Contributions to Mathematics
www.shahindp.com/locate/cm

Contrib. Math. **7** (2023) 24–33
DOI: 10.47443/cm.2023.002

Research Article
Stirling numbers and inverse factorial series

Khristo N. Boyadzhiev*

Department of Mathematics, Ohio Northern University, Ada, Ohio 45810, USA

(Received: 12 January 2023. Received in revised form: 27 February 2023. Accepted: 1 March 2023. Published online: 8 March 2023.)

Abstract

We study inverse factorial series and their relation to Stirling numbers of the first kind. We prove a special representation of the polylogarithm function in terms of series with such numbers. Using various identities for Stirling numbers of the first kind we construct a number of expansions of functions in terms of inverse factorial series where the coefficients are special numbers. These results are used to prove/reprove the asymptotic expansion of some classical functions. We also prove a binomial formula involving inverse factorials.

Keywords: inverse factorial series; Stirling numbers of the first kind; polylogarithm; logarithm antiderivatives; harmonic numbers; digamma function; Nielsen's beta function; incomplete gamma function.

2020 Mathematics Subject Classification: 05A15, 11B73, 30B50, 33B30, 40A30, 65B10, 65D05.

1. Introduction

The unsigned Stirling numbers of the first kind $\left[{n \atop k}\right]$ first appeared in James Stirling's book *Methodus Differentialis* (published in Latin, in 1730); see [26] for the English translation of this book and comments. These numbers have numerous applications in combinatorics and analysis (see [1, 3, 6, 9, 10, 12, 24]).

In this paper, we present various results involving the unsigned Stirling numbers of the first kind and inverse factorial series. These numbers are usually defined by their ordinary generating function

$$x(x+1)\dots(x+n-1) = \sum_{k=0}^{n} \left[{n \atop k} \right] x^k \tag{1}$$

or by their exponential generating function

$$\frac{[-\ln(1-x)]^k}{k!} = \sum_{n=k}^{\infty} \left[{n \atop k} \right] \frac{x^n}{n!}, \quad |x| < 1, \ k \geq 0 \tag{2}$$

(see [9, 10]). However, there is another, less known generating function coming from the original works of James Stirling

$$\frac{1}{x^{k+1}} = \sum_{n=k}^{\infty} \left[{n \atop k} \right] \frac{1}{x(x+1)\dots(x+n)} \tag{3}$$

in terms of inverse factorials (see [3] and [26, p.171]). Series of this form are called inverse factorial series. As shown by Milne-Thomson in Chapter 10 of his book [16], for appropriate functions $f(z)$ defined on a half-plane $Re\,(z) > \delta > 0$ expansions of the form

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z(z+1)\dots(z+n)}$$

are convergent and the coefficients are uniquely determined. See also Whittaker and Watson [30, pp. 142-144]. Inverse factorial series were thoroughly studied by Norlund in [18–20], Obrechkoff [22], and Watson [27], and in more recent times by Wasow in [28, Chapter 11], Weniger [29], and Karp with Prilepkina [13]. Instead of listing general conditions for convergence, in many cases, it is more practical to prove convergence by using the ratio test or the Raabe-Duhamel test.

Formula (3) brings immediately to an important result. Setting $z = m$, an integer, summing for $m = 1, 2, \dots$ and changing the order of summation, we come to the representation

$$\zeta(k+1) = \sum_{n=k}^{\infty} \left[{n \atop k} \right] \sum_{m=1}^{\infty} \frac{1}{m(m+1)\dots(m+n)},$$

*E-mail address: k-boyadzhiev@onu.edu

where $\zeta(s)$ is Riemann's zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad Re(s) > 1.$$

It is easy to compute that

$$\sum_{m=1}^{\infty} \frac{1}{m(m+1)\dots(m+n)} = \frac{1}{n!n}$$

(see, for example, [4]) and we obtain an important representation of the zeta function

$$\zeta(k+1) = \sum_{n=k}^{\infty} \left[\begin{array}{c} n \\ k \end{array} \right] \frac{1}{n!n}. \tag{4}$$

This result is not new; it was known to Charles Jordan at least in 1939 (see [12, Equation (6) on p. 166 and Equation (11) on p. 194]). Jordan even proved a more general formula (see [12, p. 343]); for another proof see [4]. Informative comments on Equation (4) can be found in Adamchik's paper [1]; see also [24, Chapter 1, Problems 46 and 47].

In this paper we extend the representation (4) to the Hurwitz zeta function

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad Re\,(s) > 1, \quad a > 0,$$

(see Proposition 2.1 in the next section). The next natural step is to replace the Riemann zeta function with the polylogarithm

$$Li_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s} \quad (|x| \le 1).$$

This is our main result (Theorem 2.1 in Section 2). It represents an important extension since $Li_s(1) = \zeta(s)$. In Section 3, we list a number of examples of functions represented by inverse factorial series. In Section 4, with the help of Proposition 4.1 and using Stirling number identities, we generate new inverse factorial representations. Then in Section 5, again with the help of Proposition 4.1, we obtain several important asymptotic expansions. Finally, in Section 6, we prove a formula relating a binomial sum to inverse factorial series with unsigned Stirling numbers of the first kind.

2. Main results

Proposition 2.1. *For the Hurwitz zeta function $\zeta(s,a)$ we have the representation*

$$\zeta(k+1,a) = \Gamma(a) \sum_{n=k}^{\infty} \left[\begin{array}{c} n \\ k \end{array} \right] \frac{1}{n\,\Gamma(n+a)} \quad (k \ge 1,\ a > 0) \tag{5}$$

which turns into (4) *when* $a = 1$.

Proof. With $x = m + a$, $m = 0, 1, 2, \dots$ we write (3) in the form

$$\frac{1}{(m+a)^{k+1}} = \sum_{n=k}^{\infty} \left[\begin{array}{c} n \\ k \end{array} \right] \frac{1}{(m+a)(m+a+1)\dots(m+a+n)}$$

then summing for $m = 0, 1, 2, \dots$ we come to

$$\zeta(k+1,a) = \sum_{n=k}^{\infty} \left[\begin{array}{c} n \\ k \end{array} \right] \left\{ \sum_{m=0}^{\infty} \frac{1}{(m+a)(m+a+1)\dots(m+a+n)} \right\}.$$

Evaluating the sum in the braces gives

$$\sum_{m=0}^{\infty} \frac{1}{(m+a)(m+a+1)\dots(m+a+n)} = \frac{1}{n\,\Gamma(n+a)}$$

and (5) is proved. □

Next, we recall the harmonic numbers

$$H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}, \quad H_0 = 0$$

which appear in the next proposition.

Proposition 2.2. *For any $n \geq 0$*

$$\sum_{p=1}^{\infty} \frac{x^{p+n}}{p(p+1)\dots(p+n)} = P_n(x) + \frac{(-1)^{n-1}}{n!}(1-x)^n \ln(1-x) \tag{6}$$

where $P_n(x)$ is the polynomial

$$P_n(x) = \frac{1}{n!}\left(H_n(x-1)^n + \sum_{j=0}^{n-1}\binom{n}{j}\frac{(x-1)^j}{n-j}\right) = \frac{1}{n!}\left(H_n(x-1)^n + \sum_{k=1}^{n}\binom{n}{k}\frac{(x-1)^{n-k}}{k}\right) \tag{7}$$

with $P_n(1) = \frac{1}{n!n}$.

Proof. Let us set

$$f_n(x) = \sum_{p=1}^{\infty} \frac{x^{p+n}}{p(p+1)\dots(p+n)}$$

for $n = 0, 1, \dots$. Differentiating this function n times we find

$$\left(\frac{d}{dx}\right)^n f_n(x) = \sum_{p=1}^{\infty} \frac{x^p}{p} = -\ln(1-x).$$

We see that $f_n(x)$ is the n-th antiderivative of $-\ln(1-x)$ satisfying the conditions (8) below. We will find a closed form evaluation for the series$f_n(x)$.

Clearly $(d/dx)f_m(x) = f_{m-1}(x)$. When $n = 0$,

$$f_0(x) = \sum_{p=1}^{\infty} \frac{x^p}{p} = -\ln(1-x)$$

and $f_1, f_2, \dots, f_n$ are obtained from here by repeated integration. The constant of integration is taken zero each time so that according to (4)

$$f_1(1) = \frac{1}{1!1},\ f_2(1) = \frac{1}{2!2},\ f_3(1) = \frac{1}{3!3}, \dots,\ f_n(1) = \frac{1}{n!n}. \tag{8}$$

The process starts this way

$$f_1(x) = \int -\ln(1-x)dx = \int \ln(1-x)d(1-x) = (1-x)\ln(1-x) + x + C = (1-x)\ln(1-x) + x$$

then

$$f_2(x) = \frac{3}{4}x^2 - \frac{x}{2} - \frac{1}{2}(1-x)^2\ln(1-x)$$

etc. We evaluate each time at $x = 1$ by taking limits,

$$\lim_{x\to 1}(1-x)^n \ln(1-x) = 0, \quad n = 1, 2, \dots.$$

Integrating repeatedly by parts, we come to the representation

$$f_n(x) = \sum_{p=1}^{\infty} \frac{x^{p+n}}{p(p+1)\dots(p+n)} = P_n(x) + \frac{(-1)^{n-1}}{n!}(1-x)^n \ln(1-x) \tag{9}$$

where $P_n(x)$ is a polynomial of degree n. For convenience, we write it in the form

$$P_n(x) = a_n(x-1)^n + a_{n-1}(x-1)^{n-1} + \dots + a_1(x-1) + a_0.$$

Obviously, $a_0 = f_n(1) = \frac{1}{n!n}$. To compute the other coefficients, we repeatedly differentiate in (9) and set $x \to 1$. The coefficient a_n requires a longer (but elementary) computation. Namely, differentiating Equation (9) n times and setting $x \to 1$ brings to

$$0 = n!a_n + \sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k}{k}$$

and now using the well-known identity

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k} = H_n\,,$$

we find that $n!a_n = H_n$ as desired. Thus we come to the representation (7), which completes the proof of the proposition. □

Setting $x = -1$ in (7) we compute that

$$P_n(-1) = \frac{(-1)^n 2^n}{n!} \sum_{k=1}^{n} \frac{1}{2^k k}$$

and we find the next result.

Corollary 2.1. *For every $n \geq 0$ we have*

$$\sum_{p=1}^{\infty} \frac{(-1)^p}{p(p+1)\dots(p+n)} = \frac{2^n}{n!} \left(\sum_{k=1}^{n} \frac{1}{2^k k} - \ln 2 \right) \tag{10}$$

Now, we come to our main result, a theorem that extends the representation (4) to polylogarithms.

Theorem 2.1. *For every integer $k \geq 0$, and $|x| < 1$, we have*

$$Li_{k+1}(x) = \sum_{n=k}^{\infty} \left[\begin{matrix} n \\ k \end{matrix} \right] \frac{1}{x^n} \left\{ P_n(x) + \frac{(-1)^{n-1}}{n!}(1-x)^n \ln(1-x) \right\} \tag{11}$$

where $P_n(x)$ is the polynomial from (7).

Note that when we set $x \to 1$ in (11) the equation turns into (4), as $P_n(1) = \frac{1}{n!n}$.

Proof. We write (3) in the form

$$\frac{1}{p^{k+1}} = \sum_{n=k}^{\infty} \left[\begin{matrix} n \\ k \end{matrix} \right] \frac{1}{p(p+1)\dots(p+n)}$$

then multiply both sides by x^p and sum for p from 1 to infinity. After changing the order of summation, we write

$$Li_{k+1}(x) = \sum_{n=k}^{\infty} \left[\begin{matrix} n \\ k \end{matrix} \right] \frac{1}{x^n} \left\{ \sum_{p=1}^{\infty} \frac{x^{p+n}}{p(p+1)\dots(p+n)} \right\}$$

and the result follows now from Proposition 2.2. The proof is completed. □

Remark 2.1. *As mentioned above, Equation (9) represents the nth antiderivative of $-\ln(1-x)$ with the conditions (8). The functions $f_2(x)$ and $f_3(x)$ can be found in the table [21] of Prudnikov et al.; f_2 is entry 5.2.6 (3) and f_3 is entry 5.2.6(12). They are also listed in Hansen's table [11], f_2 is entry 5.7.40, and f_3 is entry 5.6.29. The table [11] has some general formulas similar to (9) like 10.4.15 and 10.4.17. Hansen refers to the works of Schwatt (see [23, pp. 191-192]). The functions f_n were also studied by Mathar [14], and a similar study can be found in [15].*

We finish this section with a related result. Consider the Dirichlet series

$$H(s) = \sum_{n=1}^{\infty} \frac{H_n}{n^s} \quad (Res > 1).$$

It is reasonable to ask the same question: can we express $H(k+1)$ in terms of $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$? The affirmative answer is given in the following proposition.

Proposition 2.3. *For every $k \geq 1$,*

$$\sum_{p=1}^{\infty} \frac{H_p}{p^{k+1}} = \sum_{n=k}^{\infty} \left[\begin{matrix} n \\ k \end{matrix} \right] \frac{\psi'(n)}{n!} \tag{12}$$

and also

$$\left(\frac{k+3}{2} \right) \zeta(k+2) - \frac{1}{2} \sum_{j=1}^{k-1} \zeta(j+1)\zeta(k-j+1) = \sum_{n=k}^{\infty} \left[\begin{matrix} n \\ k \end{matrix} \right] \frac{\psi'(n)}{n!},$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function.

Proof. From (3) we have

$$\frac{H_p}{p^{k+1}} = \sum_{n=k}^{\infty} \left[\begin{matrix} n \\ k \end{matrix} \right] \frac{H_p}{p(p+1)\dots(p+n)}$$

and then we sum for $p = 1, 2, \dots$. Interchanging the summations on the right-hand side we write

$$\sum_{p=1}^{\infty} \frac{H_p}{p^{k+1}} = \sum_{n=k}^{\infty} \left[\begin{matrix} n \\ k \end{matrix} \right] \left\{ \sum_{p=1}^{\infty} \frac{H_p}{p(p+1)\dots(p+n)} \right\}.$$

It is known that for $n \geq 1$

$$\sum_{p=1}^{\infty} \frac{H_p}{p(p+1)\dots(p+n)} = \frac{1}{n!}\sum_{m=0}^{\infty} \frac{1}{(n+m)^2} = \frac{\psi'(n)}{n!}$$

(see, for instance, [4]). The Euler sums on the left hand side in (12) have the closed form evaluation [1]

$$\sum_{p=1}^{\infty} \frac{H_p}{p^{k+1}} = \binom{k+3}{2}\zeta(k+2) - \frac{1}{2}\sum_{j=1}^{k-1}\zeta(j+1)\zeta(k-j+1)$$

and this completes the proof. □

3. Examples of inverse factorial series

First, we list several known series, and then we add some new ones.

Example 3.1. Using the properties

$$\begin{bmatrix} n+1 \\ 1 \end{bmatrix} = n!, \quad \begin{bmatrix} n+1 \\ 2 \end{bmatrix} = n!H_n$$

we find from (3) with $k=1$and $k=2$ the following two expansions

$$\frac{1}{z-1} = \sum_{n=0}^{\infty} \frac{n!}{z(z+1)\dots(z+n)}, \tag{13}$$

$$\frac{1}{(z-1)^2} = \sum_{n=0}^{\infty} \frac{n!H_n}{z(z+1)\dots(z+n)},$$

by making in (3) the substitutions $n \to n+1, \quad z \to z-1$.

Both series are convergent for $Re\,(z) > 1$. This convergence can be verified by the Raabe-Duhamel test. For instance, if a_n is the general term of the series in (13) we have for real $z > 1$,

$$\lim\ n\left(\frac{a_n}{a_{n+1}} - 1\right) = z > 1.$$

The Raabe-Duhamel test also confirms the convergence in the following several expansions.

Let w be a complex number. A series extending (13) is

$$\frac{1}{z-w} = \sum_{n=0}^{\infty} \frac{w(w+1)\dots(w+n-1)}{z(z+1)\dots(z+n)} \tag{14}$$

convergent for $Re\,(z) > Re\,(w)$ (see [19, p. 222]).

Example 3.2. Another example in the same spirit is the series

$$\frac{1}{(z-1)(z-2)} = \sum_{n=0}^{\infty} \frac{n!n}{z(z+1)\dots(z+n)}, \quad Re\,(z) > 2 \tag{15}$$

(cf. entry 6.6.14 in Hansen's table [11]).

Example 3.3. Next (see [11, 6.6.39]) we have

$$\psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2} = \sum_{n=0}^{\infty} \frac{n!}{n+1}\frac{1}{z(z+1)\dots(z+n)}, \quad Re\,(z) > 0 \tag{16}$$

where again $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function.

Example 3.4. Consider the beta function

$$\beta(z) = \int_0^1 \frac{t^{z-1}}{t+1}\,dt = \int_0^{\infty} \frac{e^{-zt}}{e^{-t}+1}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+z}, \quad Re\,(z) > 0.$$

This function was studied by Niels Nielsen [17] and is often called Nielsen's beta function. For some properties and important integrals related to $\beta(z)$, see [8]. Integration by parts gives

$$\beta(z) = \frac{1}{2z} + \frac{1}{z}\int_0^1 \frac{t^z}{(t+1)}\,dt$$

and repeating this, again and again, we come to the expansion

$$\beta(z) = \sum_{n=0}^{\infty} \frac{n!}{2^{n+1}}\frac{1}{z(z+1)\dots(z+n)}. \tag{17}$$

This representation appears in Nielsen's book [17, p. 81]. It also appears in Norlund's paper [18, p. 352].

Example 3.5. Consider the lower incomplete gamma function

$$\gamma(z,\ x) = \int_0^x t^{z-1} e^{-t} dt, \quad x > 0,\ Re\,(z) > 0.$$

We have (Temme [25, p. 279])

$$\gamma(z,\ x) = x^z e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{z(z+1)\ldots(z+n)}.$$

Convergence follows from the ratio test.

Example 3.6. Jacques Binet proved the expansion for the log-gamma function

$$\ln\Gamma(z) = \left(z - \frac{1}{2}\right)\ln z - z + \ln\sqrt{2\pi} + \sum_{n=1}^{\infty} \frac{a_n}{(z+1)(z+2)\ldots(z+n)}$$

where

$$a_n = \frac{1}{n}\int_0^1 \left(t - \frac{1}{2}\right) t(t+1)\ldots(t+n-1)\, dt.$$

We can write this in the form (with $a_0 = 0$)

$$\frac{1}{z}\ln\frac{\Gamma(z)}{\sqrt{2\pi}} - \left(1 - \frac{1}{2z}\right)\ln z + 1 = \sum_{n=0}^{\infty} \frac{a_n}{z(z+1)(z+2)\ldots(z+n)}$$

(see Whittaker and Watson [30, p. 253]).

Inverse factorial series also appeared in the works of Ramanujan, as discussed by Berndt in [2]. More details are given in Section 5.

4. Generating new inverse factorial representations

The following proposition makes it possible to generate more inverse factorial representations.

Proposition 4.1. *Let $a_0, a_1, \ldots, a_n, \ldots$ be a sequence. Then we formally have*

$$\sum_{k=0}^{\infty} \frac{a_k}{z^{k+1}} = \sum_{n=0}^{\infty} \frac{1}{z(z+1)\ldots(z+n)} \left\{ \sum_{k=0}^{n} \left[\begin{matrix} n \\ k \end{matrix} \right] a_k \right\}. \tag{18}$$

Proof. For the proof, we multiply both sides in (3) by a_k, sum for k from zero to infinity and change the order of summation. We do not discuss convergence in the general case. Convergence can be checked easily in all particular examples below. □

Using the Stirling sequence transformation

$$b_n = \sum_{k=0}^{n} \left[\begin{matrix} n \\ k \end{matrix} \right] a_k$$

we can generate various inverse factorial representations. A short table of Stirling transform identities can be found in the Appendix in [5]. The entries in this table are written in terms of the (signed) Stirling numbers of the first kind $s(n,k)$ for which

$$\left[\begin{matrix} n \\ k \end{matrix} \right] = (-1)^{n-k} s(n,k) \tag{19}$$

and in terms of $s(n,k)$ formula (18) can be written in the form

$$\sum_{k=0}^{\infty} \frac{(-1)^k a_k}{z^{k+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{z(z+1)\ldots(z+n)} \left\{ \sum_{k=0}^{n} s(n,k)\, a_k \right\} \tag{20}$$

replacing a_k by $(-1)^k a_k$ and using (19).

The representations (13) follow from the identities

$$\sum_{k=0}^{n} \left[\begin{matrix} n \\ k \end{matrix} \right] = n!, \quad \sum_{k=0}^{n} \left[\begin{matrix} n \\ k \end{matrix} \right] k = n! H_n.$$

Example 4.1. In the same line, using the identity

$$\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] k^2 = n! \left(H_n + H_n^2 - H_n^{(2)} \right)$$

where $H_n^{(2)} = 1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2}$, we find from (18) the representation

$$\sum_{k=0}^{\infty} \frac{k^2}{z^{k+1}} = \sum_{n=0}^{\infty} \frac{n!}{z(z+1)\ldots(z+n)} \left(H_n + H_n^2 - H_n^{(2)} \right).$$

Computing the series on the left-hand side for $z > 1$ we come to the inverse factorial series

$$\frac{z+1}{(z-1)^3} = \sum_{n=0}^{\infty} \frac{n!}{z(z+1)\ldots(z+n)} \left(H_n + H_n^2 - H_n^{(2)} \right).$$

Example 4.2. In this example, we use the Stirling numbers of the second kind $S(n,m)$ which can be defined by the generating function

$$\sum_{n=0}^{\infty} S(n,m)\, x^n = \frac{x^m}{(1-x)(1-2x)\ldots(1-mx)}. \tag{21}$$

For any two integers $0 \le p \le n$ the following identity is true, [5, entry (A30)]

$$\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] S(k+1, p+1) = \frac{n!}{p!} \left(\begin{array}{c} n \\ p \end{array} \right).$$

According to Proposition 4.1 this implies the series identity

$$\sum_{k=0}^{\infty} \frac{S(k+1,p+1)}{z^{k+1}} = \frac{1}{p!} \sum_{n=0}^{\infty} \frac{n!}{z(z+1)\ldots(z+n)} \left(\begin{array}{c} n \\ p \end{array} \right). \tag{22}$$

In view of (21) with $x = 1/z$ we find the representation

$$\frac{1}{(z-1)(z-2)\ldots(z-p-1)} = \frac{1}{p!} \sum_{n=0}^{\infty} \frac{n!}{z(z+1)\ldots(z+n)} \left(\begin{array}{c} n \\ p \end{array} \right) \tag{23}$$

convergent for $Re\,(z) > p+1$. For $p = 0$ this is Equation (13) and for $p = 1$ this is (15).

Example 4.3. Let d_n be the sequence of Cauchy numbers of the second kind defined by the generating function

$$\frac{-x}{(1-x)\ln\,(1-x)} = \sum_{n=0}^{\infty} d_n \frac{x^n}{n!}$$

(see [9, p. 293]). The numbers d_n for $n = 0, 1, \ldots$, satisfy the identity, [5, entry (A42)]

$$\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] \frac{1}{k+1} = (-1)^n d_n$$

and from Proposition 4.1 it follows that

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)\, z^{k+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n d_n}{z(z+1)\ldots(z+n)}, \quad z > 1.$$

The series on the left is easy to recognize, and so we have the representation

$$-\ln\left(1 - \frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n d_n}{z(z+1)\ldots(z+n)}, \quad z > 1. \tag{24}$$

It is interesting to compare this to the representation

$$\ln\left(1 + \frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n c_n}{z\,(z+1)\ldots(z+n)}, \quad z > 0 \tag{25}$$

where c_n are the Cauchy numbers of the first kind defined by

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}$$

(see [6,9] , and [30, p. 144]). A similar representation can be found in [20, p. 244].

5. Asymptotic expansions

We can use Proposition 4.1 to obtain asymptotic series on the powers of z^{-n-1} for functions with inverse factorial representations. We will use this proposition to give proofs for the asymptotic representation of Nielsen's beta function $\beta(z)$ from Example 3.4, the incomplete gamma function $\gamma(z,x)$, and also $\psi'(z)$. We can use either Equation (18) or Equation (20). Setting

$$\beta(z) = \sum_{n=0}^{\infty} \frac{n!}{2^{n+1}} \frac{1}{z(z+1)\ldots(z+n)} = \sum_{n=0}^{\infty} \frac{1}{z(z+1)\ldots(z+n)} \left\{ \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] a_k \right\}$$

we solve for a_k from the equation

$$\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] a_k = \frac{n!}{2^{n+1}}.$$

Using the formulas for the Stirling transform of sequences (see the Appendix in [5]) we have

$$(-1)^n a_n = \sum_{k=0}^{n} S(n,k) k! \frac{(-1)^k}{2^{k+1}} = \frac{1}{2} \sum_{k=0}^{n} S(n,k) k! \left(-\frac{1}{2} \right)^k.$$

At this point, we involve the geometric polynomials

$$\omega_n(x) = \sum_{k=0}^{n} S(n,k) k! x^k \tag{26}$$

introduced and studied in [7]. It is known that

$$\omega_n \left(-\frac{1}{2} \right) = \frac{2}{n+1} (1 - 2^{n+1}) B_{n+1} = E_n(0)$$

where B_k are the Bernoulli numbers and $E_k(x)$ are the Euler polynomials. For the first equality, see the solution of [10, problem 6.76 p. 559]. For the second equality, see [3]. This way, we have

$$a_n = \frac{(-1)^n}{n+1} (1 - 2^{n+1}) B_{n+1} = \frac{(-1)^n}{2} E_n(0)$$

and we come to the following proposition.

Proposition 5.1. *Nielsen's beta function has the asymptotic series*

$$\beta(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (1 - 2^{n+1}) B_{n+1}}{n+1} \frac{1}{z^{n+1}} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n E_n(0) \frac{1}{z^{n+1}}. \tag{27}$$

(Compare to [2, Example 1, p. 145]). In the same way, we can use the representation of the incomplete gamma function $\gamma(z,\ x)$ from Example 3.5. Solving for the coefficients a_n from the equation

$$\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] a_k = x^n \tag{28}$$

we find that

$$a_n = (-1)^n \sum_{k=0}^{n} S(n,k) (-1)^k x^k. \tag{29}$$

This time we use the exponential polynomials

$$\varphi_n(x) = \sum_{k=0}^{n} S(n,k)\, x^k$$

(see [3, 7]). These polynomials appeared in the works of Ramanujan [2], E.T. Bell, J. Touchard, Gian-Carlo Rota, and many others. As mentioned in [3], these polynomials were used as early as 1843 by the prominent German mathematician Johann A. Grunert.

Equation (29) says that $a_n = (-1)^n \varphi_n(-x)$, so that we have

$$\gamma(z,\ x) = x^z e^{-x} \sum_{n=0}^{\infty} (-1)^n \varphi_n(-x) \frac{1}{z^{n+1}}$$

in accordance with [7, Proposition 5.1] proved there by a different method.

We notice that Equations (28) and (29) in view of Proposition 2.3 lead to the interesting formula

$$\sum_{n=0}^{\infty} \frac{x^n}{z(z+1)\dots(z+n)} = \sum_{k=0}^{\infty} (-1)^n \varphi_n(-x) \frac{1}{z^{k+1}} \tag{30}$$

which is Ramanujan's [2, Example (c), p.145] (see also [2, p. 47]). In a similar manner, using the representation from Example 3.3

$$\psi'(z) = \sum_{n=0}^{\infty} \frac{n!}{n+1} \frac{1}{z(z+1)\dots(z+n)} = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n+1} \frac{(-1)^n}{z(z+1)\dots(z+n)}.$$

We compare this to (20) and solve for a_k from the equation

$$\frac{(-1)^n n!}{n+1} = \sum_{k=0}^{n} s(n,k)\, a_k$$

to find

$$a_n = \sum_{k=0}^{n} S(n,k) \frac{(-1)^k k!}{k+1}$$

which are exactly the Bernoulli numbers [3], $a_n = B_n$. In view of (20), we find a new proof of the asymptotic expansion

$$\psi'(z) = \sum_{k=0}^{\infty} \frac{(-1)^k B_k}{z^{k-1}}.$$

6. A binomial formula with inverse factorials

In this short section, we prove a formula relating a binomial sum and an inverse factorial series with Stirling numbers of the first kind.

Proposition 6.1. *For any three integers* $p \geq 1$, $k \geq 0$, $m \geq 0$ *we have*

$$\sum_{j=0}^{m} \binom{m}{j} \frac{(-1)^j}{(j+p)^{k+1}} = (p-1)! \sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!(n+m+1)(n+m+2)\dots(n+m+p)}. \tag{31}$$

Proof. We start with the (exponential) generating function for the unsigned Stirling numbers of the first kind

$$(-1)^k \ln^k(1-x) = k! \sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^n}{n!}$$

where $|x| < 1$. We multiply both sides by $x^m(1-x)^{p-1}$ and integrate between 0 and 1 to get

$$(-1)^k \int_0^1 x^m (1-x)^{p-1} \ln^k(1-x) dx = k! \sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!} \int_0^1 x^{n+m}(1-x)^{p-1} dx. \tag{32}$$

In the integral on the left-hand side, we make the substitution $1 - x = e^{-t}$

$$(-1)^k \int_0^1 x^m (1-x)^{p-1} \ln^k(1-x) dx = \int_0^\infty (1-e^{-t})^m e^{-pt+t} t^k e^{-t} dt$$

$$= \sum_{j=0}^{m} \binom{m}{j} (-1)^j \int_0^\infty t^k e^{-(j+p)t} dt = k! \sum_{j=0}^{m} \binom{m}{j} (-1)^j \frac{1}{(j+p)^{k+1}}.$$

For the integral on the right-hand side, we use the representation

$$\int_0^1 x^{n+m}(1-x)^{p-1} dx = \frac{(p-1)!}{(n+m+1)(n+m+2)\dots(n+m+p)}$$

which comes from Euler's Beta function

$$B(u,v) = \int_0^1 x^{u-1}(1-x)^{v-1} dx = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

Equating both sides in (32), we come to the desired formula. $\square$

Example 6.1. In particular, by taking $p = 1$ in Equation (31), we have

$$\sum_{j=0}^{m} \binom{m}{j} \frac{(-1)^j}{(j+1)^{k+1}} = \sum_{n=0}^{\infty} \left[\begin{array}{c} n \\ k \end{array} \right] \frac{1}{n!\,(n+m+1)}.$$

For $m = 0$, $m = 1$, and $m = 2$ this gives correspondingly

$$1 = \sum_{n=0}^{\infty} \left[\begin{array}{c} n \\ k \end{array} \right] \frac{1}{n!(n+1)},$$

$$1 - \frac{1}{2^{k+1}} = \sum_{n=0}^{\infty} \left[\begin{array}{c} n \\ k \end{array} \right] \frac{1}{n!(n+2)}, \text{ and}$$

$$1 - \frac{1}{2^k} + \frac{1}{3^{k+1}} = \sum_{n=0}^{\infty} \left[\begin{array}{c} n \\ k \end{array} \right] \frac{1}{n!(n+3)}.$$

References

[1] V. Adamchik, On Stirling numbers and Euler sums, *J. Comput. Appl. Math.* **79** (1997) 119–130.

[2] B. C. Berndt, *Ramanujan's Notebooks*, Part 1, Springer, New York, 1983.

[3] K. N. Boyadzhiev, Close encounters with the Stirling numbers of the second kind, *Math. Mag.* **85** (2012) 252–266.

[4] K. N. Boyadzhiev Evaluation of some simple Euler-type series, *Far East J. Math. Sci.* **29** (2008) 1–11.

[5] K. N. Boyadzhiev, *Notes on the Binomial Transform, Theory and Table*, World Scientific, Singapore, 2018.

[6] K. N. Boyadzhiev, New series identities with Cauchy, Stirling, and harmonic numbers, and Laguerre polynomials, *J. Integer Seq.* **23** (2020) #20.11.7.

[7] K. N. Boyadzhiev, A Series transformation formula and related polynomials, *Int. J. Math. Math. Sci.* **2005** (2005) 3849–3866.

[8] K. N. Boyadzhiev, Louis Medina, and Victor Moll, The integrals in Gradshteyn and Ryzhik. Part 11: The incomplete beta function, *Sci. Ser. A Math. Sci.* **18** (2009) 61–75.

[9] L. Comtet, *Advanced Combinatorics*, Kluwer, Dordrecht, 1974.

[10] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics*, Addison-Wesley, New York, 1994.

[11] E. R. Hansen, *A Table of Series and Products*, Prentice-Hall, Englewood Cliffs, 1975.

[12] C. Jordan, *Calculus of Finite Differences*, Chelsea, New York, 1950.

[13] D. B. Karp, E. G. Prilepkina, An inverse factorial series for a general gamma ratio and related properties of the Norlund-Bernoulli polynomials, *J. Math. Sci.* **234** (2018) 680–696.

[14] R. J. Mathar, Series of reciprocal powers of k-almost primes, *arXiv*:0803.0900v3 [math.NT], (2009).

[15] L. A Medina, V. Moll, E. Rowland, Iterated primitives of logarithmic powers, *Int. J. Number Theory* **7** (2011) 623–634.

[16] L. M. Milne-Thomson, *The Calculus of Finite Differences*, MacMillan, London, 1951.

[17] N. Nielsen, *Die Gammafunktion*, Chelsea, New York, 1965.

[18] N. E. Nørlund, Sur les séries de facultes, *Acta Math.* **37** (1914) 327–387.

[19] N. E. Nørlund, *Leçons Sur les Séries d'Interpolation*, Gauthier-Villars, Paris, 1926.

[20] N. E. Nørlund, *Vorlesungen Uber Differenzenrechnung*, Chelsea, New York, 1954.

[21] A. P. Prudnikov, Y. A. Brychkov, O. I. Marichev, *Integrals and Series; Volume 1: Elementary Functions*, Gordon and Breach, New York, 1986.

[22] N. D. Obrechkoff, Summation by Euler's transform of the series of Dirichlet, factorial series, and the series of Newton, *Pliska Stud. Math.* **28** (2017) 7–127.

[23] I. J. Schwatt, *An Introduction to the Operations with Series*, Chelsea, New York, 1924.

[24] H. M. Srivastava, J. Choi, *Series Associated with Zeta and Related Functions*, Kluwer, Dordrecht, 2001.

[25] N. M. Temme, *Special Functions: An Introduction to the Classical Functions of Mathematical Physics*, Wiley, Chichester, 1996.

[26] I. Tweddle, *James Stirling's Methodus Differentialis: An Annotated Translation of Stirling's Text*, Springer, New York, 2003.

[27] G. N. Watson, The transformation of an asymptotic series into a convergent series of inverse factorials, *Rend. Circ. Mat. Palermo* **34** (1912) 41–88.

[28] W. R. Wasow, *Asymptotic Expansions for Ordinary Differential Equations*, Dover, New York, 2002.

[29] E. J. Weniger, Summation of divergent power series by means of factorial series, *Appl. Numer. Math.* **60** (2010) 1429–1441.

[30] E. T. Whittaker, G. N. Watson, *A Course of Modern Analysis*, Cambridge University Press, Cambridge, 1992.