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Research Article

New integrals involving a function associated with Euler-Maclaurin summation formula*

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Abstract

The authors establish novel integrals involving a certain periodic function which is associated with the Euler-Maclaurin summation formula.

Keywords: Bernoulli polynomial; Euler-Maclaurin summation formula; integrals; Riemann zeta function.

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1. Motivation

Evaluations of some integrals involving the function

$$P(t) = t - [t] - \frac{1}{2} \tag{1}$$

(where $[t]$ is the integer part of t) play an important role in evaluations of certain series of the Riemann zeta function. Note that $P(t)$ is the first periodized Bernoulli polynomial, i.e., $P(t) = B_1(t - [t]) = \{t\} - \frac{1}{2}$, with $\{t\}$ being the fractional part of t. In their excellent book on zeta functions, Srivastava and Choi [3] analyze such integrals in Chapter 6.3, referring to Rainville [2], Zhang [4] and Choi et al. [1]. They report that

$$\int_1^\infty \frac{P(t)}{t}\, dt = \frac{1}{2}\ln(2\pi) - 1 \tag{2}$$

and

$$\int_1^\infty \frac{P(t)}{t^2}\, dt = \frac{1}{2} - \gamma, \tag{3}$$

where γ is the Euler-Mascheroni constant

$$\gamma = \lim_{n\to\infty}\Big(\sum_{k=1}^{n}\frac{1}{k} - \ln n\Big) = 0,5772156649...$$

For $n \geq 3$, the following general formula can be found in [3]

$$\int_1^\infty \frac{P(t)}{t^n}\, dt = \frac{n}{2(n-1)(n-2)} - \frac{1}{n-1}\zeta(n-1) \qquad (n \neq 1, 2), \tag{4}$$

with

$$\zeta(s) = \sum_{k=1}^{\infty}\frac{1}{k^s} \qquad (\Re(s) > 1) \tag{5}$$

being the Riemann zeta function with its analytical continuation

$$\zeta(s) = (1 - 2^{1-s})^{-1}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^s} \qquad (\Re(s) > 0, s \neq 1). \tag{6}$$

We also note that

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^s} = (1 - 2^{-s})\zeta(s), \tag{7}$$

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which will be used later. Two other examples reported by Choi et al. [1, Equations (29) and (30)] are the following interesting integrals

$$\int_1^\infty \frac{P(t)}{2t+1}\,dt = -\frac{3}{4} + \frac{1}{4}\ln(2) + \frac{1}{2}\ln(3) \tag{8}$$

and

$$\int_1^\infty P(t)\ln\left(1+\frac{1}{t}\right)dt = \ln(2) - \frac{3}{4}. \tag{9}$$

In this note, we complete the above analysis by providing closed-form evaluations for the analogue of (4) involving $|P(t)|/t^n$ as an integrand.

2. Main result

Theorem 2.1.

$$\int_1^\infty \frac{|P(t)|}{t^n}\,dt = \begin{cases} \ln\left(\frac{\pi}{4}\right) + \frac{1}{2}, & n=2 \\ 2\ln(2) - \frac{5}{4}, & n=3 \\ \frac{1}{2(n-2)(n-1)}\Big((2^n-2^3)\zeta(n-2)+n-2^n\Big), & n\ge 4. \end{cases} \tag{10}$$

Proof. Starting with

$$|P(t)| = \begin{cases} \{t\} - \frac{1}{2}, & k+\frac{1}{2} \le t < k+1 \\ -\{t\} + \frac{1}{2}, & k \le t < k+\frac{1}{2} \end{cases}$$

we have

$$\begin{aligned}
\int_1^\infty \frac{|P(t)|}{t^n}\,dt &= \sum_{k=1}^\infty \frac{1}{2}\left(\int_k^{k+1/2}\frac{1}{t^n}dt - \int_{k+1/2}^{k+1}\frac{1}{t^n}dt\right) + \sum_{k=1}^\infty\left(\int_{k+1/2}^{k+1}\frac{\{t\}}{t^n}dt - \int_k^{k+1/2}\frac{\{t\}}{t^n}dt\right) \\
&= \sum_{k=1}^\infty \frac{1}{2}\left(\int_k^{k+1/2}\frac{1}{t^n}dt - \int_{k+1/2}^{k+1}\frac{1}{t^n}dt\right) + \sum_{k=1}^\infty\left(\int_{1/2}^{1}\frac{\{t+k\}}{(t+k)^n}dt - \int_0^{1/2}\frac{\{t+k\}}{(t+k)^n}dt\right) \\
&= \sum_{k=1}^\infty \frac{1}{2}\left(\int_k^{k+1/2}\frac{1}{t^n}dt - \int_{k+1/2}^{k+1}\frac{1}{t^n}dt\right) + \sum_{k=1}^\infty\left(\int_{1/2}^{1}\frac{t}{(t+k)^n}dt - \int_0^{1/2}\frac{t}{(t+k)^n}dt\right).
\end{aligned}$$

In the last step, we have used that $\{t\}=t$ for $t\in[0,1)$. Since

$$\int \frac{x}{(x+k)^n}dx = \begin{cases} x - k\ln(x+k), & n=1 \\ \frac{x}{x+k} + \ln(x+k), & n=2 \\ -\frac{1}{(n-2)(n-1)}\frac{(n-1)x+k}{(x+k)^{n-1}}, & n\ge 3, \end{cases}$$

it is clear that we have to distinguish two separate cases. The case $n=2$ is established first. Integrating, rearranging, and simplifying gives

$$\begin{aligned}
\int_1^\infty \frac{|P(t)|}{t^2}\,dt &= \frac{1}{2}\sum_{k=1}^\infty\left(\frac{1}{k}+\frac{1}{k+1}\right) - 2\sum_{k=1}^\infty \frac{1}{2k+1} + \sum_{k=1}^\infty\left(\left(\frac{k}{k+1}-\frac{2k}{2k+1}\right)+\left(1-\frac{2k}{2k+1}\right)\right) \\
&\quad + \sum_{k=1}^\infty\left(\ln\left(\frac{k+1}{k+\frac{1}{2}}\right)+\ln\left(\frac{k}{k+\frac{1}{2}}\right)\right).
\end{aligned}$$

Now,

$$\begin{aligned}\Big(\frac{k}{k+1}-\frac{2k}{2k+1}\Big)+\Big(1-\frac{2k}{2k+1}\Big) &= \frac{-k}{(k+1)(2k+1)}+\frac{1}{2k+1}\\ &= \frac{1}{(k+1)(2k+1)}\\ &= \frac{2}{2k+1}-\frac{1}{k+1},\end{aligned}$$

and

$$\begin{aligned}\int_1^\infty \frac{|P(t)|}{t^2}\,dt &= \frac{1}{2}\sum_{k=1}^\infty\Big(\frac{1}{k}-\frac{1}{k+1}\Big)+\sum_{k=1}^\infty\Big(\ln\Big(1+\frac{1}{2k+1}\Big)+\ln\Big(1-\frac{1}{2k+1}\Big)\Big)\\ &= \frac{1}{2}+\ln\Big(\prod_{k=1}^\infty\Big(1-\frac{1}{(2k+1)^2}\Big)\Big).\end{aligned}$$

The statement follows from the known result

$$\prod_{k=1}^\infty\Big(1-\frac{1}{(2k+1)^2}\Big)=\frac{\pi}{4},$$

which is a consequence of Wallis' product

$$\pi = 2\prod_{k=1}^\infty \frac{2k}{(2k+1)}\frac{2k}{(2k-1)}.$$

To show the general case, we need (7) or equivalently

$$\sum_{k=1}^\infty \frac{1}{(2k+1)^{n-1}} = (1-2^{-(n-1)})\zeta(n-1)-1. \tag{11}$$

This gives for $n\geq 3$

$$\begin{aligned}\int_1^\infty \frac{|P(t)|}{t^n}\,dt &= \frac{\zeta(n-1)}{n-1}-\frac{1}{2(n-1)}-\frac{2^{n-1}}{n-1}\Big((1-2^{-(n-1)})\zeta(n-1)-1\Big)\\ &\quad+\frac{1}{(n-2)(n-1)}\sum_{k=1}^\infty\Big(\frac{(n-1)+2k}{(k+\frac{1}{2})^{n-1}}-\frac{(n-1)+k}{(k+1)^{n-1}}-\frac{1}{k^{n-2}}\Big)\\ &= \frac{\zeta(n-1)}{n-1}(2-2^{n-1})-\frac{1}{2(n-1)}(1-2^n)\\ &\quad+\frac{2^{n-1}}{(n-2)}\Big((1-2^{-(n-1)})\zeta(n-1)-1\Big)\\ &\quad+\frac{2^{n-1}}{(n-2)(n-1)}\sum_{k=1}^\infty\Big(\frac{1}{(2k+1)^{n-2}}-\frac{1}{(2k+1)^{n-1}}\Big)\\ &\quad-\frac{1}{(n-2)}(\zeta(n-1)-1)-\frac{1}{(n-2)(n-1)}(\zeta(n-2)-1)\\ &\quad+\frac{1}{(n-2)(n-1)}(\zeta(n-1)-1)-\frac{1}{(n-2)(n-1)}\zeta(n-2).\end{aligned}$$

After applying (11) a second time and simplifying the result, we get

$$\int_1^\infty \frac{|P(t)|}{t^n}\,dt = \frac{4}{(n-2)(n-1)}(2^{n-3}-1)\zeta(n-2)+\frac{n-2^n}{2(n-2)(n-1)}.$$

This is the equation stated in the theorem and the proof is complete for $n\geq 4$. The case $n=3$ is treated separately. In this case, we use (6) to get

$$\frac{4}{(n-2)(n-1)}(2^{n-3}-1)\zeta(n-2)=\frac{2^{n-1}}{(n-2)(n-1)}\sum_{k=1}^\infty\frac{(-1)^{k+1}}{k^{n-2}},$$

or specifically for $n=3$

$$\int_1^\infty \frac{|P(t)|}{t^3}\,dt = 2\sum_{k=1}^\infty\frac{(-1)^{k+1}}{k}-\frac{5}{4}.$$

The last sum is known to be equal to $\ln(2)$. □

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