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Research Article

On graphs with prescribed chromatic number and subset index

Gary Chartrand[1], Ebrahim Salehi[2], Ping Zhang[1,*]

[1]*Department of Mathematics, Western Michigan University, Kalamazoo, Michigan 49008-5248, USA*
[2]*Department of Mathematical Sciences, University of Nevada Las Vegas, Las Vegas, Nevada 89154-4020, USA*

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Abstract

For a nontrivial graph G, a subset labeling of G is a labeling of the vertices of G with nonempty subsets of the set $[r] = \{1, 2, \ldots, r\}$ for a positive integer r such that two vertices of G have disjoint labels if and only if the vertices are adjacent. The subset index of G is the minimum positive integer r for which G has such a subset labeling from the set $[r]$. Structures of graphs with prescribed subset index are investigated. It is shown that for every two integers a and b with $2 \leq a \leq b$, there exists a connected graph with chromatic number a and subset index b.

Keywords: chromatic number; subset labeling; subset index.

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1. Introduction

While studying an article on quadratic forms, the German mathematician Martin Kneser became interested in the behavior of partitions of the family of k-element subsets of an n-element set (see [4]). For positive integers k and n with $n > 2k$, there exists a partition of the k-element subsets of the n-element set $[n] = \{1, 2, \ldots, n\}$ into $n - 2k + 2$ classes such that no pair of disjoint k-element subsets belong to the same class. Kneser asked the following question:

> *For positive integers k and n with $n > 2k$, does there exist a partition of the k-element subsets of $[n]$ into $n - 2k + 1$ classes such that no pair of disjoint k-element subsets belong to the same class?*

Kneser [4] conjectured that such a partition was impossible. In 1978 Lovász [5] verified Kneser's Conjecture using graph theory which led to a class of graphs called Kneser graphs.

For positive integers k and n with $n > 2k$, the *Kneser graph* $KG_{n,k}$ is that graph whose vertices are the k-element subsets of $[n]$ and where two vertices (k-element subsets) A and B are adjacent if and only if A and B are disjoint. Consequently, the Kneser graph $KG_{n,1}$ is the complete graph K_n, and the Kneser graph $KG_{5,2}$ is isomorphic to the Petersen graph. In terms of graph theory, Kneser's Conjecture became:

Kneser's Conjecture. *There exists no $(n - 2k + 1)$-coloring of the Kneser graph $KG_{n,k}$.*

Lovász [5] verified the conjecture by determining the chromatic number $\chi(KG_{n,k})$ of the Kneser graph $KG_{n,k}$ for positive integers k and n with $n > 2k$.

Theorem 1.1. *For every two positive integers k and n with $n > 2k$,*

$$\chi(KG_{n,k}) = n - 2k + 2.$$

In 1961, Paul Erdős, Chao Ko, and Richard Rado [3] determined the independence number $\alpha(KG_{n,k})$ of the Kneser graph $KG_{n,k}$ when $n > 2k$. This result is often referred to as the Erdős-Ko-Rado Theorem.

Theorem 1.2. *For every two positive integers k and n with $n > 2k$,*

$$\alpha(KG_{n,k}) = \binom{n-1}{k-1}.$$

*Corresponding author (ping.zhang@wmich.edu).

In other words, if G is an unlabeled graph isomorphic to the Kneser graph $KG_{n,k}$, then it is possible to label the vertices of G with distinct k-element subsets of the set $[n] = \{1, 2, \ldots, n\}$ in such a way that two vertices of G have disjoint labels if and only if the vertices are adjacent. This brings up the question of considering other familiar graphs G and determining the existence of sets $[r]$ for positive integers r such that the vertices of G can be labeled with nonempty subsets of $[r]$, not necessarily of the same cardinality, so that the labels of two vertices are disjoint if and only if these two vertices of G are adjacent. Such a labeling of a graph G is called a *subset labeling* of G, a concept introduced in [1]. For a positive integer r, the power set of $[r]$, namely the set of all subsets of $[r]$, is denoted by $\mathcal{P}([r])$, while $\mathcal{P}^*([r])$ denotes the set of all nonempty subsets of $[r]$. Thus, $|\mathcal{P}^*([r])| = 2^r - 1$. That every graph has a subset labeling was established in [1]. It is useful to include an independent proof of this fact here.

Theorem 1.3. *Every graph has a subset labeling.*

Proof. We proceed by induction on the order n of a graph. The result is immediate for small values of n, say $n \in \{2, 3, 4\}$. Assume that the statement is true for all graphs of order n for an integer $n \geq 4$ and let G be a graph of order $n + 1$. Let v be a vertex of G where $\deg_G v = p$ with $0 \leq p \leq n$ and let $G' = G - v$. Since G' is a graph of order n, it follows by the induction hypothesis that G' has a subset labeling f', say $f' : V(G) \to \mathcal{P}^*([k])$ for some positive integer k. Let $V(G') = \{v_1, v_2, \ldots, v_n\}$, where either v is an isolated vertex or $N_G(v) = \{v_1, v_2, \ldots, v_p\}$ with $1 \leq p \leq n$. Define a vertex labeling $f : V(G) \to \mathcal{P}^*([n + k + 1])$ of G by

$$f(x) = \begin{cases} f'(v_i) \cup \{k + i\} & \text{if } x = v_i \text{ for } 1 \leq i \leq n \\ \{k + p + 1, k + p + 2, \ldots, k + n + 1\} & \text{if } x = v. \end{cases}$$

Since for vertices $x, y \in V(G)$, we have $f(x) \cap f(y) = \emptyset$ if and only if $xy \in E(G)$, it follows that f is a subset labeling of G. □

The minimum positive integer r for which a graph G has such a subset labeling from the set $[r]$ is called the *subset index* of G, denoted by $\rho(G)$. We refer to the book [2] for graph theory notation and terminology not described in this paper. The subset index has been studied in [1,6], where it has been determined for paths and cycles of small order.

Theorem 1.4. *For* $3 \leq n \leq 24$,

n	$3 \leq n \leq 6$	7	$8 \leq n \leq 11$	$12 \leq n \leq 22$	23, 24
$\rho(P_n)$	$n - 1$	5	6	7	8

In particular, the smallest positive integer n for which $\rho(P_n) = 9$ is not known. The fact that $\rho(P_n) \leq \rho(P_{n+1})$ for every integer $n \geq 2$ is a consequence of the following fact [1]; while Theorem 1.5 shows that this is not the case for cycles.

Proposition 1.1. *If* H *is an induced subgraph of a graph* G*, then* $\rho(H) \leq \rho(G)$.

Theorem 1.5. *For* $3 \leq n \leq 18$,

n	3	4	5, 6	7	8	9	10	11	12, 13	14	$15 \leq n \leq 18$
$\rho(C_n)$	3	2	5	7	6	7	6	8	7	8	7

2. On graphs with a given subset index

For a given nontrivial connected graph G, there is a class of graphs associated with G that was constructed in [1] (by the means of the composition of graphs), all of which have the same subset index as G. More precisely, let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let H be the graph obtained from G by replacing each vertex v_i $(1 \leq i \leq n)$ of G with the empty graph $\overline{K}_{q_i}$ of order q_i. Hence, the vertex set of H is $\cup_{i=1}^{n} V(\overline{K}_{q_i})$ and two vertices u and w of H are adjacent in H if $u \in V(\overline{K}_{q_i})$ and $w \in V(\overline{K}_{q_j})$ where $v_i v_j \in E(G)$. The graph H is referred to as the *composition graph* of G and $\overline{K}_{q_1}, \overline{K}_{q_2}, \ldots, \overline{K}_{q_n}$ and is often denoted by $G[\overline{K}_{q_1}, \overline{K}_{q_2}, \ldots, \overline{K}_{q_n}]$. The following result was established in [1].

Theorem 2.1. *For a nontrivial connected graph* G *with vertex set* $\{v_1, v_2, \ldots, v_n\}$*, let* H *be the composition graph of* G *and* $\overline{K}_{q_1}, \overline{K}_{q_2}, \ldots, \overline{K}_{q_n}$*. Then*

$$\rho(H) = \rho(G).$$

By Theorem 2.1, a composition graph can be constructed from P_4, K_3, or the corona $\text{cor}(K_3)$ of K_3 (obtained from K_3 by adding a pendant edge at each vertex of K_3), all of which have subset index 3, by replacing each vertex v_i by an empty graph, resulting in another graph having subset index 3. For example, for $F = \text{cor}(K_3)$ where $V(F) = \{v_1, v_2, \ldots, v_6\}$, let $\mathcal{H}$ be the set of all composition graphs $F[\overline{K}_{q_1}, \overline{K}_{q_2}, \ldots, \overline{K}_{q_6}]$, where $q_1, q_2, \ldots, q_6$ are positive integers. Then $\rho(H) = 3$ for every graph $H \in \mathcal{H}$.

For a positive integer n, let F_n be the graph of order $2^n - 1$ whose vertices are labeled with nonempty subsets of $[n]$ such that two vertices of F_n have disjoint labels if and only if the vertices are adjacent. Thus, the vertex labeled $[n]$ is an isolated vertex of F_n. The graphs F_3 and F_4 are shown in Figure 1. (For simplicity, we write the set $\{a\}$ as a, $\{a, b\}$ as ab, $\{a, b, c\}$ as abc, and so on.) For $n \geq 2$, $F_n = G_n + K_1$, where G_n is a connected graph of order $2^n - 2$. For example, $G_3 = \text{cor}(K_3)$.

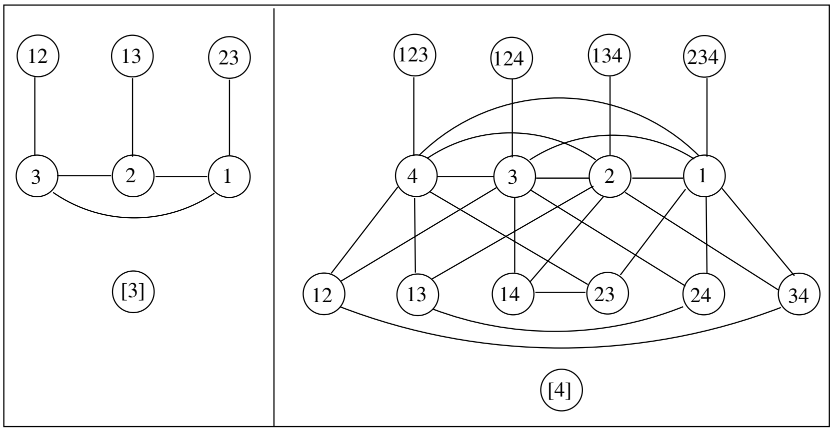

Figure 1: The graphs F_3 and F_4.

Let $\mathcal{F}_1 = \{K_1\}$ and for $n \geq 2$, let $\mathcal{F}_n$ denote the set of all graphs that are isomorphic to an induced subgraph of F_n but not to an induced subgraph of F_{n-1}. In particular, $G_n, F_n \in \mathcal{F}_n$. Thus, $\mathcal{F}_2 = \{K_2, K_1 + K_2\}$. If we let $A = \{\text{cor}(K_3), K_3, H_1, H_2, P_4\}$, where H_1 and H_2 are the graphs shown in Figure 2, and $B = \{G + K_1 : G \in A\}$, then $\mathcal{F}_3 = A \cup B$.

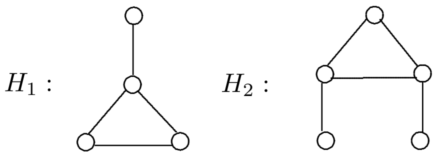

Figure 2: The graphs H_1 and H_2.

A graph H is called a *magnified copy of a graph* G (or simply a *magnified* G) where $V(G) = \{v_1, v_2, \ldots, v_n\}$ if H is isomorphic to a graph obtained from G by replacing each vertex v_i of G by $\overline{K}_{q_i}$ for some positive integer q_i in a composition of G. If H is a magnified G, then $\rho(H) = \rho(G)$. If $H \cong G$, then H is a *trivially magnified* G. If the only graph of which G is a magnified graph is G itself, then G is called a *basis graph*. The set $\mathcal{F}_n^*$ consists of all graphs that are magnified graphs of the graphs in $\mathcal{F}_n$. This set $\mathcal{F}_n^*$ is therefore the set of all graphs F with $\rho(F) = n$. In the definition of $\mathcal{F}_n$, the term *induced subgraph* cannot be replaced by *subgraph*. For example, P_5 is a subgraph of F_3 but not an induced subgraph of F_3. We have seen that $\rho(P_5) \neq 3$; in fact, $\rho(P_5) = 4$. We can now describe all those graphs having subset index 2 or 3 (see [1]).

Proposition 2.1. *A connected graph G has subset index 2 if and only if G is a complete bipartite graph.*

Proof. Since $F_2 = K_2 + K_1$, the only induced subgraph of F_2 without isolated vertices is K_2. Therefore, the only nontrivial component of G is a complete bipartite graph. □

Corollary 2.1. *A graph G has subset index 2 if and only if the only nontrivial component of G is a complete bipartite graph.*

Proposition 2.2. *A connected graph G has subset index 3 if and only if G is a magnified $\text{cor}(K_3)$, a magnified K_3, a magnified P_4, a magnified H_1, or a magnified H_2, where H_1 and H_2 are shown in Figure 2. Consequently, every complete 3-partite graph has subset index 3.*

Proof. Since $F_3 = \text{cor}(K_3) + K_1$, the only induced subgraphs of F_3 without isolated vertices (that are not induced subgraphs of F_2) are $\text{cor}(K_3)$, K_3, H_1, or H_2, P_4, which gives the desired result. Since a magnified K_3 is a complete 3-partite graph, every complete 3-partite graph has subset index 3. By Theorem 2.1, a magnified P_4 has subset index 3. □

Corollary 2.2. *A graph G has subset index 3 if and only if the only nontrivial component of G is a magnified $\mathrm{cor}(K_3)$, a magnified K_3, a magnified P_4, a magnified H_1, or a magnified H_2, where H_1 and H_2 are shown in Figure 2.*

We now describe some properties of the graph F_n for a given positive integer n.

Theorem 2.2. *For each positive integer n, $\omega(F_n) = n$.*

Proof. Let $f : V(F_n) \to \mathcal{P}^*([n])$ be a subset labeling of F_n and let $S = \{u_1, u_2, \ldots, u_n\}$ be the set of those vertices u_i, $1 \le i \le n$, for which $f(u_i) = \{i\}$. Since $f(u_i) \cap f(u_j) = \emptyset$ for each pair i, j of integers with $1 \le i < j \le n$, it follows that $F_n[S] = K_n$ and so $\omega(F_n) \ge n$. It remains to show that $\omega(F_n) \le n$. Let $A = \{v_1, v_2, \ldots, v_{n+1}\}$ be an arbitrary set of $n+1$ vertices of F_n. Suppose that $f(v_i) = S_i \in \mathcal{P}^*([n])$ for $1 \le i \le n+1$. Let a_i be the minimum element of $[n]$ belonging to S_i where $1 \le i \le n+1$. We may assume that $a_i \le a_{i+1}$ for $1 \le i \le n$. Thus,

$$1 \le a_1 \le a_2 \le \cdots \le a_{n+1} \le n.$$

Hence, there is an integer j with $1 \le j \le n$ such that $a_j = a_{j+1}$. Since $a_j \in S_j \cap S_{j+1}$, it follows that S_j and S_{j+1} are not disjoint and so $v_j v_{j+1} \notin E(F_n)$. Hence, $F_n[A]$ is not a clique of F_n. Therefore, $\omega(F_n) \le n$ and so $\omega(F_n) = n$. □

Since the subgraph of F_n induced by S is K_n and $\omega(F_{n-1}) = n-1$ by Theorem 2.2, it follows that K_n is not an induced subgraph of F_{n-1}. Therefore, $\rho(K_n) = n$.

Proposition 2.3. *For each integer $n \ge 2$, every complete n-partite graph has subset index n.*

Proof. We have seen that $\omega(F_n) = n$ by Theorem 2.2. Thus, the complete graph K_n is an induced subgraph of F_n but not an induced subgraph of F_{n-1}. Since a magnified K_n is a complete n-partite graph, it follows that every complete n-partite graph has subset index n. □

Theorem 2.3. *For each positive integer n, $\chi(F_n) = n$.*

Proof. The statement is immediate for $n = 1, 2, 3$. Thus, we may assume that $n \ge 4$. Since $\omega(F_n) = n$ by Theorem 2.2, it follows that $\chi(F_n) \ge n$. It remains to show that $\chi(F_n) \le n$. Let $V(F_n) = \{v_1, v_2, \ldots, v_{2^n-1}\}$ and let $f : V(F_n) \to \mathcal{P}^*([n])$ be a subset labeling of F_n where $f(v_i) = A_i$ for $1 \le i \le 2^n - 1$. Next, let a_i be the minimum element of $[n]$ belonging to A_i where $1 \le i \le 2^n - 1$. For $j = 1, 2, \ldots, n$, let $V_j = \{v_i : a_i = j\}$. Thus, $|V_n| = 1$. If v_r and v_s are distinct vertices of V_j where $1 \le j \le n$, then $j \in f(v_r) \cap f(v_s) = A_r \cap A_s$ and so $v_r v_s \notin E(F_n)$. Hence, V_j is a set of independent vertices of F_n for $1 \le j \le n$. Assigning the color j to all vertices in V_j ($1 \le j \le n$) produces a proper n-coloring of F_n. Therefore, $\chi(F_n) \le n$ and so $\chi(F_n) = n$. □

If G is a graph with $\chi(G) = k$, then G is not a magnified graph of any subgraph of F_n where $n < k$. Thus, $\rho(G) \ge k$. For example, $\chi(P_4) = 2$ but $\rho(P_4) = 3$. Each of the graphs G_1 and G_2 in Figure 3 belongs to $\mathcal{F}_4$ but not to $\mathcal{F}_3$. Thus, $\rho(G_i) = 4$ for $i = 1, 2$, while $\chi(G_i) = 3$ for $i = 1, 2$.

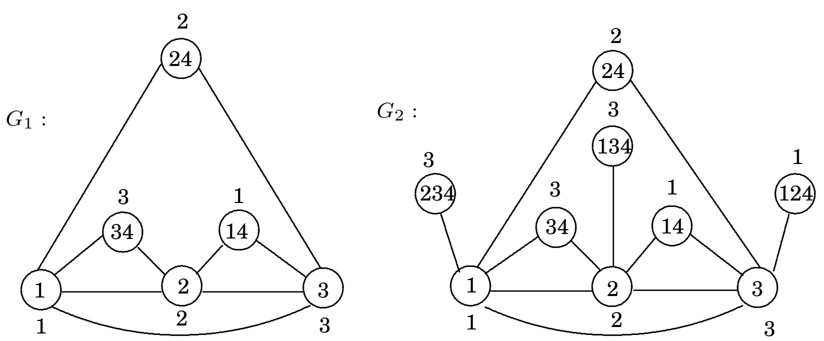

Figure 3: The graphs G_1 and G_2.

Since $\rho(C_{18}) = 7$, it follows that $C_{18} \in \mathcal{F}_7$ but $C_{18} \notin \mathcal{F}_6$. Since $\rho(C_n) > 7$ for $n \ge 19$, the induced cycle of greatest length in $\mathcal{F}_7$ is C_{18}.

3. Chromatic number and subset index

In this section, we investigate the relationship between the chromatic number $\chi(G)$ and the subset index $\rho(G)$ of a connected graph G. The following result was obtained in [1]. We include a proof here for completion.

Theorem 3.1. *If G is a nontrivial connected graph, then $\chi(G) \le \rho(G)$.*

Proof. Let $\rho(G) = k \ge 2$ and let $f : V(G) \to \mathcal{P}^*([k])$ be a subset labeling of G. Define the vertex coloring $c : V(G) \to [k]$ by

$$c(x) = \min\{i \in [k] : \ i \in f(x)\}.$$

Let u and v be two adjacent vertices of G. Since $f(u) \cap f(v) = \emptyset$, it follows that $c(u) \ne c(v)$. Thus, c is a proper coloring of G using at most k colors. Therefore, $\chi(G) \le k = \rho(G)$. □

By Proposition 2.3, for each integer $n \ge 2$, every complete n-partite graph has subset index n. Since $\chi(G) = n$ for each such graph G, it follows that $\chi(G) = \rho(G)$. Furthermore, $\chi(F_n) = \rho(F_n)$ for each integer $n \ge 2$ by Theorem 2.3. Therefore, there are infinite classes of connected graphs G for which $\chi(G) = \rho(G)$. Hence, we have the following observation.

Observation 3.1. *For each integer $n \ge 2$, there is a connected graph G such that*

$$\chi(G) = \rho(G) = n.$$

In particular, $\chi(K_n) = \rho(K_n) = n$.

On the other hand, the value of $\rho(G) - \chi(G)$ can be arbitrarily large for a connected graph G. The following result was established in [1].

Theorem 3.2. *If $n \ge 3$, then $\rho(P_n) \le \rho(P_{n+1}) \le \rho(P_n) + 1$. Furthermore, $\lim_{n\to\infty} \rho(P_n) = \infty$.*

By Theorem 3.2, for each integer $p \ge 2$ there exists an integer n_p such that $\rho(P_{n_p}) = p$. For an integer $a \ge 2$, let G be the graph obtained from the complete graph K_a of order a and the path P_{n_p} by joining a vertex of K_a and an end-vertex of P_{n_p}. Then $\chi(G) = a$. Since P_{n_p} is an induced subgraph of G, it follows by Observation 1.1 that $\rho(G) \ge p$. Since $\lim_{n\to\infty} \rho(P_n) = \infty$, it follows that the value of $\rho(G) - \chi(G)$ can be arbitrarily large for this graph G. In fact, more can be said about the subset indices of this class of graphs. First, we introduce some additional definitions and notation. For integers $a \ge 3$ and $\ell \ge 1$, let $F(a, \ell)$ be the graph obtained from the complete graph K_a and the path P_ℓ by identifying a vertex of K_a with an end-vertex of P_ℓ. Thus, $F(a, 1) = K_a$ and $F(a, 2)$ is the graph obtained by adding a pendant edge at a vertex of K_a. For $\ell \ge 3$, the graph $F(a, \ell)$ is obtained by subdividing the pendant edge of $F(a, 2)$ exactly $\ell - 2$ times. Then $\rho(F(a, 1) = \rho(F(a, 2) = \rho(K_a) = a$. Next, we show that $\rho(F(a, 3)) = 2a - 1$.

Proposition 3.1. *For an integer $a \ge 3$, $\rho(F(a, 3)) = 2a - 1$.*

Proof. Let $G = F(a, 3)$, let $V(K_a) = \{v_1, v_2, \ldots, v_a\}$, and let $P_3 = (v, u, w)$, where G is obtained from K_a and the path P_3 by identifying the end-vertex w of P_3 and the vertex v_1 of K_n, denoting the identified vertex by v_1 in G. The subset labeling $g : V(G) \to \mathcal{P}^*([2a - 1])$ is defined by

$$\begin{aligned} g(v_1) &= \{1\} \\ g(v_i) &= \{i, a + (i - 1)\} \text{ for } 2 \le i \le a \\ g(v) &= [a] \\ g((u) &= [a + 1, 2a - 1]. \end{aligned}$$

Thus, $\rho(G) \le 2a - 1$. Next, we show that $\rho(G) \ge 2a - 1$. Assume, to the contrary, that there is a subset labeling $f : V(G) \to \mathcal{P}^*([2a - 2])$ of G. Then $f(v_i) \cap f(v_j) = \emptyset$ for $1 \le i < j \le a$. Since $f(v) \cap f(v_i) \ne \emptyset$ for $1 \le i \le a$, we may assume that $i \in f(v) \cap f(v_i)$ for $1 \le i \le a$. Thus, $[a] \subseteq f(v)$. Since $f(u) \cap f(v_i) \ne \emptyset$ for $2 \le i \le a$, there is $t_i \in [2a - 2] - [a]$ such that $t_i \in f(u) \cap f(v_i)$ for $2 \le i \le a$. Thus, $t_2, t_3, \ldots, t_a$ are $a - 1$ distinct elements in $[a + 1, \ 2a - 2]$, which is impossible. Therefore, $\rho(G) \ge 2a - 1$ and so $\rho(G) = 2a - 1$. □

For $\ell \ge 4$, we have the following.

Theorem 3.3. *For integers a, ℓ with $a \ge 3$ and $\ell \ge 4$,*

$$\rho(F(a, \ell)) \le \rho(F(a, \ell + 1)) \le \rho(F(a, \ell)) + 1.$$

Proof. Since $F(a,\ell)$ is an induced subgraph of $F(a,\ell+1)$, it follows that $\rho(F(a,\ell)) \le \rho(F(a,\ell+1))$. Thus, it remains to show that $\rho(F(a,\ell+1)) \le \rho(F(a,\ell))+1$. Let $G' = F(a,\ell)$ and $G = F(a,\ell+1)$. Suppose that $\rho(G') = k$. Hence, there exists a subset labeling $g : V(G') \to \mathcal{P}^*([k])$ of G'. Let v be the end-vertex of G and let (v,u,w,z) be a subpath of P_ℓ. The labeling $f : V(G) \to \mathcal{P}^*([k])$ of G is defined by

$$\begin{aligned} f(v) &= [k+1] - g(u) \\ f(z) &= g(z) \cup \{k+1\} \\ f(x) &= g(x) \text{ if } x \in V(G) - \{v, z\}. \end{aligned}$$

We show that f is a subset labeling of G. Let x and y be two distinct vertices of G. If $x, y \in V(G')$, then $f(x) \cap f(y) = g(x) \cap g(y)$. Since z is the only vertex of G' that contains $k+1$, it follows that $f(x) \cap f(y) = \emptyset$ if and only if $xy \in E(T')$. Thus, we may assume that $x = v$.

- ⋆ If $y = u$, then $uv \in E(G)$ and $f(v) \cap f(u) = \emptyset$ by the definition of f.
- ⋆ If $y = w$, then $f(w) = g(w) \subseteq [k] - g(u) \subseteq [k+1] - g(u) = f(v)$ and so $f(v) \cap f(w) \neq \emptyset$.
- ⋆ If $y = z$, $k+1 \in f(v) \cap f(z)$ and so $f(v) \cap f(z) \neq \emptyset$.
- ⋆ If $y \in V(G') - \{u, w, z\}$, then y is not adjacent to w. This implies that $f(y) \cap f(w) = g(y) \cap g(w) \neq \emptyset$. Since $f(w) \subseteq f(v)$, it follows that $f(y) \cap f(w) \subseteq f(y) \cap f(v)$. Therefore, $f(y) \cap f(v) \neq \emptyset$.

Hence, f is a subset labeling of G and so $\rho(G) \le k+1 = \rho(G')+1$. □

With the aid of Proposition 3.1 and Theorem 3.3, we have the following realization result.

Theorem 3.4. *For each pair a, b of integers with $a \ge 4$ and $b \ge 2a-1$, there exists a connected graph G with $\chi(G) = a$ and $\rho(G) = b$.*

Proof. Since $F(a,3)$ is an induced subgraph of $F(a,\ell)$, it follows that $\rho(F(a,\ell)) \ge 2a-1$ by Proposition 3.1. Since

$$\lim_{\ell \to \infty} \rho(P_\ell) = \infty$$

and P_ℓ is an induced subgraph of $F(a,\ell)$, it follows that $\lim_{\ell\to\infty} \rho(F(a,\ell)) = \infty$. Thus, there is an integer ℓ_0 such that $\rho(F(a,\ell_0)) = N > b$. It then follows by Theorem 3.3 that there is an integer ℓ such that $\rho(F(a,\ell)) = b$. Since $\chi(F(a,\ell)) = a$, the graph $F(a,\ell)$ has the desired property. □

By Theorem 3.1, if G is a connected graph with $\rho(G) = 3$, then $\chi(G) = 2$ or $\chi(G) = 3$. Therefore, there are graphs G with $\rho(G) = 3$ for which $\chi(G) = k$ where $k = 2$ or $k = 3$. Similarly, if G is a connected graph with $\rho(G) = 4$, then for each integer $k \in \{2,3,4\}$, there exists a connected graph G with $\rho(G) = 4$ and $\chi(G) = k$. Furthermore, by Theorem 3.4, for each pair a, b of integers with $a \ge 4$ and $b \ge 2a-1$, there exists a connected graph G with $\chi(G) = a$ and $\rho(G) = b$. We now establish a more general result.

First, we present some definitions and notation. For two vertex-disjoint graphs G and H, the *join* $G \vee H$ has $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$. For integers $a \ge 3$ and $t \ge 1$, let $G_0 = K_{a-1} \vee \overline{K}_t$ be the join of the complete graph K_{a-1} and the empty graph $\overline{K}_t$, where $V(K_{a-1}) = \{v_1, v_2, \ldots, v_{a-1}\}$ and $V(\overline{K}_t) = \{u_1, u_2, \ldots, u_t\}$. Since G_0 is a magnified K_a, it follows that $\chi(G_0) = \rho(G_0) = a$. Let G_1 be the graph obtained by adding the pendant edge u_1w_1 at the vertex u_1 of G_0. For each integer i with $2 \le i \le t$, let G_i be the graph obtained by adding the pendant edge u_iw_i at the vertex u_i of G_{i-1}. The graph G_t is shown in Figure 4. Equivalently, $G_{i-1} = G_i - w_i$ for $1 \le i \le t$. Since G_{i-1} is an induced subgraph of G_i for $1 \le i \le t$, it follows that

$$a = \rho(G_0) \le \rho(G_1) \le \rho(G_2) \le \cdots \le \rho(G_t). \tag{1}$$

Next, we show that $\rho(G_i)$ exceeds $\rho(G_{i-1})$ by at most 1 for $1 \le i \le t$.

Theorem 3.5. *Let a and t be integers with $a \ge 3$ and $t \ge 1$. For $1 \le i \le t$,*

$$\rho(G_{i-1}) \le \rho(G_i) \le \rho(G_{i-1}) + 1.$$

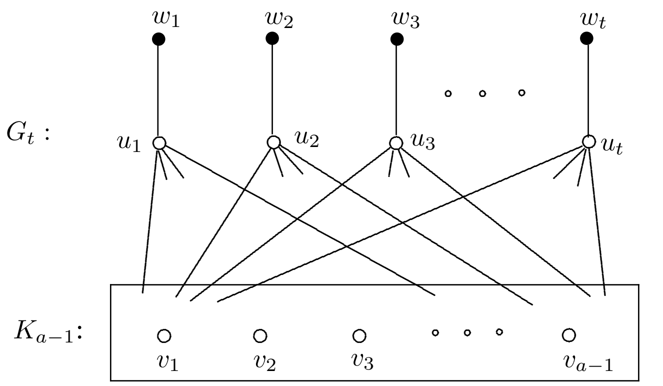

Figure 4: The graph G_t.

Proof. Suppose that $\rho(G_{i-1}) = k$ for some integer $k \geq a$ where $1 \leq i \leq t$. Then there is a subset labeling $g : V(G_{i-1}) \to \mathcal{P}^*([k])$ of G_{i-1}. The graph G_i is obtained by adding the pendant edge u_iw_i at the vertex u_i of G_{i-1}. The graph G_3 where $3 < t$ is shown in Figure 5. We now extend the subset labeling g of G_{i-1} to a labeling $f : V(G_i) \to \mathcal{P}^*([k+1])$ of G_i by defining

$$\begin{aligned} f(w_i) &= [k+1] - g(u_i) \\ f(u_j) &= g(u_j) \cup \{k+1\} \text{ for } 1 \leq j \leq t \text{ and } j \neq i \\ f(x) &= g(x) \text{ if } x \neq u_j \text{ for } 1 \leq j \leq t \text{ and } j \neq i \text{ and } x \neq w_i. \end{aligned}$$

We show that f is a subset labeling of G_i. To simplify notation, we let

$$U = \{u_1, u_2, \ldots, u_t\} - \{u_i\}.$$

Let x and y be two distinct vertices of G_i. If $x, y \in V(G_{i-1})$, then

$$f(x) \cap f(y) = \begin{cases} g(x) \cap g(y) & \text{if } x \notin U \text{ or } y \notin U \\ (g(x) \cap g(y)) \cup \{k+1\} & \text{if } x, y \in U. \end{cases}$$

Thus, $f(x) \cap f(y) = \emptyset$ if and only if $xy \in E(G_{i-1})$. Thus, we may assume that $x = w_i$.

- ⋆ If $y = u_i$, then $u_iw_i \in E(G_i)$ and $f(u_i) \cap f(w_i) = \emptyset$ by the definition of f.
- ⋆ If $y = w_j$, where then $1 \leq j \leq i-1$ and $i \geq 2$, say $y = w_1$, then $w_1w_i \notin E(G_i)$. Since $w_1v_1 \notin E(G_{i-1})$, it follows that $g(w_1) \cap g(v_1) \neq \emptyset$. Because $g(w_1) \cap g(v_1) \subseteq [k] - g(u_i) \subseteq [k+1] - g(u_i) = f(w_i)$, it follows that $f(w_1) \cap f(w_i) \neq \emptyset$.
- ⋆ If $y \in U$, then $k+1 \in f(w_i) \cap f(y)$ and so $f(w_i) \cap f(y) \neq \emptyset$.
- ⋆ If $y \in V(K_{a-1})$, then y is adjacent to u_i and so $f(y) = g(y) \subseteq [k] - g(u_i) \subseteq [k+1] - g(u_i) = f(w_i)$. Therefore, $f(y) \cap f(w_i) \neq \emptyset$.

Hence, f is a subset labeling of G_i and so $\rho(G_i) \leq k+1 = \rho(G_{i-1}) + 1$. □

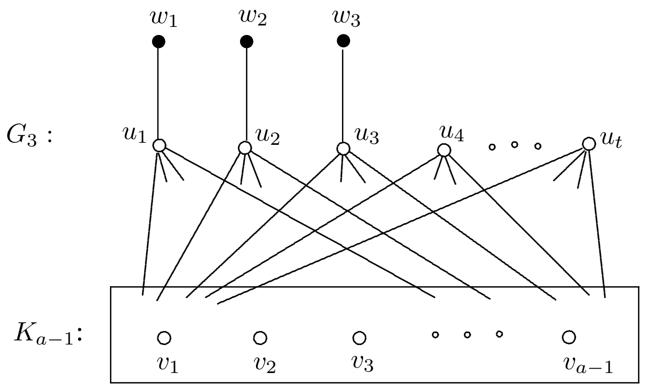

Figure 5: The graph G_3.

For integers $a \geq 3$ and $t \geq 1$, let G_0 and G_t be defined as above. That is, $G_0 = K_{a-1} \vee \overline{K}_t$ is the join of the complete graph K_{a-1} and the empty graph $\overline{K}_t$, where $V(K_{a-1}) = \{v_1, v_2, \dots, v_{a-1}\}$ and $V(\overline{K}_t) = \{u_1, u_2, \dots, u_t\}$. The graph G_t is the graph obtained by adding the pendant edge $u_i w_i$ at the vertex u_i of $K_{a-1} \vee \overline{K}_t$ for $1 \leq i \leq t$.

Proposition 3.2. $\lim_{t\to\infty} \rho(G_t) = \infty$.

Proof. Let $N \geq 2$ be an arbitrary integer. We show that $\rho(G_t) > N$ for all integers $t > 2^N$. Suppose that $\rho(G_t) = k$ and let $f : V(G_t) \to \mathcal{P}^*([k])$ be a subset labeling of G_t. Since $f(v_i) \cap f(v_j) = \emptyset$ for $1 \leq i < j \leq a-1$, we may assume that $i \in f(v_i)$ for $1 \leq i \leq a-1$. Thus, $f(u_i)$ is a subset of $[k] - [a-1]$ for $1 \leq i \leq t$. Since $N(u_i) \neq N(u_j)$ for $1 \leq i < j \leq t$, it follows that $f(u_1, f(u_2), \dots, f(u_t)$ are distinct subsets of $[k] - [a-1]$. This implies that $t \leq 2^{k-a+1} < 2^k$ and so $\log_2 t < k$. Thus, if $t > 2^N$, then $\log_2 t > N$ and so $\rho(G_t) = k > \log_2 t > N$. Therefore, $\lim_{n\to\infty} \rho(G_t) = \infty$. □

We are now prepared to prove that every two integers a and b with $2 \leq a \leq b$ are realizable as the chromatic number and subset index, respectively, of some connected graph.

Theorem 3.6. *For every pair a, b of integers with $2 \leq a \leq b$, there is a connected graph G such that $\chi(G) = a$ and $\rho(G) = b$.*

Proof. If $a = b \geq 2$, then let $G = K_a$. Then $\chi(G) = \rho(G) = a$ by Observation 3.1. If $a = 2$ and $b \geq 3$, then there exists an integer n_b such that $\rho(P_{n_b}) = b$ by Theorem 3.2. Since $\chi(P_{n_b}) = 2$, the graph $G = P_{n_b}$ has the desired properties. Thus, we may assume that $3 \leq a < b$. For integers $a \geq 3$ and $t \geq 1$, again let G_0 be defined as above, namely $G_0 = K_{a-1} \vee \overline{K}_t$ is the join of the complete graph K_{a-1} and the empty graph $\overline{K}_t$, where

$$V(K_{a-1}) = \{v_1, v_2, \dots, v_{a-1}\} \text{ and } V(\overline{K}_t) = \{u_1, u_2, \dots, u_t\}.$$

Since G_0 is a magnified K_a, it follows that $\chi(G_0) = \rho(G_0) = a$. Let G_1 be the graph obtained by adding the pendant edge $u_1 w_1$ at the vertex u_1 of G_0. For each integer i with $2 \leq i \leq t$, let G_i be the graph obtained by adding the pendant edge $u_i w_i$ at the vertex u_i of G_{i-1}. By Proposition 3.2, $\lim_{t\to\infty} \rho(G_t) = \infty$. Thus, there is an integer t_0 such that $\rho(G_{t_0}) = N > b$. It then follows by Theorem 3.5 that there is an integer i with $1 \leq i \leq t_0$ such that $\rho(G_i) = b$. Since $\chi(G_i) = a$, the graph G_i has the desired property. □

As an illustration of Theorem 3.6 and its proof, we determine the subset indices of the graphs G_0, G_1, G_2, G_3, and G_4 for $a = 5$. Thus, $G_0 = K_4 \vee \overline{K}_4$ and the graph G_4 is shown in Figure 6. Hence, $G_{i-1} = G_i - w_i$ for $1 \leq i \leq 4$ and $\chi(G_i) = 5$ for $1 \leq i \leq 4$. We saw that $\chi(G_0) = \rho(G_0) = 5$.

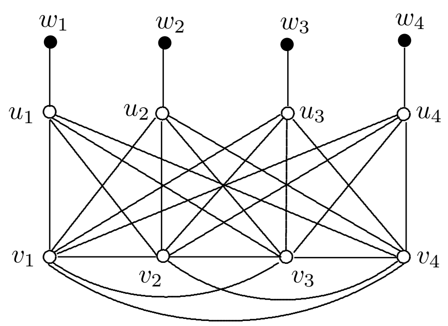

Figure 6: The graph G_4 for $a = 5$.

Example 3.1. $\rho(G_1) = 6$*,* $\rho(G_2) = \rho(G_3) = 7$*, and* $\rho(G_4) = 8$.

Proof. First, we make some observations. For $1 \leq i \leq 4$, let f_i be a subset labeling of G_i. Then $f_i(x) \neq f_i(y)$ for every two distinct vertices x and y of G_i. Furthermore, if $2 \leq i \leq 4$, then $|f_i(u_j)| \geq 2$ and $|f_i(w_j)| \geq 2$ for all integers j with $2 \leq j \leq i$. A subset labeling $f_0 : V(G_0) \to \mathcal{P}^*([5])$ of G_0 is defined by

$$\begin{aligned} f_0(v_j) &= \{j\} \text{ for } 1 \leq j \leq 4 \\ f_0(u_j) &= \{5\} \text{ for } 1 \leq j \leq 4. \end{aligned}$$

For $1 \leq i \leq 4$, define a subset labeling f_i of G_i recursively as follows.

⋆ The subset labeling $f_1 : V(G_1) \to \mathcal{P}^*([6])$ of G_1 is defined in terms of f_0 by

$$f_1(v_j) = f_0(v_j) = \{j\} \text{ for } 1 \leq j \leq 4$$

$$\begin{aligned} f_1(u_1) &= f_0(u_1) = \{5\} \\ f_1(u_j) &= f_0(u_i) \cup \{6\} = \{5,6\} \text{ for } 2 \le j \le 4 \\ f_1(w_1) &= [4] \cup \{6\}. \end{aligned}$$

Thus, $\rho(G_1) \le 6$. We show that $\rho(G_1) \ne 5$. Assume, to the contrary, that there is a subset labeling $g_1 : V(G_1) \to \mathcal{P}^*([5])$ of G_1. We may assume that $j \in g_1(v_j)$ for $1 \le j \le 4$. This forces $g_1(v_j) = \{j\}$ and $g_1(u_j) = \{5\}$ for $1 \le j \le 4$. However then, $g_1(w_1) = [4]$ and so $g_1(w_1) \cap g_1(u_2) = \emptyset$, a contradiction. Therefore, $\rho(G_1) = 6$.

⋆ The subset labeling $f_2 : V(G_2) \to \mathcal{P}^*([7])$ of G_2 is defined in terms of f_1 by

$$\begin{aligned} f_2(v_j) &= f_1(v_j) = \{j\} \text{ for } 1 \le j \le 4 \\ f_2(u_1) &= f_1(u_1) \cup \{7\} = \{5,7\} \\ f_2(u_2) &= f_1(u_2) = \{5,6\} \\ f_2(u_j) &= f_1(u_i) \cup \{7\} = \{5,6,7\} \text{ for } j = 3,4 \\ f_2(w_j) &= [4] \cup \{5+j\} \text{ for } j = 1,2. \end{aligned}$$

Thus, $\rho(G_2) \le 7$. We show that $\rho(G_2) \ne 6$. Assume, to the contrary, that there is a subset labeling $g_2 : V(G_2) \to \mathcal{P}^*([6])$ of G_2. We may assume that $j \in g_2(v_j)$ for $1 \le j \le 4$. Since $|g_2(u_j)| \ge 2$ for $1 \le j \le 4$, this forces $g_2(v_j) = \{j\}$ and $g_2(u_j) = \{5,6\}$. However then, $g_2(w_1) = [4]$ and so $g_2(w_1) \cap g_2(u_2) = \emptyset$, a contradiction. Therefore, $\rho(G_2) = 7$.

⋆ The subset labeling $f_3 : V(G_3) \to \mathcal{P}^*([7])$ of G_3 is defined in terms of f_2 by $f_3(w_3) = [4] \cup \{6,7\}$ and $f_3(x) = f_2(x)$ for $x \in V(G_2)$. Thus, $\rho(G_3) \le 7$. Since $7 \le \rho(G_2) \le \rho(G_3)$, it follows that $\rho(G_3) = 7$.

⋆ The subset labeling $f_4 : V(G_4) \to \mathcal{P}^*([8])$ of G_4 is defined in terms of f_3 by

$$\begin{aligned} f_4(w_4) &= [4] \cup \{8\} \\ f_4(u_j) &= f_3(u_i) \cup \{8\} \text{ for } 1 \le j \le 3 \\ f_4(x) &= f_3(x) \text{ if } x \notin \{u_1, u_2, u_3, w_4\}. \end{aligned}$$

Thus, $\rho(G_4) \le 8$. We show that $\rho(G_4) \ne 7$. Assume, to the contrary, that there is a subset labeling $g_4 : V(G_4) \to \mathcal{P}^*([7])$ of G_4. We may assume that $j \in g_4(v_j)$ for $1 \le j \le 4$. Since $2 \le |g_4(u_j)| \le 3$ for $1 \le j \le 4$, this forces $g_4(v_j) = \{j\}$ and so $g_4(u_j) \subseteq \{5,6,7\}$. Since there are only three 2-element subsets of $\{5,6,7\}$, it follows that $|f_4(u_j)| = 3$ for exactly one integer j with $1 \le j \le 4$. We may assume that $g_4(u_1) = \{5,6,7\}$. This forces $f_4(w_1) = [4]$ and so $f_4(w_1) \cap g_4(u_2) = \emptyset$, a contradiction. Therefore, $\rho(G_4) = 8$.

□

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