended the concept of Fibonacci numbers to the complex plane [2]. Furthermore, Taşcı studied in [10–13] complex Fibonacci p numbers, Gaussian Padovan and Gaussian Pell-Padovan sequences, Gaussian Mersenne numbers, Gauss balancing numbers, and Gauss Lucas-balancing numbers.

Fibonacci and Lucas numbers are defined by recursively, for $n \geq 1$:

$$F_{n+1} = F_n + F_{n-1},$$

with the initial conditions $F_0 = 0, F_1 = 1$ and

$$L_{n+1} = L_n + L_{n-1},$$

with the initial conditions $L_0 = 2, L_1 = 1$, respectively (see [1]). Gaussian Fibonacci numbers GF_n are also defined recursively:

$$GF_n = GF_{n-1} + GF_{n-2},\ n \geq 2, \tag{1}$$

with the initial values $GF_0 = i, GF_1 = 1$. It is clear that these numbers are closely related to Fibonacci numbers:

$$GF_n = F_n + iF_{n-1}, \tag{2}$$

where $i = \sqrt{-1}$. Similarly, Gaussian Lucas numbers GL_n are defined recursively, for $n \geq 2$:

$$GL_n = GL_{n-1} + GL_{n-2}, \tag{3}$$

with the initial conditions $GL_0 = 2 + i, GL_1 = 1 + 2i$. Leonardo numbers are studied by Catarino and Burgers in [3–5]. They defined these numbers by the second order inhomogeneous recurrence relation:

$$Le_n = Le_{n-1} + Le_{n-2} + 1,\ n \geq 2, \tag{4}$$

with the initial conditions $Le_0 = Le_1 = 1$. Also, these numbers can be defined as:

$$Le_{n+1} = 2Le_n - Le_{n-2},\ n \geq 2 \tag{5}$$

The purpose of the present study is to introduce Gaussian Leonardo numbers and investigate their properties.

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2. Gaussian Leonardo numbers

We begin with the definition of the Gaussian Leonardo numbers.

Definition 2.1. *Gaussian Leonardo numbers* GLe_n *are defined as*

$$GLe_n = GLe_{n-1} + GLe_{n-2} + (1+i),\ n \geq 2, \tag{6}$$

with the initial conditions $GLe_0 = 1-i,\ GLe_1 = 1+i$. *Some Gaussian Leonardo numbers are* $1-i, 1+i, 3+i, 5+3i, 9+5i, \cdots$.

It is clear that Gaussian Leonardo numbers are closely related to Leonardo numbers:

$$GLe_n = Le_n + iLe_{n-1}, \tag{7}$$

where Le_n denotes the n-th Leonardo number.

Lemma 2.1. *For* $n \geq 1$, *it holds that*

$$GLe_{n+2} = 2\,GLe_{n+1} - GLe_{n-1}. \tag{8}$$

Proof. Using Equations (5) and (7), we have

$$\begin{aligned} GLe_{n+2} &= Le_{n+2} + i\,Le_{n+1} \\ &= 2Le_{n+1} - Le_{n-1} + i(2Le_n - Le_{n-2}) \\ &= 2\,(Le_{n+1} + iLe_n) - (Le_{n-1} + iLe_{n-2}) \\ &= 2\,GLe_{n+1} - GLe_{n-1}. \end{aligned}$$

□

The next result gives a relation between the Gaussian Leonardo and Gaussian Fibonacci numbers.

Theorem 2.1. *For* $n \geq 0$, *the following identity holds*

$$GLe_n = 2\,GF_{n+1} - (1+i), \tag{9}$$

where GF_{n+1} *denotes the* $(n+1)$*-th Gaussian Fibonacci number and* GLe_n *denotes the* n*-th Gaussian Leonardo number.*

Proof. Consider Identities (2) and (7), together with

$$Le_n = 2\,F_{n+1} - 1,$$

where Le_n denotes the n-th Leonardo number and F_{n+1} denotes the $(n+1)$-th Fibonacci number. Then, we have

$$\begin{aligned} GLe_n &= Le_n + i\,Le_{n-1} \\ &= (2\,F_{n+1} - 1) + i(2\,F_n - 1) \\ &= 2(\,F_{n+1} + iF_n) - (1+i) \\ &= 2\,GF_{n+1} - (1+i). \end{aligned}$$

□

Lemma 2.2. *For* $n \geq 0$, *it holds that* $GLe_{n+1} - GLe_n = 2\,GF_n$.

Proof. In view of Theorem 2.1, we write

$$\begin{aligned} GLe_{n+1} - GLe_n &= [2\,GF_{n+2} - (1+i)] - [2\,GF_{n+1} - (1+i)] \\ &= 2(GF_{n+2} - GF_{n+1}) \\ &= 2\,GF_n. \end{aligned}$$

□

Lemma 2.3. *For $n \geq 1$, the following identities hold:*

(a). $GF_n = \dfrac{GL_{n-1} + GL_{n+1}}{5}$,

(b). $GL_n = GF_n + 2GF_{n-1}$,

where GF_n denotes n-th Gaussian Fibonacci number and GL_n denotes n-th Gaussian Lucas number.

Proof. We prove Part (a) by the induction on n. Part (b) can be proved in a similar way. For $n = 1$, we have

$$GF_1 = \frac{GL_0 + GL_2}{5} = \frac{2 - i + 3 + i}{5} = 1$$

and the induction starts. Now, suppose that the desired identity holds for every k satisfying $1 < k \leq n$. Using

$$GF_{n+1} = GF_n + GF_{n-1}$$

and the induction hypothesis, we have

$$\begin{aligned} GF_{n+1} &= GF_n + GF_{n-1} \\ &= \frac{GL_{n-1} + GL_{n+1}}{5} + \frac{GL_{n-2} + GL_n}{5} \\ &= \frac{1}{5}\left[(GL_{n-1} + GL_{n-2}) + (GL_{n+1} + GL_n)\right] \\ &= \frac{1}{5}\left(GL_n + GL_{n+2}\right). \end{aligned}$$

Thus, the required identity holds for $n + 1$, as desired. □

The next result gives relationships among Gaussian Leonardo, Gaussian Lucas and Gaussian Fibonacci numbers.

Theorem 2.2. *For the n-th Gaussian Leonardo number GLe_n with $n \geq 0$, the following identities hold:*

(a). $GLe_n = 2\left(\dfrac{GL_n + GL_{n+2}}{5}\right) - (1 + i)$,

(b). $GLe_n = GL_{n+2} - GF_{n+2} - (1 + i)$.

Proof. The required identities follow directly from Theorem 2.1 and Lemma 2.2. □

Theorem 2.3. *For $1 - 2x + x^3 \neq 0$, the generating function of the Gaussian Leonardo sequence is*

$$\sum_{n=0}^{\infty} GLe_n\, x^n = \frac{(1 - i) + (-1 + 3i)x + (1 - i)x^2}{1 - 2x + x^3}.$$

Proof. Consider the Gaussian Leonardo sequence $\{GLe_n\}_{n=0}^{\infty}$ and let $g(x)$ be its generating function, i.e.,

$$g(x) = \sum_{n=0}^{\infty} GLe_n\, x^n.$$

By using Lemma 2.1 and the initial conditions of the Gaussian Leonardo sequence, we have

$$\begin{aligned} g(x) = \sum_{n=0}^{\infty} GLe_n\, x^n &= GLe_0 + GLe_1 x + GLe_2 x^2 + \sum_{n=3}^{\infty} GLe_n\, x^n \\ &= GLe_0 + GLe_1 x + GLe_2 x^2 + \sum_{n=3}^{\infty} (2GLe_{n-1} - GLe_{n-3})\, x^n \\ &= GLe_0 + GLe_1 x + GLe_2 x^2 + 2x\sum_{n=3}^{\infty} GLe_{n-1} x^{n-1} - x^3 \sum_{n=3}^{\infty} GLe_{n-3} x^{n-3} \\ &= GLe_0 + GLe_1 x + GLe_2 x^2 + 2x\left(\sum_{n=0}^{\infty} GLe_n x^n - GLe_0 - GLe_1 x\right) - x^3\left(\sum_{n=0}^{\infty} GLe_n x^n\right) \\ &= GLe_0 + GLe_1 x + GLe_2 x^2 + 2x\sum_{n=0}^{\infty} GLe_n x^n - 2GLe_0 x - 2GLe_1 x^2 - x^3 \sum_{n=0}^{\infty} GLe_n x^n \end{aligned}$$

Therefore,

$$(1-2x+x^3)\sum_{n=0}^{\infty} GLe_n x^n = GLe_0 + (GLe_1 - 2GLe_0)x + (GLe_2 - 2GLe_1)x^2$$

and the result immediately follows. (We remark that $GLe_0 = 1-i, GLe_1 = 1+i$, and $GLe_2 = 3+i$.) □

Theorem 2.4. *Binet's formula for the Gaussian Leonardo numbers is*

$$GLe_n = 2\frac{(\alpha^{n+1}-\beta^{n+1}) + i(\alpha^n - \beta^n)}{\alpha-\beta} - (1+i),$$

with $n \geq 0$*, where*

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \textit{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Proof. We know that the Binet formula for the Fibonacci numbers is

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

It is easily seen that

$$\begin{aligned} GF_n &= F_n + iF_{n-1} \\ &= \frac{\alpha^n - \beta^n}{\alpha-\beta} + i\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}. \end{aligned}$$

Now, using

$$GLe_n = 2GF_{n+1} - (1+i),$$

we have

$$GLe_n = 2\frac{(\alpha^{n+1}-\beta^{n+1}) + i(\alpha^n - \beta^n)}{\alpha-\beta} - (1+i).$$

□

Theorem 2.5. *Cassini's Identity for the Gaussian Leonardo sequence is*

$$GLe_{n+1}GLe_{n-1} - (GLe_n)^2 = (-1)^{n+1}(8-4i) + 2(1+i)(2GF_{n+1} - GF_{n+2} - GF_n).$$

Proof. Considering Theorem 2.1 and using

$$GF_{n+1}GF_{n-1} - GF_n^2 = (-1)^n(2-i),$$

we have

$$\begin{aligned} GLe_{n+1}GLe_{n-1} - (GLe_n)^2 &= [2GF_{n+2} - (1+i)]\,[2GF_n - (1+i)] - [2GF_{n+1} - (1+i)]^2 \\ &= 4GF_{n+2}GF_n - 2(1+i)GF_{n+2} - 2(1+i)GF_n + (1+i)^2 \\ &\quad - 4(GF_{n+1})^2 + 4(1+i)GF_{n+1} - (1+i)^2 \\ &= 4\left[GF_{n+2}GF_n - (GF_{n+1})^2\right] + 2(1+i)(2GF_{n+1} - GF_{n+2} - GF_n) \\ &= (-1)^{n+1}(8-4i) + 2(1+i)(2GF_{n+1} - GF_{n+2} - GF_n) \end{aligned}$$

and the result follows. □

The next result gives the sum formulae for the Gaussian Leonardo numbers.

Theorem 2.6. *For* $n \geq 0$*, it holds that*

$$\sum_{j=0}^{n} GLe_j = GLe_{n+2} - (n+2)(1+i).$$

Proof. If we consider

$$\sum_{j=0}^{n} GF_j = GF_{n+2} - 1$$

and Theorem 2.1, then we have

$$\begin{aligned}
\sum_{j=0}^{\infty} GLe_j &= \sum_{j=0}^{n} \Big(2GF_{j+1} - (1+i) \Big) \\
&= 2\sum_{j=0}^{n} GF_{j+1} - (n+1)(1+i) \\
&= 2\left(\sum_{j=0}^{n+1} GF_j - GF_0 \right) - (n+1)(1+i) \\
&= 2(GF_{n+3} - 1 - i) - (n+1)(1+i) \\
&= 2GF_{n+3} - (1+i) - (1+i) - (n+1)(1+i) \\
&= 2GF_{n+3} - (1+i) - (n+2)(1+i) \\
&= 2GLe_{n+2} - (n+2)(1+i).
\end{aligned}$$

□

Theorem 2.7. *For $n \geq 0$, it holds that*

$$\sum_{j=0}^{n} GLe_{2j} = GLe_{2n+1} - n - (n+2)i.$$

Proof. Using

$$\sum_{j=0}^{n} GF_j = GF_{n+2} - 1$$

and considering Theorem 2.1, we have

$$\begin{aligned}
\sum_{j=0}^{n} GLe_{2j} &= \sum_{j=0}^{n} \Big(2GF_{2j+1} - (\ 1+i) \Big) \\
&= 2\sum_{j=0}^{n} GF_{2j+1} - (n+1)(\ 1+i) \\
&= 2\sum_{j=1}^{n+1} GF_{2j-1} - (n+1)(\ 1+i) \\
&= 2(GF_{2n+2} - i) - (n+1)(\ 1+i) \\
&= 2GF_{(2n+1)+1} - 2i - (n+1)(\ 1+i) \\
&= GLe_{2n+1} - n - (n+2)\ i.
\end{aligned}$$

□

Theorem 2.8. *For $n \geq 0$, it holds that*

$$\sum_{j=0}^{n} GLe_{2j+1} = GLe_{2n+2} - (n+2) - n\ i.$$

Proof. Using the relations between the Gaussian Leonardo and Gaussian Fibonacci numbers, we have

$$\begin{aligned}\sum_{j=0}^{n} GLe_{2j+1} &= \sum_{j=0}^{n}\Big(2GF_{2j+2} - (1+i)\Big)\\ &= 2\sum_{j=0}^{n} GF_{2j+2} - (n+1)(1+i)\\ &= 2\left(\sum_{j=0}^{n+1} GF_{2j} - i\right) - (n+1)(1+i)\\ &= 2\sum_{j=0}^{n+1} GF_{2j} - 2i - (n+1)(1+i)\\ &= 2(GF_{2n+3} - 1 + i) - 2i - (n+1)(1+i)\\ &= 2\ GF_{(2n+2)+1} - 2 + 2i - 2i - (n+1)(1+i)\\ &= 2\ GF_{(2n+2)+1} - (1+i) - 2 - n(1+i)\\ &= GLe_{2n+2} - (n+2) - n\ i\end{aligned}$$

and the proof is completed. □

Theorem 2.9. *For $n \geq 0$, the following identities hold:*

(a). $\sum_{j=0}^{n}(GLe_j + GF_j) = GLe_{n+2} + GF_{n+2} - (n+3) - (n+2)i$,

(b). $\sum_{j=0}^{n}(GLe_j + GL_j) = GLe_{n+2} + GLe_{n+2} - (n+3) - (n+4)i$.

Proof. **(a).** Considering Theorem 2.6 and the identity

$$\sum_{j=0}^{n} GF_j = GF_{n+2} - 1,$$

the required identity is obtained.

(b). Using Theorem 2.6 and

$$\sum_{j=0}^{n} GL_j = GFL_{n+2} - (1+2i),$$

we get the desired result. □

Theorem 2.10. *For $n \geq 1$, the following identity holds:*

$$GLe_{-n} = (-1)^{n-1} i (GLe_{n-1} + 1 - i) - (1+i).$$

Proof. Using Theorem 2.1 and

$$GF_{-n} = (-1)^{n}\ i\ GF_{n+1},$$

we write

$$\begin{aligned}GLe_{-n} &= 2GF_{-n+1} - (1+i)\\ &= 2(-1)^{n-1}\ i\ GF_n - (1+i)\\ &= (-1)^{n-1}\ i\ 2GF_n - (1+i)\\ &= (-1)^{n-1}\ i\ (GLe_{n-1} + 1 + i) - (1+i).\end{aligned}$$

□

Next, we give a matrix representation of the Gaussian Leonardo numbers.

Theorem 2.11. *Let n be a positive integer and let GLe_n be the n-th Gaussian Leonardo number. If*

$$Q = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} GLe_3 & GLe_2 & GLe_1 \\ GLe_2 & GLe_1 & GLe_0 \\ GLe_1 & GLe_0 & GLe_{-1} \end{bmatrix}, \quad \text{and} \quad D_n = \begin{bmatrix} GLe_{n+3} & GLe_{n+2} & GLe_{n+1} \\ GLe_{n+2} & GLe_{n+1} & GLe_n \\ GLe_{n+1} & GLe_n & GLe_{n-1} \end{bmatrix},$$

then

$$KQ^n = D_n. \tag{10}$$

Proof. We prove the result by the induction on n. For $n = 1$, we have

$$\begin{aligned} KQ &= \begin{bmatrix} GLe_3 & GLe_2 & GLe_1 \\ GLe_2 & GLe_1 & GLe_0 \\ GLe_1 & GLe_0 & GLe_{-1} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2GLe_3 - GLe_1 & GLe_3 & GLe_2 \\ 2GLe_2 - GLe_0 & GLe_2 & GLe_1 \\ 2GLe_1 - GLe_{-1} & GLe_1 & GLe_0 \end{bmatrix} \\ &= \begin{bmatrix} GLe_4 & GLe_3 & GLe_2 \\ GLe_3 & GLe_2 & GLe_1 \\ GLe_2 & GLe_1 & GLe_0 \end{bmatrix} \\ &= D_1. \end{aligned}$$

Thus, Equation (10) holds holds for $n = 1$. Now, suppose that Equation (10) holds for every k satisfying $1 < k \leq n$. We prove that Equation (10) remains valid when $k = n + 1$. Note that

$$\begin{aligned} KQ^{n+1} &= KQ^n.Q \\ &= \begin{bmatrix} GLe_{n+3} & GLe_{n+2} & GLe_{n+1} \\ GLe_{n+2} & GLe_{n+1} & GLe_n \\ GLe_{n+1} & GLe_n & GLe_{n-1} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2GLe_{n+3} - GLe_{n+1} & GLe_{n+3} & GLe_{n+2} \\ 2GLe_{n+2} - GLe_n & GLe_{n+2} & GLe_{n+1} \\ 2GLe_{n+1} - GLe_{n-1} & GLe_{n+1} & GLe_n \end{bmatrix} \\ &= \begin{bmatrix} GLe_{n+4} & GLe_{n+3} & GLe_{n+2} \\ GLe_{n+3} & GLe_{n+2} & GLe_{n+1} \\ GLe_{n+2} & GLe_{n+1} & GLe_n \end{bmatrix} \\ &= D_n, \end{aligned}$$

and thus the result is verified. □

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