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Research Article

On the remainder of a series representation for π^3

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Abstract

The main motivation for obtaining the results reported in the present paper comes from the following existing identity:

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)^3 16^n} = \frac{7\pi^3}{216}.$$

Let

$$R_n = \frac{7\pi^3}{216} - \sum_{k=0}^{n} \frac{\binom{2k}{k}}{(2k+1)^3 16^k}.$$

We obtain the asymptotic expansion of the remainder R_n as given below:

$$R_n \sim \frac{\binom{2n}{n}}{(2n+1)^3 16^n} \left\{ \frac{1}{3} - \frac{14}{9n} + \frac{59}{9n^2} - \frac{527}{18n^3} + \cdots \right\}, \quad n \to \infty.$$

We also give a recursive relation for determining the coefficients involved in the obtained expansion. Moreover, we establish an upper bound and a lower bound on the remainder R_n. As an application of the obtained bounds, we give an approximate value of π.

Keywords: asymptotic expansion; the constant π; inequality.

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1. Introduction

Throughout this paper, $\mathbb{N}$ represents the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. There exist many formulas in literature for the representation of π and a collection of such formulas can be found in [8,9]. Ramanujan [6] provided seventeen series for $1/\pi$. The following formula (known as Leibniz series)

$$\frac{\pi}{4} = \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1}$$

is due to Gottfried Wilhelm Leibniz. Recently, Alzer [2] presented a series representation for π which relates π to the partial sums of the Leibniz series,

$$T_k = \sum_{j=0}^{k} \frac{(-1)^j}{2j+1}, \quad k \in \mathbb{N}.$$

More precisely, Alzer [2] obtained the following result:

$$\pi = 32 \sum_{k=0}^{\infty} (-1)^{k+1} \frac{4k^2+8k+1}{(2k-1)(2k+1)(2k+3)(2k+5)} T_k^2.$$

For additional information on the topic under consideration, see [1,3–5].

Consider the following identity (see [7,10])

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)^3 16^n} = \frac{7\pi^3}{216} \tag{1}$$

and let

$$S_n = \sum_{k=0}^{n} \frac{\binom{2k}{k}}{(2k+1)^3 16^k} \tag{2}$$

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be the partial sums of the series (1). We now consider the remainder R_n defined as

$$R_n = \frac{7\pi^3}{216} - S_n = \sum_{k=n+1}^{\infty} \frac{\binom{2k}{k}}{(2k+1)^3 16^k}. \tag{3}$$

At the start of Section 2, using the Maple software, we derive the asymptotic expansion of the remainder R_n as given below:

$$R_n \sim \frac{\binom{2n}{n}}{(2n+1)^3 16^n} \left\{ \frac{1}{3} - \frac{14}{9n} + \frac{59}{9n^2} - \frac{527}{18n^3} + \cdots \right\}, \quad n \to \infty. \tag{4}$$

In Theorem 2.1, we give a recursive relation for determining the coefficients involved in the obtained expansion. We establish an upper bound and a lower bound on the remainder R_n in Theorem 2.2. In Remark 2.1, as an application of the obtained bounds, we give an approximate value of π.

We end this section with the remark that all the numerical calculations presented in this study are performed by using the Maple software for symbolic computations.

2. Results

Using the Maple software, we here give a derivation of (4). We find, as $n \to \infty$,

$$\frac{\binom{2(n+1)}{n+1}}{(2(n+1)+1)^3 16^{n+1}} = \frac{\binom{2n}{n}}{(2n+1)^3 16^n} \frac{(2n+1)^4}{8(2n+3)^3(n+1)} = \frac{\binom{2n}{n}}{(2n+1)^3 16^n} \left\{ \frac{1}{4} - \frac{7}{8n} + \frac{19}{8n^2} - \frac{45}{8n^3} + \dots \right\}, \tag{5}$$

$$\frac{\binom{2(n+2)}{n+2}}{(2(n+2)+1)^3 16^{n+2}} = \frac{\binom{2n}{n}}{(2n+1)^3 16^n} \frac{(2n+3)(2n+1)^4}{64(2n+5)^3(n+1)(n+2)} = \frac{\binom{2n}{n}}{(2n+1)^3 16^n} \left\{ \frac{1}{16} - \frac{7}{16n} + \frac{139}{64n^2} - \frac{585}{64n^3} + \dots \right\}, \tag{6}$$

$$\frac{\binom{2(n+3)}{n+3}}{(2(n+3)+1)^3 16^{n+3}} = \frac{\binom{2n}{n}}{(2n+1)^3 16^n} \left\{ \frac{1}{64} - \frac{21}{128n} + \frac{303}{256n^2} - \frac{3663}{512n^3} + \dots \right\}, \tag{7}$$

$$\frac{\binom{2(n+4)}{n+4}}{(2(n+4)+1)^3 16^{n+4}} = \frac{\binom{2n}{n}}{(2n+1)^3 16^n} \left\{ \frac{1}{256} - \frac{7}{128n} + \frac{265}{512n^2} - \frac{261}{64n^3} + \dots \right\}, \tag{8}$$

$$\frac{\binom{2(n+5)}{n+5}}{(2(n+5)+1)^3 16^{n+5}} = \frac{\binom{2n}{n}}{(2n+1)^3 16^n} \left\{ \frac{1}{1024} - \frac{35}{2048n} + \frac{205}{1024n^2} - \frac{7965}{4096n^3} + \dots \right\}, \tag{9}$$

and so on. In view of (5) to (9), we find the sums of the following series:

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} + \dots = \sum_{j=0}^{\infty} \frac{1}{4 \cdot 4^j} = \frac{1}{3},$$

$$\frac{7}{8} + \frac{7}{16} + \frac{21}{128} + \frac{7}{128} + \frac{35}{2048} + \dots = \sum_{j=0}^{\infty} \frac{7+7j}{8 \cdot 4^j} = \frac{14}{9},$$

$$\frac{19}{8} + \frac{139}{64} + \frac{303}{256} + \frac{265}{512} + \frac{205}{1024} + \dots = \sum_{j=0}^{\infty} \frac{76+139j+63j^2}{32 \cdot 4^j} = \frac{59}{9},$$

$$\frac{45}{8} + \frac{585}{64} + \frac{3663}{512} + \frac{261}{64} + \frac{7965}{4096} + \dots = \sum_{j=0}^{\infty} \frac{3(120+313j+270j^2+77j^3)}{64 \cdot 4^j} = \frac{527}{18}.$$

Summing the expansions (5) to (9) side by side, we obtain the asymptotic expansion (4).

Theorem 2.1. *The remainder R_n, defined by (3), has the following asymptotic expansion:*

$$R_n \sim \frac{\binom{2n}{n}}{(2n+1)^3 16^n} \sum_{k=0}^{\infty} \frac{r_k}{n^k} = \frac{\binom{2n}{n}}{(2n+1)^3 16^n} \left\{ \frac{1}{3} - \frac{14}{9n} + \frac{59}{9n^2} - \frac{527}{18n^3} + \cdots \right\} \tag{10}$$

as $n \to \infty$, where the coefficients r_k (with $k \in \mathbb{N}_0$) are given by the following recursive relation:

$$\begin{aligned} r_0 &= \frac{1}{3}, \\ r_k &= \frac{4}{3} \Bigg\{ (-1)^{k-1} \left[\frac{1}{8} - \left(\frac{10}{27} + \frac{2}{9}k + \frac{2}{27}k^2 \right) \left(\frac{3}{2} \right)^k \right] + \frac{1}{4} \sum_{p=0}^{k-1} r_p (-1)^{k-p} \binom{k-1}{k-p} \\ &\quad + \sum_{\ell=1}^{k} (-1)^{\ell-1} \left[\frac{1}{8} - \left(\frac{10}{27} + \frac{2}{9}\ell + \frac{2}{27}\ell^2 \right) \left(\frac{3}{2} \right)^{\ell} \right] \sum_{p=0}^{k-\ell} r_p (-1)^{k-\ell-p} \binom{k-\ell-1}{k-\ell-p} \Bigg\}, \end{aligned} \tag{11}$$

for $k \in \mathbb{N}$.

Proof. Let

$$T_n = \frac{\binom{2n}{n}}{(2n+1)^3 16^n} \sum_{k=0}^{\infty} \frac{r_k}{n^k},$$

where r_k (with $k \in \mathbb{N}_0$) are the real numbers to be determined. In view of (4), we can let $R_n \sim T_n$ and

$$\Delta R_n := R_{n+1} - R_n \sim T_{n+1} - T_n =: \Delta T_n, \qquad n \to \infty.$$

Direct computation yields

$$\begin{aligned} \Delta R_n &= -\frac{\binom{2(n+1)}{n+1}}{(2(n+1)+1)^3 16^{n+1}} = -\frac{\binom{2n}{n}}{(2n+1)^3 16^n} \frac{(2n+1)^4}{8(2n+3)^3(n+1)} \\ &= -\frac{\binom{2n}{n}}{(2n+1)^3 16^n} \left\{ \frac{1}{4} + \frac{1}{8n} \frac{1}{1+\frac{1}{n}} - \frac{1}{n} \frac{1}{1+\frac{3}{2n}} + \frac{1}{n^2} \frac{1}{(1+\frac{3}{2n})^2} - \frac{1}{2n^3} \frac{1}{(1+\frac{3}{2n})^3} \right\} \\ &= -\frac{\binom{2n}{n}}{(2n+1)^3 16^n} \Bigg\{ \frac{1}{4} + \frac{1}{8n} \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{n} \right)^k - \frac{1}{n} \sum_{k=0}^{\infty} (-1)^k \left(\frac{3}{2n} \right)^k \\ &\quad + \frac{1}{n^2} \sum_{k=0}^{\infty} (-1)^k (k+1) \left(\frac{3}{2n} \right)^k - \frac{1}{2n^3} \sum_{k=0}^{\infty} \frac{(-1)^k (k+2)(k+1)}{2} \left(\frac{3}{2n} \right)^k \Bigg\} \\ &= -\frac{\binom{2n}{n}}{(2n+1)^3 16^n} \sum_{k=0}^{\infty} \frac{a_k}{n^k}, \qquad n \to \infty, \end{aligned}$$

which can be written as

$$\left(\frac{\binom{2n}{n}}{(2n+1)^3 16^n} \right)^{-1} \Delta R_n = \sum_{k=0}^{\infty} (-a_k) n^{-k}, \tag{12}$$

where

$$a_0 = \frac{1}{4}, \quad a_k = (-1)^{k-1} \left[\frac{1}{8} - \left(\frac{10}{27} + \frac{2}{9}k + \frac{2}{27}k^2 \right) \left(\frac{3}{2} \right)^k \right], \qquad k \geq 1.$$

Also, we have

$$\begin{aligned} \Delta T_n &= \frac{\binom{2(n+1)}{n+1}}{(2(n+1)+1)^3 16^{n+1}} \sum_{k=0}^{\infty} \frac{r_k}{(n+1)^k} - \frac{\binom{2n}{n}}{(2n+1)^3 16^n} \sum_{k=0}^{\infty} \frac{r_k}{n^k}, \\ &= \frac{\binom{2n}{n}}{(2n+1)^3 16^n} \left\{ \frac{(2n+1)^4}{8(2n+3)^3(n+1)} \sum_{k=0}^{\infty} \frac{r_k}{(n+1)^k} - \sum_{k=0}^{\infty} \frac{r_k}{n^k} \right\} \\ &= \frac{\binom{2n}{n}}{(2n+1)^3 16^n} \left\{ \sum_{k=0}^{\infty} \frac{a_k}{n^k} \sum_{k=0}^{\infty} \frac{r_k}{(n+1)^k} - \sum_{k=0}^{\infty} \frac{r_k}{n^k} \right\} \end{aligned} \tag{13}$$

and

$$\sum_{k=0}^{\infty}\frac{r_k}{(n+1)^k}=\sum_{k=0}^{\infty}\frac{r_k}{n^k}\left(1+\frac{1}{n}\right)^{-k}=\sum_{k=0}^{\infty}\frac{r_k}{n^k}\sum_{j=0}^{\infty}\binom{-k}{j}\frac{1}{n^j}$$

$$=\sum_{k=0}^{\infty}\frac{r_k}{n^k}\sum_{j=0}^{\infty}(-1)^j\binom{k+j-1}{j}\frac{1}{n^j}=\sum_{k=0}^{\infty}\frac{b_k}{n^k},$$

where

$$b_0=r_0,\qquad b_k=\sum_{p=0}^{k}r_p(-1)^{k-p}\binom{k-1}{k-p},\qquad k\geq 1.$$

We obtain from (13), as $n\to\infty$,

$$\left(\frac{\binom{2n}{n}}{(2n+1)^3 16^n}\right)^{-1}\Delta T_n=\sum_{k=0}^{\infty}\frac{a_k}{n^k}\sum_{k=0}^{\infty}\frac{b_k}{n^k}-\sum_{k=0}^{\infty}\frac{r_k}{n^k}=\sum_{k=0}^{\infty}\left\{\sum_{j=0}^{k}a_j b_{k-j}-r_k\right\}n^{-k}. \tag{14}$$

Equating coefficients of the term n^{-k} (with $k\in\mathbb{N}_0$) on the right-hand sides of (12) and (14), we obtain

$$-a_k=\sum_{j=0}^{k}a_j b_{k-j}-r_k,\quad k\in\mathbb{N}_0. \tag{15}$$

For $k=0$, from (15) it follows that (noting $a_0=\frac{1}{4}$, $b_0=r_0$)

$$-\frac{1}{4}=a_0b_0-r_0=\frac{1}{4}r_0-r_0\Longrightarrow r_0=\frac{1}{3}.$$

For $k\in\mathbb{N}$, we obtain from (15) that

$$-a_k=\frac{1}{4}b_k+\sum_{\ell=1}^{k}a_\ell b_{k-\ell}-r_k,$$

$$-a_k=\frac{1}{4}\sum_{p=0}^{k}r_p(-1)^{k-p}\binom{k-1}{k-p}+\sum_{\ell=1}^{k}a_\ell b_{k-\ell}-r_k,$$

$$-a_k=\frac{1}{4}\sum_{p=0}^{k-1}r_p(-1)^{k-p}\binom{k-1}{k-p}+\frac{1}{4}r_k+\sum_{\ell=1}^{k}a_\ell b_{k-\ell}-r_k,$$

$$\frac{3}{4}r_k=a_k+\frac{1}{4}\sum_{p=0}^{k-1}r_p(-1)^{k-p}\binom{k-1}{k-p}+\sum_{\ell=1}^{k}a_\ell b_{k-\ell},$$

which implies that

$$r_k=\frac{4}{3}\left\{a_k+\frac{1}{4}\sum_{p=0}^{k-1}r_p(-1)^{k-p}\binom{k-1}{k-p}+\sum_{\ell=1}^{k}a_\ell b_{k-\ell}\right\}$$

$$=\frac{4}{3}\left\{(-1)^{k-1}\left[\frac{1}{8}-\left(\frac{10}{27}+\frac{2}{9}k+\frac{2}{27}k^2\right)\left(\frac{3}{2}\right)^k\right]+\frac{1}{4}\sum_{p=0}^{k-1}r_p(-1)^{k-p}\binom{k-1}{k-p}\right.$$

$$\left.+\sum_{\ell=1}^{k}(-1)^{\ell-1}\left[\frac{1}{8}-\left(\frac{10}{27}+\frac{2}{9}\ell+\frac{2}{27}\ell^2\right)\left(\frac{3}{2}\right)^\ell\right]\sum_{p=0}^{k-\ell}r_p(-1)^{k-\ell-p}\binom{k-\ell-1}{k-\ell-p}\right\},$$

which yields the required formula (11).

□

Next, by utilizing (11), we demonstrate how straightforwardly one can find r_k given in (10). We give the values of r_k for $k = 0, 1, 2, 3,$ as follows:

$$\begin{aligned}
r_0 &= \frac{1}{3}, \\
r_1 &= -\frac{7}{6} - \frac{7}{6}r_0 = -\frac{14}{9}, \\
r_2 &= \frac{19}{6} + \frac{19}{6}r_0 - \frac{3}{2}r_1 = \frac{59}{9}, \\
r_3 &= -\frac{15}{2} - \frac{15}{2}r_0 + \frac{14}{3}r_1 - \frac{11}{6}r_2 = -\frac{527}{18}.
\end{aligned}$$

Theorem 2.2. *For all $n \in \mathbb{N}$, the following inequality holds:*

$$L_n < R_n < U_n, \tag{16}$$

where

$$L_n = \frac{\binom{2n}{n}}{(2n+1)^3 16^n}\left(\frac{1}{3} - \frac{14}{9n}\right) \quad \textit{and} \quad U_n = \frac{\binom{2n}{n}}{(2n+1)^3 16^n}\left(\frac{1}{3} - \frac{14}{9n} + \frac{59}{9n^2}\right).$$

Proof. For $n \in \mathbb{N}$, let

$$\xi_n = R_n - L_n, \quad \eta_n = R_n - U_n.$$

Then, we have

$$\lim_{n\to\infty} \xi_n = 0, \quad \lim_{n\to\infty} \eta_n = 0.$$

To prove (16), it is sufficient to prove that the sequence $\{\xi_n\}$ is strictly decreasing and the sequence $\{\eta_n\}$ is strictly increasing. By elementary calculations, we have

$$\begin{aligned}
\xi_n - \xi_{n+1} &= \frac{\binom{2n}{n}}{(2n+1)^3 16^n}\frac{(2n+1)^4}{8(2n+1)^3(n+1)} + L_{n+1} - L_n \\
&= \frac{\binom{2n}{n}}{(2n+1)^3 16^n}\frac{(2n+1)^4}{8(2n+1)^3(n+1)} + \frac{\binom{2n}{n}}{(2n+1)^3 16^n}\frac{(2n+1)^4}{8(2n+1)^3(n+1)}\left(\frac{1}{3} - \frac{14}{9(n+1)}\right) - \frac{\binom{2n}{n}}{(2n+1)^3 16^n}\left(\frac{1}{3} - \frac{14}{9n}\right) \\
&= \frac{\binom{2n}{n}}{(2n+1)^3 16^n}\frac{1416n^4 + 5476n^3 + 8278n^2 + 5723n + 1512}{36n(2n+3)^3(n+1)^2} > 0
\end{aligned}$$

and

$$\begin{aligned}
\eta_n - \eta_{n+1} &= \frac{\binom{2n}{n}}{(2n+1)^3 16^n}\frac{(2n+1)^4}{8(2n+3)^3(n+1)} + U_{n+1} - U_n \\
&= \frac{\binom{2n}{n}}{(2n+1)^3 16^n}\frac{(2n+1)^4}{8(2n+3)^3(n+1)} + \frac{\binom{2n}{n}}{(2n+1)^3 16^n}\frac{(2n+1)^4}{8(2n+3)^3(n+1)}\left(\frac{1}{3} - \frac{14}{9(n+1)} + \frac{59}{9(n+1)^2}\right) \\
&\quad - \frac{\binom{2n}{n}}{(2n+1)^3 16^n}\left(\frac{1}{3} - \frac{14}{9n} + \frac{59}{9n^2}\right) \\
&= -\frac{\binom{2n}{n}}{(2n+1)^3 16^n}\frac{12648n^5 + 58868n^4 + 115486n^3 + 117159n^2 + 60696n + 12744}{72n^2(2n+3)^3(n+1)^3} < 0.
\end{aligned}$$

Thus, for $n \in \mathbb{N}$, we have $\xi_n > \xi_{n+1}$ and $\eta_n < \eta_{n+1}$.

□

Remark 2.1. We now apply (16) to give an approximate value of π. Write (16) as

$$\alpha_n < \pi < \beta_n, \tag{17}$$

where

$$\alpha_n = \left[\frac{216}{7}(L_n + S_n)\right]^{1/3} \quad \text{and} \quad \beta_n = \left[\frac{216}{7}(U_n + S_n)\right]^{1/3}.$$

For $n = 10$, we have

$$\alpha_{10} = 3.14159265358\cdots,$$
$$\beta_{10} = 3.14159265359\cdots.$$

From (17), we get an approximate value of π,

$$\pi \approx 3.1415926535.$$

The choice $n = 1000$ in (17) gives

$$\pi \approx 3.1415926535897932384626433832795028841971693993751058209749445923078 1.$$

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