On graphs with prescribed chromatic number and subset index

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Abstract

For a nontrivial graph $G$, a subset labeling of $G$ is a labeling of the vertices of $G$ with nonempty subsets of the set $[r] = \{1, 2, \ldots, r\}$ for a positive integer $r$ such that two vertices of $G$ have disjoint labels if and only if the vertices are adjacent. The subset index of $G$ is the minimum positive integer $r$ for which $G$ has such a subset labeling from the set $[r]$. Structures of graphs with prescribed subset index are investigated. It is shown that for every two integers $a$ and $b$ with $2 \leq a \leq b$, there exists a connected graph with chromatic number $a$ and subset index $b$.

Keywords: chromatic number; subset labeling; subset index.

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1. Introduction

While studying an article on quadratic forms, the German mathematician Martin Kneser became interested in the behavior of partitions of the family of $k$-element subsets of an $n$-element set (see [4]). For positive integers $k$ and $n$ with $n > 2k$, there exists a partition of the $k$-element subsets of the $n$-element set $[n] = \{1, 2, \ldots, n\}$ into $n - 2k + 2$ classes such that no pair of disjoint $k$-element subsets belong to the same class. Kneser asked the following question:

For positive integers $k$ and $n$ with $n > 2k$, does there exist a partition of the $k$-element subsets of $[n]$ into $n - 2k + 1$ classes such that no pair of disjoint $k$-element subsets belong to the same class?

Kneser [4] conjectured that such a partition was impossible. In 1978 Lovász [5] verified Kneser’s Conjecture using graph theory which led to a class of graphs called Kneser graphs.

For positive integers $k$ and $n$ with $n > 2k$, the Kneser graph $KG_{n,k}$ is that graph whose vertices are the $k$-element subsets of $[n]$ and where two vertices $(k$-element subsets) $A$ and $B$ are adjacent if and only if $A$ and $B$ are disjoint. Consequently, the Kneser graph $KG_{n,1}$ is the complete graph $K_n$, and the Kneser graph $KG_{5,2}$ is isomorphic to the Petersen graph. In terms of graph theory, Kneser’s Conjecture became:

Kneser’s Conjecture. There exists no $(n - 2k + 1)$-coloring of the Kneser graph $KG_{n,k}$.

Lovász [5] verified the conjecture by determining the chromatic number $\chi(KG_{n,k})$ of the Kneser graph $KG_{n,k}$ for positive integers $k$ and $n$ with $n > 2k$.

Theorem 1.1. For every two positive integers $k$ and $n$ with $n > 2k$,

$$\chi(KG_{n,k}) = n - 2k + 2.$$ 

In 1961, Paul Erdős, Chao Ko, and Richard Rado [3] determined the independence number $\alpha(KG_{n,k})$ of the Kneser graph $KG_{n,k}$ when $n > 2k$. This result is often referred to as the Erdős-Ko-Rado Theorem.

Theorem 1.2. For every two positive integers $k$ and $n$ with $n > 2k$,

$$\alpha(KG_{n,k}) = \binom{n - 1}{k - 1}.$$
In other words, if $G$ is an unlabeled graph isomorphic to the Kneser graph $KG_{n,k}$, then it is possible to label the vertices of $G$ with distinct $k$-element subsets of the set $[n] = \{1, 2, \ldots, n\}$ in such a way that two vertices of $G$ have disjoint labels if and only if the vertices are adjacent. This brings up the question of considering other familiar graphs $G$ and determining the existence of sets $[r]$ for positive integers $r$ such that the vertices of $G$ can be labeled with nonempty subsets of $[r]$, not necessarily of the same cardinality, so that the labels of two vertices are disjoint if and only if these two vertices of $G$ are adjacent. Such a labeling of a graph $G$ is called a subset labeling of $G$, a concept introduced in [1]. For a positive integer $r$, the power set of $[r]$, namely the set of all subsets of $[r]$, is denoted by $\mathcal{P}([r])$, while $\mathcal{P}^*([r])$ denotes the set of all nonempty subsets of $[r]$. Thus, $|\mathcal{P}^*([r])| = 2^r - 1$. That every graph has a subset labeling was established in [1]. It is useful to include an independent proof of this fact here.

**Theorem 1.3.** Every graph has a subset labeling.

**Proof.** We proceed by induction on the order $n$ of a graph. The result is immediate for small values of $n$, say $n \in \{2, 3, 4\}$. Assume that the statement is true for all graphs of order $n$ for an integer $n \geq 4$ and let $G$ be a graph of order $n + 1$. Let $v$ be a vertex of $G$ where $\deg_G v = p$ with $0 \leq p \leq n$ and let $G' = G - v$. Since $G'$ is a graph of order $n$, it follows by the induction hypothesis that $G'$ has a subset labeling $f'$, say $f' : V(G) \to \mathcal{P}^*([k])$ for some positive integer $k$. Let $V(G') = \{v_1, v_2, \ldots, v_n\}$, where either $v$ is an isolated vertex or $N_G(v) = \{v_1, v_2, \ldots, v_p\}$ with $1 \leq p \leq n$. Define a vertex labeling $f : V(G) \to \mathcal{P}^*([n + k + 1])$ of $G$ by

$$f(x) = \begin{cases} f'(v_i) \cup \{k + i\} & \text{if } x = v_i \text{ for } 1 \leq i \leq n \\ \{k + p + 1, k + p + 2, \ldots, k + n + 1\} & \text{if } x = v. \end{cases}$$

Since for vertices $x, y \in V(G)$, we have $f(x) \cap f(y) = \emptyset$ if and only if $xy \in E(G)$, it follows that $f$ is a subset labeling of $G$. \qed

The minimum positive integer $r$ for which a graph $G$ has such a subset labeling from the set $[r]$ is called the subset index of $G$, denoted by $\rho(G)$. We refer to the book [2] for graph theory notation and terminology not described in this paper. The subset index has been studied in [1, 6], where it has been determined for paths and cycles of small order.

**Theorem 1.4.** For $3 \leq n \leq 24$,

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In particular, the smallest positive integer $n$ for which $\rho(P_n) = 9$ is not known. The fact that $\rho(P_n) \leq \rho(P_{n+1})$ for every integer $n \geq 2$ is a consequence of the following fact [1]; while Theorem 1.5 shows that this is not the case for cycles.

**Proposition 1.1.** If $H$ is an induced subgraph of a graph $G$, then $\rho(H) \leq \rho(G)$.

**Theorem 1.5.** For $3 \leq n \leq 18$,

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2. On graphs with a given subset index

For a given nontrivial connected graph $G$, there is a class of graphs associated with $G$ that was constructed in [1] (by the means of the composition of graphs), all of which have the same subset index as $G$. More precisely, let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $H$ be the graph obtained from $G$ by replacing each vertex $v_i$ ($1 \leq i \leq n$) of $G$ with the empty graph $K_{q_i}$ of order $q_i$. Hence, the vertex set of $H$ is $\bigcup_{i=1}^n V(K_{q_i})$ and two vertices $u$ and $w$ of $H$ are adjacent in $H$ if $u \in V(K_{q_i})$ and $w \in V(K_{q_j})$ where $v_i, v_j \in E(G)$. The graph $H$ is referred to as the composition graph of $G$ and $K_{q_1}, K_{q_2}, \ldots, K_{q_n}$ and is often denoted by $G[K_{q_1}, K_{q_2}, \ldots, K_{q_n}]$. The following result was established in [1].

**Theorem 2.1.** For a nontrivial connected graph $G$ with vertex set $\{v_1, v_2, \ldots, v_n\}$, let $H$ be the composition graph of $G$ and $K_{q_1}, K_{q_2}, \ldots, K_{q_n}$. Then

$$\rho(H) = \rho(G).$$
By Theorem 2.1, a composition graph can be constructed from $P_3, K_3$, or the corona $\text{cor}(K_3)$ of $K_3$ (obtained from $K_3$ by adding a pendant edge at each vertex of $K_3$), all of which have subset index 3, by replacing each vertex $v_i$ by an empty graph, resulting in another graph having subset index 3. For example, for $F = \text{cor}(K_3)$ where $V(F) = \{v_1, v_2, \ldots, v_6\}$, let $\mathcal{H}$ be the set of all composition graphs $F[K_{q_1}, K_{q_2}, \ldots, K_{q_6}]$, where $q_1, q_2, \ldots, q_6$ are positive integers. Then $\rho(H) = 3$ for every graph $H \in \mathcal{H}$.

For a positive integer $n$, let $F_n$ be the graph of order $2^n - 1$ whose vertices are labeled with nonempty subsets of $[n]$ such that two vertices of $F_n$ have disjoint labels if and only if the vertices are adjacent. Thus, the vertex labeled $[v]$ is an isolated vertex of $F_n$. The graphs $F_3$ and $F_4$ are shown in Figure 1. (For simplicity, we write the set $[a]$ as $a$, $[a, b]$ as $ab$, $[a, b, c]$ as $abc$, and so on.) For $n \geq 2$, $F_n = G_n + K_1$, where $G_n$ is a connected graph of order $2^n - 2$. For example, $G_3 = \text{cor}(K_3)$.

Let $F_1 = \{K_1\}$ and for $n \geq 2$, let $F_n$ denote the set of all graphs that are isomorphic to an induced subgraph of $F_n$ but not to an induced subgraph of $F_{n-1}$. In particular, $G_n, F_n \in F_n$. Thus, $F_2 = \{K_2, K_1 + K_2\}$. If we let $A = \{\text{cor}(K_3), K_3, H_1, H_2, P_4\}$, where $H_1$ and $H_2$ are the graphs shown in Figure 2, and $B = \{G + K_1 : G \in A\}$, then $F_3 = A \cup B$.

A graph $H$ is called a magnified copy of a graph $G$ (or simply a magnified $G$) where $V(G) = \{v_1, v_2, \ldots, v_n\}$ if $H$ is isomorphic to a graph obtained from $G$ by replacing each vertex $v_i$ of $G$ by $K_q$, for some positive integer $q$, in a composition of $G$. If $H$ is a magnified $G$, then $\rho(H) = \rho(G)$. If $H \cong G$, then $H$ is a trivially magnified $G$. If the only graph of which $G$ is a magnified graph is $G$ itself, then $G$ is called a basis graph. The set $F_n$ consists of all graphs that are magnified graphs of the graphs in $F_n$. This set $F_n$ is therefore the set of all graphs $F$ with $\rho(F) = n$. In the definition of $F_n$, the term induced subgraph cannot be replaced by subgraph. For example, $P_3$ is a subgraph of $F_3$ but not an induced subgraph of $F_3$. We have seen that $\rho(P_3) \neq 3$; in fact, $\rho(P_3) = 4$. We can now describe all those graphs having subset index 2 or 3 (see [1]).

**Proposition 2.1.** A connected graph $G$ has subset index 2 if and only if $G$ is a complete bipartite graph.

**Proof:** Since $F_2 = K_2 + K_1$, the only induced subgraph of $F_2$ without isolated vertices is $K_2$. Therefore, the only nontrivial component of $G$ is a complete bipartite graph. 

**Corollary 2.1.** A graph $G$ has subset index 2 if and only if the only nontrivial component of $G$ is a complete bipartite graph.

**Proposition 2.2.** A connected graph $G$ has subset index 3 if and only if $G$ is a magnified $\text{cor}(K_3)$, a magnified $K_3$, a magnified $P_4$, a magnified $H_1$, or a magnified $H_2$, where $H_1$ and $H_2$ are shown in Figure 2. Consequently, every complete 3-partite graph has subset index 3.

**Proof:** Since $F_3 = \text{cor}(K_3) + K_1$, the only induced subgraphs of $F_3$ without isolated vertices (that are not induced subgraphs of $F_2$) are $\text{cor}(K_3)$, $K_3$, $H_1$, or $H_2$, $P_4$, which gives the desired result. Since a magnified $K_3$ is a complete 3-partite graph, every complete 3-partite graph has subset index 3. By Theorem 2.1, a magnified $P_4$ has subset index 3.
Corollary 2.2. A graph \(G\) has subset index 3 if and only if the only nontrivial component of \(G\) is a magnified \(\text{cor}(K_3)\), a magnified \(K_3\), a magnified \(P_n\), a magnified \(H_1\), or a magnified \(H_2\), where \(H_1\) and \(H_2\) are shown in Figure 2.

We now describe some properties of the graph \(F_n\) for a given positive integer \(n\).

**Theorem 2.2.** For each positive integer \(n\), \(\omega(F_n) = n\).

**Proof.** Let \(f : V(F_n) \to \mathcal{P}^+(\{n\})\) be a subset labeling of \(F_n\) and let \(S = \{u_1, u_2, \ldots, u_n\}\) be the set of those vertices \(u_i, 1 \leq i \leq n\), for which \(f(u_i) = \{i\}\). Since \(f(u_i) \cap f(u_j) = \emptyset\) for each pair \(i, j\) of integers with \(1 \leq i < j \leq n\), it follows that \(F_n[S] = K_n\) and so \(\omega(F_n) \geq n\). It remains to show that \(\omega(F_n) \leq n\). Let \(A = \{v_1, v_2, \ldots, v_{n+1}\}\) be an arbitrary set of \(n + 1\) vertices of \(F_n\). Suppose that \(f(v_i) = S_i \in \mathcal{P}^+(\{n\})\) for \(1 \leq i \leq n + 1\). Let \(a_i\) be the minimum element of \([n]\) belonging to \(S_i\) where \(1 \leq i \leq n + 1\). We may assume that \(a_i \leq a_{i+1}\) for \(1 \leq i \leq n\). Thus,

\[
1 \leq a_1 \leq a_2 \leq \cdots \leq a_{n+1} \leq n.
\]

Hence, there is an integer \(j\) with \(1 \leq j \leq n\) such that \(a_j = a_{j+1}\). Since \(a_j \in S_j \cap S_{j+1}\), it follows that \(S_j\) and \(S_{j+1}\) are not disjoint and so \(v_jv_{j+1} \notin E(F_n)\). Hence, \(F_n[A]\) is not a clique of \(F_n\). Therefore, \(\omega(F_n) \leq n\) and so \(\omega(F_n) = n\). \(\square\)

Since the subgraph of \(F_n\) induced by \(S\) is \(K_n\) and \(\omega(F_{n-1}) = n - 1\) by Theorem 2.2, it follows that \(K_n\) is not an induced subgraph of \(F_{n-1}\). Therefore, \(\rho(K_n) = n\).

**Proposition 2.3.** For each integer \(n \geq 2\), every complete \(n\)-partite graph has subset index \(n\).

**Proof.** We have seen that \(\omega(F_n) = n\) by Theorem 2.2. Thus, the complete graph \(K_n\) is an induced subgraph of \(F_n\) but not an induced subgraph of \(F_{n-1}\). Since a magnified \(K_n\) is a complete \(n\)-partite graph, it follows that every complete \(n\)-partite graph has subset index \(n\). \(\square\)

**Theorem 2.3.** For each positive integer \(n\), \(\chi(F_n) = n\).

**Proof.** The statement is immediate for \(n = 1, 2, 3\). Thus, we may assume that \(n \geq 4\). Since \(\omega(F_n) = n\) by Theorem 2.2, it follows that \(\chi(F_n) \geq n\). It remains to show that \(\chi(F_n) \leq n\). Let \(V(F_n) = \{v_1, v_2, \ldots, v_{2^n-1}\}\) and let \(f : V(F_n) \to \mathcal{P}^+(\{n\})\) be a subset labeling of \(F_n\) where \(f(v_i) = A_i\) for \(1 \leq i \leq 2^n - 1\). Next, let \(a_i\) be the minimum element of \([n]\) belonging to \(A_i\) where \(1 \leq i \leq 2^n - 1\). For \(j = 1, 2, \ldots, n\), let \(V_j = \{v_i : a_i = j\}\). Thus, \(|V_n| = 1\). If \(v_r\) and \(v_s\) are distinct vertices of \(V_j\) where \(1 \leq j \leq n\), then \(j \in f(v_r) \cap f(v_s) = A_r \cap A_s\) and so \(v_rv_s \notin E(F_n)\). Hence, \(V_j\) is a set of independent vertices of \(F_n\) for \(1 \leq j \leq n\). Assigning the color \(j\) to all vertices in \(V_j\) (\(1 \leq j \leq n\)) produces a proper \(n\)-coloring of \(F_n\). Therefore, \(\chi(F_n) \leq n\) and so \(\chi(F_n) = n\). \(\square\)

If \(G\) is a graph with \(\chi(G) = k\), then \(G\) is not a magnified graph of any subgraph of \(F_n\) where \(n < k\). Thus, \(\rho(G) \geq k\). For example, \(\chi(P_4) = 2\) but \(\rho(P_4) = 3\). Each of the graphs \(G_1\) and \(G_2\) in Figure 3 belongs to \(\mathcal{F}_4\) but not to \(\mathcal{F}_3\). Thus, \(\rho(G_i) = 4\) for \(i = 1, 2\), while \(\chi(G_i) = 3\) for \(i = 1, 2\).

![Figure 3: The graphs G1 and G2.](image)

Since \(\rho(C_{18}) = 7\), it follows that \(C_{18} \in \mathcal{F}_7\) but \(C_{18} \notin \mathcal{F}_6\). Since \(\rho(C_n) > 7\) for \(n \geq 19\), the induced cycle of greatest length in \(\mathcal{F}_7\) is \(C_{18}\).
3. Chromatic number and subset index

In this section, we investigate the relationship between the chromatic number \( \chi(G) \) and the subset index \( \rho(G) \) of a connected graph \( G \). The following result was obtained in [1]. We include a proof here for completion.

**Theorem 3.1.** If \( G \) is a nontrivial connected graph, then \( \chi(G) \leq \rho(G) \).

**Proof.** Let \( \rho(G) = k \geq 2 \) and let \( f : V(G) \to \mathcal{P}^*(\mathcal{C}) \) be a subset labeling of \( G \). Define the vertex coloring \( c : V(G) \to \mathcal{C} \) by
\[
c(x) = \min\{i \in [\mathcal{C}] : i \in f(x)\}.
\]
Let \( u \) and \( v \) be two adjacent vertices of \( G \). Since \( f(u) \cap f(v) = \emptyset \), it follows that \( c(u) \neq c(v) \). Thus, \( c \) is a proper coloring of \( G \) using at most \( k \) colors. Therefore, \( \chi(G) \leq k = \rho(G) \).

By Proposition 2.3, for each integer \( n \geq 2 \), every complete \( n \)-partite graph has subset index \( n \). Since \( \chi(G) = n \) for each such graph \( G \), it follows that \( \chi(G) = \rho(G) \). Furthermore, \( \chi(F_n) = \rho(F_n) \) for each integer \( n \geq 2 \) by Theorem 2.3. Therefore, there are infinite classes of connected graphs \( G \) for which \( \chi(G) = \rho(G) \). Hence, we have the following observation.

**Observation 3.1.** For each integer \( n \geq 2 \), there is a connected graph \( G \) such that \( \chi(G) = \rho(G) = n \).

In particular, \( \chi(K_n) = \rho(K_n) = n \).

On the other hand, the value of \( \rho(G) - \chi(G) \) can be arbitrarily large for a connected graph \( G \). The following result was established in [1].

**Theorem 3.2.** If \( n \geq 3 \), then \( \rho(P_n) \leq \rho(P_{n+1}) \leq \rho(P_n) + 1 \). Furthermore, \( \lim_{n \to \infty} \rho(P_n) = \infty \).

By Theorem 3.2, for each integer \( p \geq 2 \) there exists an integer \( n_p \) such that \( \rho(P_{n_p}) = p \). For an integer \( a \geq 2 \), let \( G \) be the graph obtained from the complete graph \( K_a \) of order \( a \) and the path \( P_{n_p} \) by joining a vertex of \( K_a \) and an end-vertex of \( P_{n_p} \). Then \( \chi(G) = a \). Since \( P_{n_p} \) is an induced subgraph of \( G \), it follows by Observation 1.1 that \( \rho(G) \geq p \). Since \( \lim_{n \to \infty} \rho(P_{n_p}) = \infty \), it follows that the value of \( \rho(G) - \chi(G) \) can be arbitrarily large for this graph \( G \). In fact, more can be said about the subset indices of this class of graphs. First, we introduce some additional definitions and notation. For integers \( a \geq 3 \) and \( \ell \geq 1 \), let \( F(a, \ell) \) be the graph obtained from the complete graph \( K_a \) and the path \( P_t \) by identifying a vertex of \( K_a \) with an end-vertex of \( P_t \). Thus, \( F(a, 1) = K_a \) and \( F(a, 2) \) is the graph obtained by adding a pendant edge at a vertex of \( K_a \). For \( \ell \geq 3 \), the graph \( F(a, \ell) \) is obtained by subdividing the pendant edge of \( F(a, 2) \) exactly \( \ell - 2 \) times. Then \( \rho(F(a, 1)) = \rho(F(a, 2)) = \rho(K_a) = a \).

Next, we show that \( \rho(F(a, 3)) = 2a - 1 \).

**Proposition 3.1.** For an integer \( a \geq 3 \), \( \rho(F(a, 3)) = 2a - 1 \).

**Proof.** Let \( G = F(a, 3) \), let \( V(K_a) = \{v_1, v_2, \ldots, v_a\} \), and let \( P_3 = (v, u, w) \), where \( G \) is obtained from \( K_a \) and the path \( P_3 \) by identifying the end-vertex \( w \) of \( P_3 \) and the vertex \( v_1 \) of \( K_a \), denoting the identified vertex by \( v_1 \) in \( G \). The subset labeling \( g : V(G) \to \mathcal{P}^*([2a - 1]) \) is defined by
\[
\begin{align*}
g(v_1) &= \{1\} \\
g(v_i) &= \{i, a + (i - 1)\} \text{ for } 2 \leq i \leq a \\
g(v) &= \{a\} \\
g(u) &= \{a + 1, 2a - 1\}.
\end{align*}
\]
Thus, \( \rho(G) \leq 2a - 1 \). Next, we show that \( \rho(G) \geq 2a - 1 \). Assume, to the contrary, that there is a subset labeling \( f : V(G) \to \mathcal{P}^*([2a - 2]) \) of \( G \). Then \( f(v_1) \cap f(v_j) = \emptyset \) for \( 1 \leq i < j \leq a \). Since \( f(v) \cap f(v_i) \neq \emptyset \) for \( 1 \leq i \leq a \), we may assume that \( i \in f(v) \cap f(v_i) \) for \( 1 \leq i \leq a \). Thus, \( \{a\} \subseteq f(v) \). Since \( f(u) \cap f(v_i) \neq \emptyset \) for \( 2 \leq i \leq a \), there is \( t_i \in [2a - 2] - [a] \) such that \( t_i \in f(u) \cap f(v_i) \) for \( 2 \leq i \leq a \). Thus, \( t_1, t_2, \ldots, t_a \) are \( a - 1 \) distinct elements in \( [a + 1, 2a - 2] \), which is impossible. Therefore, \( \rho(G) \geq 2a - 1 \) and so \( \rho(G) = 2a - 1 \).

For \( \ell \geq 4 \), we have the following.

**Theorem 3.3.** For integers \( a, \ell \) with \( a \geq 3 \) and \( \ell \geq 4 \),
\[
\rho(F(a, \ell)) \leq \rho(F(a, \ell + 1)) \leq \rho(F(a, \ell)) + 1.
\]
Since $F(a, \ell)$ is an induced subgraph of $F(a, \ell + 1)$, it follows that $\rho(F(a, \ell)) \leq \rho(F(a, \ell + 1))$. Thus, it remains to show that $\rho(F(a, \ell + 1)) \leq \rho(F(a, \ell)) + 1$. Let $G' = F(a, \ell)$ and $G = F(a, \ell + 1)$. Suppose that $\rho(G') = k$. Hence, there exists a subset labeling $g : V(G') \to \mathcal{P}^*(|k|)$ of $G'$. Let $v$ be the end-vertex of $G$ and let $(v, u, w, z)$ be a subpath of $P_t$. The labeling $f : V(G) \to \mathcal{P}^*(|k|)$ of $G$ is defined by

$$f(v) = [k + 1] - g(u),$$
$$f(z) = g(z) \cup \{k + 1\},$$
$$f(x) = g(x) \text{ if } x \in V(G) - \{v, z\}.$$

We show that $f$ is a subset labeling of $G$. Let $x$ and $y$ be two distinct vertices of $G$. If $x, y \in V(G')$, then $f(x) \cap f(y) = g(x) \cap g(y)$. Since $z$ is the only vertex of $G'$ that contains $k + 1$, it follows that $f(x) \cap f(y) = \emptyset$ if and only if $xy \in E(T')$. Thus, we may assume that $x = v$.

- If $y = u$, then $uv \in E(G)$ and $f(v) \cap f(u) = \emptyset$ by the definition of $f$.
- If $y = w$, then $f(w) = g(w) \subseteq [k] - g(u) \subseteq [k + 1] - g(u) = f(v)$ and so $f(v) \cap f(w) \neq \emptyset$.
- If $y = z$, then $f(z) \subseteq f(v)$ and $f(v) \cap f(z) \neq \emptyset$.
- If $y \in V(G') - \{u, v, w, z\}$, then $y$ is not adjacent to $w$. This implies that $f(y) \cap f(w) = g(y) \cap g(w) \neq \emptyset$. Since $f(w) \subseteq f(v)$, it follows that $f(y) \cap f(w) \subseteq f(y) \cap f(v)$. Therefore, $f(y) \cap f(v) \neq \emptyset$.

Hence, $f$ is a subset labeling of $G$ and so $\rho(G) \leq k + 1 = \rho(G') + 1$.

With the aid of Proposition 3.1 and Theorem 3.3, we have the following realization result.

**Theorem 3.4.** For each pair $a, b$ of integers with $a \geq 4$ and $b \geq 2a - 1$, there exists a connected graph $G$ with $\chi(G) = a$ and $\rho(G) = b$.

**Proof:** Since $F(a, 3)$ is an induced subgraph of $F(a, \ell)$, it follows that $\rho(F(a, \ell)) \geq 2a - 1$ by Proposition 3.1. Since

$$\lim_{\ell \to \infty} \rho(P_\ell) = \infty$$

and $P_\ell$ is an induced subgraph of $F(a, \ell)$, it follows that $\lim_{\ell \to \infty} \rho(F(a, \ell)) = \infty$. Thus, there is an integer $\ell_0$ such that $\rho(F(a, \ell_0)) = N > b$. It then follows by Theorem 3.3 that there is an integer $\ell$ such that $\rho(F(a, \ell)) = b$. Since $\chi(F(a, \ell)) = a$, the graph $F(a, \ell)$ has the desired property.

By Theorem 3.1, if $G$ is a connected graph with $\rho(G) = 3$, then $\chi(G) = 2$ or $\chi(G) = 3$. Therefore, there are graphs $G$ with $\rho(G) = 3$ for which $\chi(G) = k$ where $k = 2$ or $k = 3$. Similarly, if $G$ is a connected graph with $\rho(G) = 4$, then for each integer $k \in \{2, 3, 4\}$, there exists a connected graph $G$ with $\rho(G) = 4$ and $\chi(G) = k$. Furthermore, by Theorem 3.4, for each pair $a, b$ of integers with $a \geq 4$ and $b \geq 2a - 1$, there exists a connected graph $G$ with $\chi(G) = a$ and $\rho(G) = b$. We now establish a more general result.

First, we present some definitions and notation. For two vertex-disjoint graphs $G$ and $H$, the *join* $G \vee H$ has $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$. For integers $a \geq 3$ and $t \geq 1$, let $G_0 = K_{a-1} \vee K_t$ be the join of the complete graph $K_{a-1}$ and the empty graph $K_t$, where $V(K_{a-1}) = \{v_1, v_2, \ldots, v_{a-1}\}$ and $V(K_t) = \{u_1, u_2, \ldots, u_t\}$. Since $G_0$ is a magnified $K_a$, it follows that $\chi(G_0) = \rho(G_0) = a$. Let $G_i$ be the graph obtained by adding the pendant edge $u_1w_1$ at the vertex $v_1$ of $G_0$. For each integer $i$ with $2 \leq i \leq t$, let $G_i$ be the graph obtained by adding the pendant edge $u_iw_i$ at the vertex $u_i$ of $G_{i-1}$. The graph $G_i$ is shown in Figure 4. Equivalently, $G_{i-1} = G_i - w_i$ for $1 \leq i \leq t$. Since $G_{i-1}$ is an induced subgraph of $G_i$ for $1 \leq i \leq t$, it follows that

$$a = \rho(G_0) \leq \rho(G_1) \leq \rho(G_2) \leq \cdots \leq \rho(G_i).$$

Next, we show that $\rho(G_i)$ exceeds $\rho(G_{i-1})$ by at least 1 for $1 \leq i \leq t$.

**Theorem 3.5.** Let $a$ and $t$ be integers with $a \geq 3$ and $t \geq 1$. For $1 \leq i \leq t$,

$$\rho(G_{i-1}) \leq \rho(G_i) \leq \rho(G_{i-1}) + 1.$$
We show that $f$ is a subset labeling of $G$. Chartrand, E. Salehi, and P. Zhang

**Proof.** Suppose that $\rho(G_{i-1}) = k$ for some integer $k \geq a$ where $1 \leq i \leq t$. Then there is a subset labeling $g : V(G_{i-1}) \rightarrow \mathcal{P}^*(k)$ of $G_{i-1}$. The graph $G_i$ is obtained by adding the pendant edge $u_i w_i$ at the vertex $u_i$ of $G_{i-1}$. The graph $G_3$ where $3 < t$ is shown in Figure 5. We now extend the subset labeling $g$ of $G_{i-1}$ to a labeling $f : V(G_i) \rightarrow \mathcal{P}^*([k + 1])$ of $G_i$ by defining

$$
f(w_i) = [k + 1] - g(u_i)
$$

$$
f(u_j) = g(u_j) \cup \{k + 1\} \text{ for } 1 \leq j \leq t \text{ and } j \neq i
$$

$$
f(x) = g(x) \text{ if } x \neq u_j \text{ for } 1 \leq j \leq t \text{ and } j \neq i \text{ and } x \neq w_i.
$$

We show that $f$ is a subset labeling of $G_i$. To simplify notation, we let

$$U = \{u_1, u_2, \ldots, u_t\} - \{u_i\}.$$

Let $x$ and $y$ be two distinct vertices of $G_i$. If $x, y \in V(G_{i-1})$, then

$$f(x) \cap f(y) = \begin{cases} g(x) \cap g(y) & \text{if } x \notin U \text{ or } y \notin U \\ (g(x) \cap g(y)) \cup \{k + 1\} & \text{if } x, y \in U. \end{cases}$$

Thus, $f(x) \cap f(y) = \emptyset$ if and only if $xy \in E(G_{i-1})$. Thus, we may assume that $x = w_i$.

* If $y = u_i$, then $u_i w_i \in E(G_i)$ and $f(u_i) \cap f(w_i) = \emptyset$ by the definition of $f$.

* If $y = w_j$, where then $1 \leq j \leq i - 1$ and $i \geq 2$, say $y = w_1$, then $w_1 w_i \notin E(G_i)$. Since $w_1 w_i \notin E(G_{i-1})$, it follows that $g(w_1) \cap g(v_1) \neq \emptyset$. Because $g(w_1) \cap g(v_1) \subseteq [k] - g(u_i) \subseteq [k + 1] - g(u_i) = f(w_i)$, it follows that $f(w_1) \cap f(v_1) \neq \emptyset$.

* If $y \in U$, then $k + 1 \in f(w_i) \cap f(y)$ and so $f(w_i) \cap f(y) \neq \emptyset$.

* If $y \in V(K_{a-1})$, then $y$ is adjacent to $u_i$ and so $f(y) = g(y) \subseteq [k] - g(u_i) \subseteq [k + 1] - g(u_i) = f(w_i)$. Therefore, $f(y) \cap f(w_i) \neq \emptyset$.

Hence, $f$ is a subset labeling of $G_i$ and so $\rho(G_i) \leq k + 1 = \rho(G_{i-1}) + 1$. \qed

Figure 4: The graph $G_i$.

Figure 5: The graph $G_3$.
For integers $a \geq 3$ and $t \geq 1$, let $G_0$ and $G_t$ be defined as above. That is, $G_0 = K_{a-1} \vee \overline{K}_t$ is the join of the complete graph $K_{a-1}$ and the empty graph $\overline{K}_t$, where $V(K_{a-1}) = \{ v_1, v_2, \ldots, v_{a-1} \}$ and $V(\overline{K}_t) = \{ u_1, u_2, \ldots, u_t \}$. The graph $G_t$ is the graph obtained by adding the pendant edge $u_iw_i$ at the vertex $u_i$ of $K_{a-1} \vee \overline{K}_t$ for $1 \leq i \leq t$.

**Proposition 3.2.** $\lim_{t \to \infty} \rho(G_t) = \infty$.

**Proof.** Let $N \geq 2$ be an arbitrary integer. We show that $\rho(G_t) > N$ for all integers $t > 2^N$. Suppose that $\rho(G_t) = k$ and let $f : V(G_t) \to 2^{[k]}$ be a subset labeling of $G_t$. Since $f(v_i) \cap f(v_j) = \emptyset$ for $1 \leq i < j \leq a - 1$, we may assume that $i \in f(v_i)$ for $1 \leq i \leq a - 1$. Thus, $f(u_i)$ is a subset of $[k] - [a-1]$ for $1 \leq i \leq t$. Since $N(u_i) \neq N(u_j)$ for $1 \leq i < j \leq t$, it follows that $f(u_1), f(u_2), \ldots, f(u_t)$ are distinct subsets of $[k] - [a-1]$. This implies that $t \leq 2^{k-a+1} < 2^k$ and so $\log_2 t < k$. Thus, if $t > 2^N$, then $\log_2 t > N$ and so $\rho(G_t) = k > \log_2 t > N$. Therefore, $\lim_{t \to \infty} \rho(G_t) = \infty$. $\Box$

We are now prepared to prove that every two integers $a$ and $b$ with $2 \leq a \leq b$ are realizable as the chromatic number and subset index, respectively, of some connected graph.

**Theorem 3.6.** For every pair $a, b$ of integers with $2 \leq a \leq b$, there is a connected graph $G$ such that $\chi(G) = a$ and $\rho(G) = b$.

**Proof.** If $a = b \geq 2$, then let $G = K_a$. Then $\chi(G) = \rho(G) = a$ by Observation 3.1. If $a = 2$ and $b \geq 3$, then there exists an integer $n_0$ such that $\rho(P_{n_0}) = b$ by Theorem 3.2. Since $\chi(P_{n_0}) = 2$, the graph $G = P_{n_0}$ has the desired properties. Thus, we may assume that $3 \leq a < b$. For integers $a \geq 3$ and $t \geq 1$, again let $G_0$ be defined as above, namely $G_0 = K_{a-1} \vee \overline{K}_t$ is the join of the complete graph $K_{a-1}$ and the empty graph $\overline{K}_t$, where $V(K_{a-1}) = \{ v_1, v_2, \ldots, v_{a-1} \}$ and $V(\overline{K}_t) = \{ u_1, u_2, \ldots, u_t \}$.

Since $G_0$ is a magnified $K_a$, it follows that $\chi(G_0) = \rho(G_0) = a$. Let $G_1$ be the graph obtained by adding the pendant edge $u_1w_1$ at the vertex $u_1$ of $G_0$. For each integer $i$ with $2 \leq i \leq t$, let $G_i$ be the graph obtained by adding the pendant edge $u_iw_i$ at the vertex $u_i$ of $G_{i-1}$. By Proposition 3.2, $\lim_{t \to \infty} \rho(G_t) = \infty$. Thus, there is an integer $t_0$ such that $\rho(G_{t_0}) = N > b$. It then follows by Theorem 3.5 that there is an integer $i$ with $1 \leq i \leq t_0$ such that $\rho(G_i) = b$. Since $\chi(G_i) = a$, the graph $G_i$ has the desired property. $\Box$

As an illustration of Theorem 3.6 and its proof, we determine the subset indices of the graphs $G_0$, $G_1$, $G_2$, $G_3$, and $G_4$ for $a = 5$. Thus, $G_0 = K_4 \vee \overline{K}_4$ and the graph $G_4$ is shown in Figure 6. Hence, $G_{i-1} = G_i - w_i$ for $1 \leq i \leq 4$ and $\chi(G_i) = 5$ for $1 \leq i \leq 4$. We saw that $\chi(G_0) = \rho(G_0) = 5$.

![Figure 6: The graph $G_4$ for $a = 5$.](image)

**Example 3.1.** $\rho(G_1) = 6$, $\rho(G_2) = \rho(G_3) = 7$, and $\rho(G_4) = 8$.

**Proof.** First, we make some observations. For $1 \leq i \leq 4$, let $f_i$ be a subset labeling of $G_i$. Then $f_i(x) \neq f_i(y)$ for every two distinct vertices $x$ and $y$ of $G_i$. Furthermore, if $2 \leq i \leq 4$, then $|f_i(u_j)| \geq 2$ and $|f_i(v_j)| \geq 2$ for all integers $j$ with $2 \leq j \leq i$. A subset labeling $f_0 : V(G_0) \to 2^{[5]}$ of $G_0$ is defined by

$$f_0(v_i) = \{ j \} \text{ for } 1 \leq j \leq 4$$

$$f_0(u_i) = \{ 5 \} \text{ for } 1 \leq j \leq 4$$

For $1 \leq i \leq 4$, define a subset labeling $f_i$ of $G_i$ recursively as follows.

* The subset labeling $f_1 : V(G_1) \to 2^{[6]}$ of $G_1$ is defined in terms of $f_0$ by

$$f_1(v_i) = f_0(v_j) = \{ j \} \text{ for } 1 \leq j \leq 4$$
Thus, $\rho(G_1) \leq 6$. We show that $\rho(G_1) \neq 5$. Assume, to the contrary, that there is a subset labeling $g_1 : V(G_1) \to \mathcal{P}^*(\{5\})$ of $G_1$. We may assume that $j \in g_1(v_j)$ for $1 \leq j \leq 4$. This forces $g_1(v_j) = \{j\}$ and $g_1(u_j) = \{5\}$ for $1 \leq j \leq 4$. However then, $g_1(w_1) = [4]$ and so $g_1(w_1) \cap g_1(u_2) = \emptyset$, a contradiction. Therefore, $\rho(G_1) = 6$.

* The subset labeling $f_2 : V(G_2) \to \mathcal{P}^*(\{7\})$ of $G_2$ is defined in terms of $f_1$ by

$$
\begin{align*}
f_2(v_j) &= f_1(v_j) = \{j\} \text{ for } 1 \leq j \leq 4 \\
f_2(u_1) &= f_1(u_1) \cup \{7\} = \{5, 7\} \\
f_2(u_2) &= f_1(u_2) = \{5, 6\} \\
f_2(u_3) &= f_1(u_3) \cup \{7\} = \{5, 6, 7\} \text{ for } j = 3, 4 \\
f_2(w_j) &= [4] \cup \{5 + j\} \text{ for } j = 1, 2.
\end{align*}
$$

Thus, $\rho(G_2) \leq 7$. We show that $\rho(G_2) \neq 6$. Assume, to the contrary, that there is a subset labeling $g_2 : V(G_2) \to \mathcal{P}^*(\{6\})$ of $G_2$. We may assume that $j \in g_2(v_j)$ for $1 \leq j \leq 4$. Since $|g_2(u_j)| \geq 2$ for $1 \leq j \leq 4$, this forces $g_2(v_j) = \{j\}$ and $g_2(u_j) = \{5, 6\}$. However then, $g_2(w_1) = [4]$ and so $g_2(w_1) \cap g_2(u_2) = \emptyset$, a contradiction. Therefore, $\rho(G_2) = 7$.

* The subset labeling $f_3 : V(G_3) \to \mathcal{P}^*(\{7\})$ of $G_3$ is defined in terms of $f_2$ by $f_3(w_3) = [4] \cup \{6, 7\}$ and $f_3(x) = f_2(x)$ for $x \in V(G_2)$. Thus, $\rho(G_3) \leq 7$. Since $7 \leq \rho(G_2) \leq \rho(G_3)$, it follows that $\rho(G_3) = 7$.

* The subset labeling $f_4 : V(G_4) \to \mathcal{P}^*(\{8\})$ of $G_4$ is defined in terms of $f_3$ by

$$
\begin{align*}
f_4(w_4) &= [4] \cup \{8\} \\
f_4(u_j) &= f_3(u_j) \cup \{8\} \text{ for } 1 \leq j \leq 3 \\
f_4(x) &= f_3(x) \text{ if } x \notin \{u_1, u_2, u_3, u_4\}.
\end{align*}
$$

Thus, $\rho(G_4) \leq 8$. We show that $\rho(G_4) \neq 7$. Assume, to the contrary, that there is a subset labeling $g_4 : V(G_4) \to \mathcal{P}^*(\{7\})$ of $G_4$. We may assume that $j \in g_4(v_j)$ for $1 \leq j \leq 4$. Since $2 \leq |g_4(u_j)| \leq 3$ for $1 \leq j \leq 4$, this forces $g_4(v_j) = \{j\}$ and $g_4(u_j) \subseteq \{5, 6, 7\}$. Since there are only three 2-element subsets of $\{5, 6, 7\}$, it follows that $|f_4(u_j)| = 3$ for exactly one integer $j$ with $1 \leq j \leq 4$. We may assume that $g_4(u_1) = \{5, 6, 7\}$. This forces $f_4(w_1) = [4]$ and so $f_4(w_1) \cap g_4(u_2) = \emptyset$, a contradiction. Therefore, $\rho(G_4) = 8$.

\[\square\]

References