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Research Article

Irregular domination graphs

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Abstract

A set S of vertices in a connected graph G is an irregular dominating set if the vertices of S can be labeled with distinct positive integers in such a way that for every vertex u of G, there is a vertex $v \in S$ such that the distance from u to v is the label assigned to v. If for every vertex $v \in S$, there is a vertex u of G such that v is the only vertex of S whose distance to u is the label of v, then S is a minimal irregular dominating set. A graph H is an irregular domination graph if there exists a graph G with a minimal irregular dominating set S such that H is isomorphic to the subgraph $G[S]$ of G induced by S. We determine all paths and cycles that are irregular domination graphs as well as the familiar graphs of ladders and prisms, which are Cartesian products of K_2 with paths and cycles, respectively. Other results and problems are also presented on this topic.

Keywords: domination; distance; irregular domination; irregular domination graph.

2020 Mathematics Subject Classification: 05C12, 05C38, 05C69, 05C76.

1. Introduction

In recent decades, domination in graphs has grown in popularity in graph theory. While this area evidently began with the work of Berge [3] in 1958 and Ore [12] in 1962, it did not become an active area of research until 1977 with the appearance of a survey paper by Cockayne and Hedetniemi [9]. Since then, a large number of variations and applications of domination have surfaced. A vertex v in a graph G is said to *dominate* a vertex u if either $u = v$ or u is adjacent to v in G. That is, v dominates itself and all vertices in its neighborhood $N(v)$. A set S of vertices in G is a *dominating set* of G if every vertex of G is dominated by some vertex in S. The minimum cardinality of a dominating set of G is the *domination number* $\gamma(G)$ of G.

In their 2022 book, Haynes, Hedetniemi, and Henning [10] presented the major results that have been obtained on what they refer to as the three core concepts of graph domination. One of these is the standard domination. A second is independent domination. A set S of vertices in a graph G is an *independent dominating set* of G if it is both an independent set (no two vertices in S are adjacent) and a dominating set of G. The third core concept is total domination, introduced by Cockayne, Dawes and Hedetniemi [8] in 1977. In total domination, a vertex u (totally) dominates a vertex v in a graph G if $uv \in E(G)$. That is, v does not dominate itself in total domination. A set S of vertices in a graph G is a *total dominating set* of G if every vertex v of G is totally dominated by some vertex of S. The minimum cardinality of a total dominating set of G is the *total domination number* $\gamma_t(G)$ of G. A graph G has a total dominating set if and only if G has no isolated vertices. The 2013 book by Henning and Yeo [11] deals exclusively with total domination in graphs.

As with some other types of domination, total domination can be described by means of distance in graphs. A vertex u in a graph G totally dominates a vertex v if the distance $d(u, v)$ from u to v is 1. Consequently, for a total dominating set S in a graph G without isolated vertices, one can think of assigning the label 1 to each vertex of S and assigning no label to all other vertices of G. Thus, if $u \in S$, then u is labeled 1, indicating that u (totally) dominates all vertices v of G for which $d(u, v) = 1$. Therefore, every vertex of G has distance 1 from at least one vertex of S. Looking at total domination in this manner led to a domination concept called irregular domination. (In the book [1] various "regularity" concepts are discussed, describing how this can lead to concepts that are in a sense opposite to these, resulting in "irregularity" concepts.) A set S of vertices in a nontrivial connected graph G is called an *irregular dominating set* if the vertices of S can be assigned distinct positive integers in such a way that for every vertex v in G, there is at least one vertex $u \in S$ such that $d(u, v)$ is the label $\ell(u)$ assigned to u. The vertex u is then said to *dominate* all vertices v for which $d(u, v) = \ell(u)$. This concept was introduced and studied in [5] and studied further in [2, 4, 7].

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When considering an irregular dominating set S in a connected graph G, it is assumed that the vertices of S have been assigned distinct positive integer labels, where the largest label is necessarily at least as large as $|S|$. As expected, some nontrivial connected graphs have irregular dominating sets and some do not. For example, the path P_3 of order 3 does not have an irregular dominating set but P_4 does. In fact, no connected graph of diameter at most 2 has an irregular dominating set. Furthermore, no vertex transitive graph has an irregular dominating set (see [7]). For a graph G possessing an irregular dominating set, the minimum cardinality of an irregular dominating set in G is the *irregular domination number* $\tilde{\gamma}(G)$ of G. If S is an irregular dominating set of a connected graph G but no proper subset T of S is an irregular dominating set of G (where the label of each vertex of T is that in S), then S is a *minimal irregular dominating set*. Equivalently, an irregular dominating set S in a graph is *minimal* if for every vertex $u \in S$, there is a vertex v of G such that v is dominated by u only. Figure 1 shows three different minimal irregular dominating sets of the path P_9 of order 9. The irregular dominating sets have cardinalities 6, 7, and 8 from top to bottom. It can be shown that $\tilde{\gamma}(P_9) = 6$. Since the diameter of P_9 is 8, these are the only possible cardinalities of minimal irregular dominating sets of P_9.

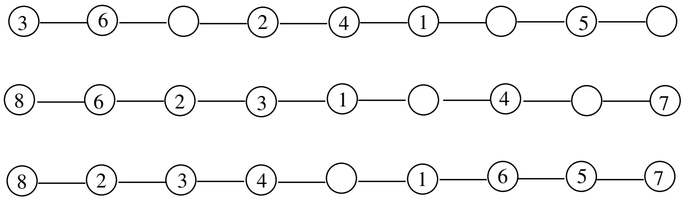

Figure 1: Three minimal irregular dominating sets of P_9.

2. Irregular domination graphs

For a graph G with a minimal irregular dominating set S, the subgraph $G[S]$ induced by S provides some information on the structural relationship among the vertices of S. This subgraph is called the *irregular domination subgraph* of G induced by the minimal irregular dominating set S. A graph H is an *irregular domination graph* if there exists a graph G with a minimal irregular dominating set S such that $G[S] \cong H$. For example, in the graph $G = P_4$ of Figure 2, the set $S = \{v_1, v_2, v_3\}$ is a minimal irregular dominating set of G, where the corresponding labeling assigns the label i to v_i for $1 \leq i \leq 3$. Since $G[S] \cong K_2 + K_1$, it follows that $K_2 + K_1$ is an irregular domination graph.

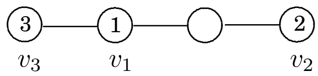

Figure 2: A minimal irregular dominating set in P_4.

In fact, $K_2 + K_1$ is the only irregular domination graph of order 3. To verify this, we first present two observations, the first of which is a consequence of a result obtained by Chartrand, Henning, and Schultz in [6].

Observation 2.1. *If G is a connected graph with an irregular dominating set, then $\tilde{\gamma}(G) \geq 3$. Furthermore, if S is an irregular dominating set of cardinality 3 in G, then the three vertices of S are labeled by $1, 2, 3$.*

Observation 2.2. *No two vertices in an irregular dominating set of a connected graph dominate each other.*

Proposition 2.1. *The graph $K_2 + K_1$ is the only irregular domination graph of order 3.*

Proof. That $K_2 + K_1$ is an irregular domination graph of order 3 is shown in Figure 2. Assume, to the contrary, that there is an irregular domination graph H of order 3 such that $H \not\cong K_2 + K_1$. Then there exists a graph G with a minimal irregular dominating set $S = \{u, v, w\}$ such that $G[S] \cong H$. We may assume, by Observation 2.1, that S has an irregular dominating labeling f such that $f(u) = 1$, $f(v) = 2$, and $f(w) = 3$. First, suppose that $H \cong \overline{K}_3$. Then no vertex of S is dominated by u. Thus, v and w must dominate each other, which is impossible by Observation 2.2. Next, suppose that H is connected. Hence, the distance between every two vertices of S in G is either 1 or 2. Therefore, no vertex of S is dominated by w and so u and v must dominate each other. Once again, this is impossible by Observation 2.2. □

We mentioned earlier that no graph of diameter at most 2 has an irregular dominating set. We have the following corresponding result.

Proposition 2.2. *No connected graph of diameter at most 2 is an irregular domination graph.*

Proof. Assume, to the contrary, that there exists a connected graph H with $\text{diam}(H) \le 2$ that is an irregular domination graph. Then there exists a graph G with a minimal irregular dominating set S such that $G[S] \cong H$. Since $K_2 + K_1$ is the only irregular domination graph of order 3, it follows that the order n of H is at least 4 and so $|S| \ge 4$. Since the distance between every two vertices of S in G is 1 or 2, every vertex of S must be dominated by a vertex of S labeled 1 or 2. Because no vertex of S dominates itself, there are two vertices $u, v \in S$ such that u is labeled 1 and v is labeled 2. However then, u and v must dominate each other, a contradiction by Observation 2.2. □

By Proposition 2.2, there is no irregular domination graph of order $n \ge 3$ having a vertex of degree $n-1$. There is, however, for each pair Δ, n of integers with $0 \le \Delta \le n-2$ and $n \ge 3$, an irregular domination graph of order n having maximum degree Δ. This is a consequence of the following result.

Theorem 2.1. *If H is a graph of order 4 or more having an isolated vertex, then H is an irregular domination graph.*

Proof. Let H be a graph of order 4 or more having an isolated vertex. Then $H = F + K_1$, where the order n of F is at least 3. If $n = 3$, then $F \in \{\overline{K}_3, P_2 + K_1, P_3, K_3\}$. For each graph in Figure 3, a minimal irregular dominating labeling is also shown in that figure. Thus, if F is one of $\overline{K}_3$, $P_2 + K_1$, P_3, K_3, then $F + K_1$ is an irregular domination graph. Hence, we may assume that F is a graph of order $n \ge 4$ and show that there is a graph G_n having a minimal irregular dominating set S_n such that $G_n[S_n] = F + K_1$. We consider two cases.

Case 1. *$F = \overline{K_n}$ is the empty graph of order n.* Then $F + K_1 = \overline{K}_{n+1}$ is the empty graph of order $n+1$. For $n = 4, 5, 6, 7$, the graphs G_n shown in Figure 4 have a minimal irregular dominating set S_n (also shown in Figure 4 for each graph) where $G[S_n] = \overline{K}_{n+1}$.

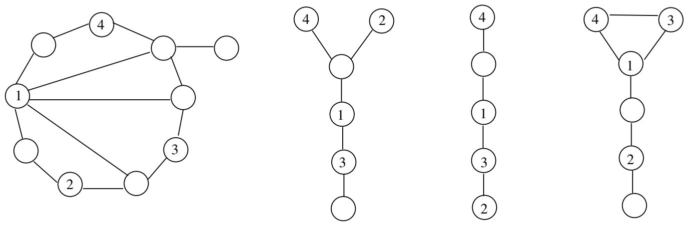

Figure 3: A step in the proof of Proposition 2.1.

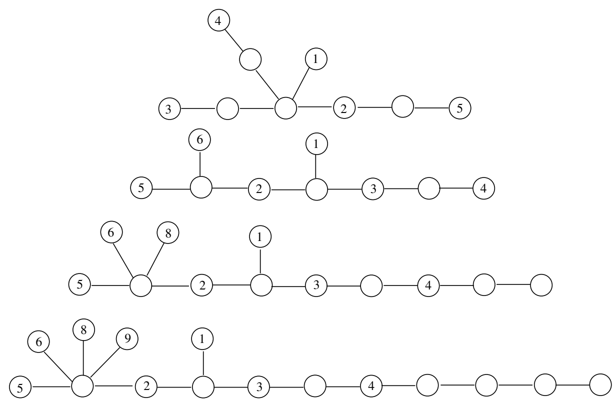

Figure 4: The graphs G_n for $n = 4, 5, 6, 7$.

For $n \ge 8$, let G_n be the graph obtained from the path $P_{n+3} = (v_1, v_2, \ldots, v_{n+3})$ of order $n+3$ by adding the pendant edge uv_3 at v_3 and $n-3$ pendant edges u_iv_1 at v_1 for $1 \le i \le n-3$. Then $\text{diam}(G_n) = n+3$. We show that $S_n = \{u, u_1, u_2, \ldots, u_{n-3}, v_2, v_4, v_6\}$ is a minimal irregular dominating set of G_n. Define the labeling $f : S_n \to [n+3]$ by assigning the label 1 to u, the label 2 to v_2, the label 3 to v_4, the label 4 to v_6, and the $n-3$ distinct labels in $[n+3] - \{1, 2, 3, 4, 7, 10\}$ arbitrarily to the remaining $n-3$ vertices of $S_n - \{u, v_2, v_4, v_6\} = \{u_1, u_2, \ldots, u_{n-3}\}$. Observe that the vertex v_3 is only dominated by the vertex u labeled 1, both v_4 and u_i where $1 \le i \le n-3$ are only dominated by the vertex v_2 labeled 2, both v_7 and v_1 are only dominated by the vertex v_4 labeled 3, both v_{10} and v_2 are only dominated by the vertex v_6 labeled 4, and for $i \in [n+3] - \{1, 2, 3, 4, 7, 10\}$, the vertex v_i is only dominated by the vertex labeled i. Since for each labeled vertex $x \in S_n$, there is a vertex y of G_n that is dominated only by x, it follows that S_n is a minimal irregular dominating set and $G_n[S_n] = \overline{K}_{n+1}$.

Case 2. $F \ne \overline{K_n}$. Let $V(F) = \{u_1, u_2, \ldots, u_n\}$. We may assume that $u_1u_2 \in E(F)$. Let G be the graph obtained from F, the n-path $P_n = (v_1, v_2, \ldots, v_n)$, and the 3-path $P_3 = (w_1, w_2, w_3)$ by adding two new vertices x and y and joining (1) the vertex x to each vertex in $(V(F) - \{u_2\}) \cup \{v_1, w_1\}$ and (2) the vertex y to each vertex in $\{u_1, v_2, w_2\}$. Then $\text{diam}(G) = n+1$. Define the labeling $f : V(F) \cup \{v_1\} \to [n+1]$ by assigning the label 1 to u_1, the label 2 to v_1, and the label i to u_{i-1} for $3 \le i \le n+1$. Observe that x and u_2 are only dominated by the vertex u_1 labeled 1, the vertex u_1 is only dominated by the vertex v_1 labeled 2, the vertex v_1 is only dominated by the vertex u_2 labeled 3, the vertex w_3 is only dominated by the vertex u_3 labeled 4, and for $5 \le i \le n+1$, the vertex v_{i-1} is only dominated by the vertex u_{i-1} labeled i. Thus, $S_n = V(F) \cup \{v_1\}$ is a minimal irregular dominating set of G. Since $G[S] \cong F + K_1$, it follows that $F + K_1$ is an irregular domination graph. □

Corollary 2.1. *For each pair Δ, n of integers with $0 \le \Delta \le n-2$ and $n \ge 3$, there exists an irregular domination graph of order n having maximum degree Δ.*

With this information, all irregular domination graphs of order 4 and 5 are determined.

Proposition 2.3. *A graph H of order 4 or 5 is an irregular domination graph if and only if H is disconnected or H is connected and* $\operatorname{diam}(H) \geq 3$*.*

Proof. First, suppose that H is a graph of order 4. We show that H is an irregular domination graph if and only if H is disconnected or $H = P_4$. By Proposition 2.2, if H is a connected graph of order 4 with $\operatorname{diam}(H) \leq 2$, then H is not an irregular domination graph. Thus, it remains to verify the converse. By Theorem 2.1, if H is one of $\overline{K}_4$, $P_2 + 2K_1$, $P_3 + K_1$, $K_3 + K_1$, then H is an irregular domination graph. For each graph in Figure 5, a minimal irregular dominating labeling is also shown in that figure. Thus, $2P_2$ and P_4 are irregular domination graphs.

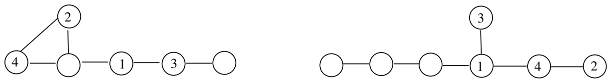

Figure 5: A step in the proof of Proposition 2.3.

Next, suppose that H is a graph of order 5. We show that H is an irregular domination graph if and only if H is disconnected or H is connected and $\operatorname{diam}(H) \geq 3$. By Proposition 2.2, if H is a (connected) graph of order 5 with $\operatorname{diam}(H) \leq 2$, then H is not an irregular domination graph. Thus, it remains to verify the converse. Suppose that H is a graph of order 5 that is either disconnected or is a connected graph of diameter at least 3. We show that there is a graph G with a minimal irregular dominating set S such that $G[S] = H$. By Theorem 2.1, we may assume that H does not contain isolated vertices. Thus, H is one of the eight graphs $H_1, H_2, \ldots, H_8$ in Figure 6.

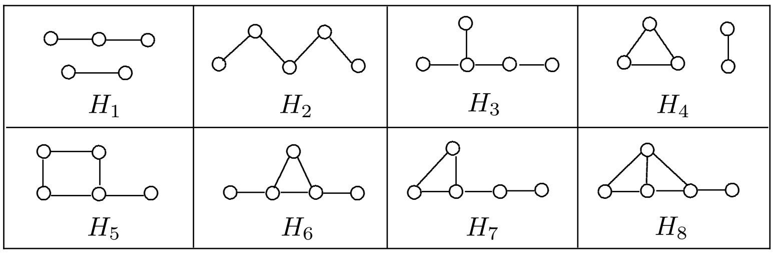

Figure 6: Eight graphs of order 5.

If $H = H_i$ where $1 \leq i \leq 8$, then the graph G_i in Figure 7 has a minimal irregular dominating set (also shown in Figure 7) that induces a subgraph of G_i isomorphic to H_i. □

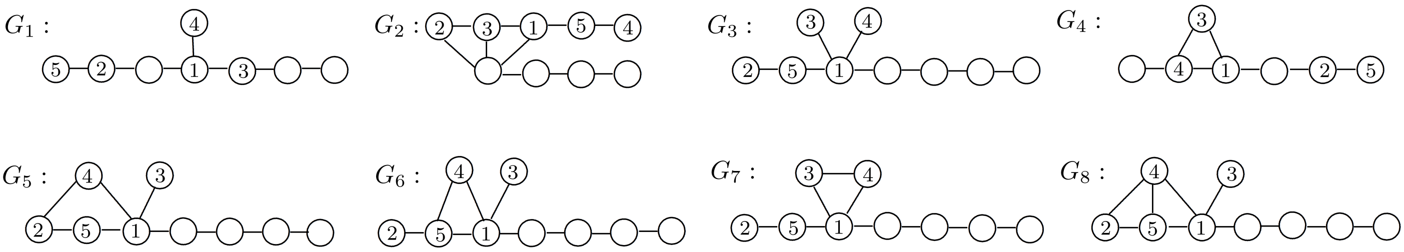

Figure 7: The eight graphs G_i for $1 \leq i \leq 8$ in the proof of Proposition 2.3.

3. Irregular domination paths and cycles

We now determine all paths and cycles that are irregular domination graphs, beginning with paths. By Proposition 2.2, the paths P_2 and P_3 are not irregular domination graphs. These are, however, the exceptions for paths.

Theorem 3.1. *For each integer $n \geq 4$, the path P_n is an irregular domination graph.*

Proof. By Proposition 2.3, both P_4 and P_5 are irregular domination graphs. Since the graph G in Figure 8 has a minimal irregular domination set $\{u_1, u_2, \ldots, u_6\}$ that induces P_6, it follows that P_6 is an irregular domination graph.

Next, we show that for each integer $n \geq 7$, there is a graph G_n having a minimal irregular dominating set S_n such that $G_n[S_n] \cong P_n$. First, suppose that $n = 11$ and let G_{11} be the graph shown in Figure 9, where $\operatorname{diam}(G_{11}) = d(w_1, v_5) = 13$. Let $S_{11} = \{u_1, u_2, \ldots, u_{11}\}$ with the corresponding irregular dominating labeling f_{11} shown in Figure 9. Observe that the vertex x is only dominated by the vertex u_5 labeled 1, the vertex u_5 is only dominated by the vertex u_3 labeled 2, the vertex u_3 is only dominated by the vertex u_6 labeled 3, the vertex v_1 is only dominated by the vertex u_4 labeled 4, the vertex u_2 is only dominated by the vertex u_7 labeled 5, the vertex u_7 is only dominated by the vertex u_1 labeled 6, the vertex w_1 is

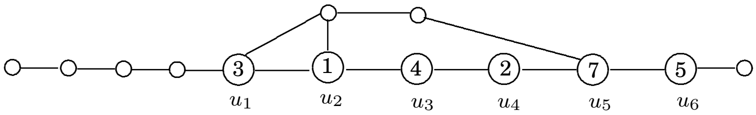

Figure 8: The graph G in the proof of Theorem 3.1.

only dominated by the vertex u_2 labeled 7, and for $8 \leq i \leq 11$, the vertex v_{i-6} is only dominated by the vertex u_i labeled i. Furthermore, every vertex of G is dominated by at least one vertex in S_{11}. Since S_{11} is a minimal irregular dominating set of G_{11} and $G_{11}[S_{11}] = P_{11}$, it follows that P_{11} is an irregular domination graph.

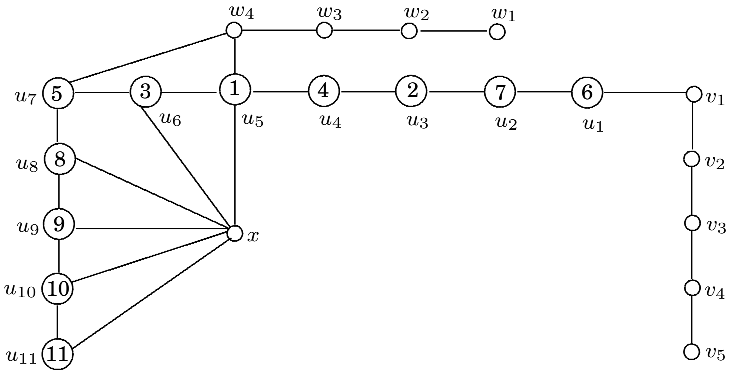

Figure 9: The graph G_{11} in the proof of Theorem 3.1.

It remains to show that if $n \geq 7$ and $n \neq 11$, then P_n is an irregular domination graph. We consider two cases, according to whether $7 \leq n \leq 10$ or $n \geq 12$.

Case 1. $7 \leq n \leq 10$. Beginning with G_{11}, we construct the graph G_n from the graph G_{n+1} recursively as follows. For $n = 10$, let G_{10} be the graph obtained from G_{11} by deleting the vertices u_{11} and v_5, for $n = 9$, let G_9 be the graph obtained from G_{10} by deleting the vertices u_{10} and v_4, for $n = 8$, let G_8 be the graph obtained from G_9 by deleting the vertices u_9, and v_3, and for $n = 7$, let G_7 be the graph obtained from G_8 by deleting the vertices u_8, x, and v_2. For $n = 7, 8, 9, 10$, let $S_n = \{u_1, u_2, \ldots, u_n\}$ and let $f_n(u_i) = f_{11}(u_i)$ for $1 \leq i \leq n$. Then every vertex of G_n is dominated by at least one vertex in S_n. Furthermore, G_n $(7 \leq n \leq 10)$ is a distance-preserving subgraph of G_{11}. Thus, for each $u \in S_n$, there is a vertex of G_n that is only dominated by u. Therefore, S_n is a minimal irregular dominating set of G_n and $G_n[S_n] = P_n$.

Case 2. $n \geq 12$. Let G_n be the graph obtained from G_{11}, the path $(u_{12}, u_{13}, \ldots, u_n)$ of order $n - 11$, and the path $(v_6, v_7, \ldots, v_{n-5})$ of order $n - 10$ by (1) joining each vertex u_i to x for $12 \leq i \leq n$ and (2) joining v_5 to v_6. Then $\operatorname{diam}(G_n) = d(w_1, v_{n-5}) = 8 + (n - 5) = n + 3$. Let $S_n = \{u_1, u_2, \ldots, u_n\}$. Define a labeling $f_n : S_n \to [n + 1]$ by $f_n(u_i) = f_{11}(u_i)$ for $1 \leq i \leq 11$ and $f_n(u_i) = i + 1$ for $12 \leq i \leq n$. First, every vertex of G_n is dominated by at least one vertex in S_n and so S_n is an irregular dominating set of G_n. Since G_{11} is a distance-preserving subgraph of G_n, it follows that if $u_i \in S_n$, where $1 \leq i \leq 11$, then there is a vertex of $G_{11} \subseteq G_n$ that is only dominated by u_i. Furthermore, for $7 \leq i \leq n - 5$, the vertex v_i is only dominated by u_{i+5} labeled $i + 6$. Therefore, S_n is a minimal irregular dominating set of G_n and $G_n[S_n] \cong P_n$. □

By Theorem 3.1, we now know exactly which paths are irregular domination graphs.

Corollary 3.1. *A path P_n of order $n \geq 2$ is an irregular domination graph if and only if $n \notin \{2, 3\}$.*

Next, we turn our attention to cycles. Since $\operatorname{diam}(C_n) \leq 2$ for $n = 3, 4, 5$, it follows by Proposition 2.2 that C_n is not an irregular domination graph if $n = 3, 4, 5$. Before proceeding further with our discussion of cycles, we present the following result.

Proposition 3.1. *Let H be an r-regular graph, $r \geq 2$, of diameter 3 with the property that for each vertex x of H, there is exactly one vertex y such that $d(x, y) = 3$. Then H is not an irregular domination graph.*

Proof. Assume, to the contrary, that such a graph H is an irregular domination graph. Then there exists a graph G with a minimal irregular dominating set S such that $G[S] \cong H$. Let f be an irregular dominating labeling of S. Since the distance between every two vertices of S in G is 1, 2, or 3, every vertex of S must be dominated by a vertex of S labeled 1, 2 or 3. We consider two cases.

Case 1. *No vertex of S is dominated by a vertex labeled* 3. Thus, every vertex of S is dominated by a vertex of S labeled 1 or 2. Since no vertex of S can dominate itself, there are vertices $u, v \in S$ such that $f(u) = 1$ and $f(v) = 2$. However then, u and v dominate each other, which is impossible by Observation 2.2.

Case 2. There is a vertex of S that is dominated by a vertex labeled 3. Let $w, z \in S$ such that $f(w) = 3$ and z is dominated by w. Thus, $d_G(w, z) = 3$ and z is the only vertex of S that is dominated by w. Therefore, every vertex of $S - \{z\}$ is dominated by a vertex labeled 1 or 2. Since no vertex of S can dominate itself, there are vertices $u, v \in S$ such that $f(u) = 1$ and $f(v) = 2$. If neither u nor v is z, then u and v dominate each other, which is impossible by Observation 2.2. Thus, either $u = z$ or $v = z$. First, suppose that $u = z$, that is, $f(z) = 1$ and $f(v) = 2$. Since only v can dominate w, it follows that $d(v, w) = 2$ and there is a $v - w$ geodesic (v, x, w) in G. However then, x cannot be dominated by any of the three labeled vertices v, w, z of S, which is impossible. Next, suppose that $v = z$, that is, $f(u) = 1$ and $f(z) = 2$. Since z is the only vertex of S that is dominated by w, it follows that w cannot dominate any neighbor of z in S. Furthermore, z cannot dominate any of its neighbor. This implies that u must dominate each of the r neighbors of z in S as well as the vertex w. Thus, u must dominate at least $r + 1$ vertices of S, which says that u is adjacent to at least $r + 1$ vertices of S. Since $G[S]$ is r-regular, this is impossible. □

By Proposition 3.1, the 6-cycle C_6 is not an irregular domination graph. The 7-cycle C_7 has diameter 3 but for each vertex x of C_7, there are two vertices y such that $d(x, y) = 3$. Not only is C_7 an irregular domination graph, C_n is an irregular domination graph for every integer $n \geq 7$.

Theorem 3.2. *For each integer $n \geq 7$, the cycle C_n is an irregular domination graph.*

Proof. We show for each integer $n \geq 7$ that there is a graph G_n having a minimal irregular dominating set S_n such that $G_n[S_n] \cong C_n$. First, suppose that $n = 7$. Let G_7 be the graph shown in Figure 10. Then $\text{diam}(G_7) = d(u_3, v_6) = d(u_5, v_6) = 8$. Let $S_7 = \{u_1, u_2, \dots, u_7\}$ with the corresponding irregular dominating labeling f_7 as shown in Figure 10. Observe that the vertex x is only dominated by the vertex u_2 labeled 1, the vertex v_2 is only dominated by the vertex u_7 labeled 2, the vertex v_1 is only dominated by the vertex u_3 labeled 3, the vertex v_3 is only dominated by the vertex u_4 labeled 4, the vertex v_4 is only dominated by the vertex u_6 labeled 5, the vertex v_6 is only dominated by the vertex u_1 labeled 6, and the vertex v_5 is only dominated by the vertex u_5 labeled 7. Furthermore, every vertex of G_7 is dominated by at least one vertex in S_7. Therefore, S_7 is a minimal irregular dominating set of G_7 and $G_7[S_7] \cong C_7$.

Next, suppose that $n = 8$. Let G_8 be the graph shown in Figure 11, where $\text{diam}(G_8) = d(u_7, v_6) = 10$. Let $S_8 = \{u_1, u_2, \dots, u_8\}$ with the corresponding irregular dominating labeling f_8 shown in Figure 11. Observe that the vertex x is only dominated by the vertex u_5 labeled 1, the vertex v_1 is only dominated by the vertex u_4 labeled 2, the vertex v_2 is only dominated by the vertex u_2 labeled 3, the vertex u_3 is only dominated by the vertex u_7 labeled 4, the vertex v_5 is only dominated by the vertex u_3 labeled 5, the vertex v_3 is only dominated by the vertex u_6 labeled 6, the vertex v_4 is only dominated by the vertex u_8 labeled 7, and the vertex v_6 is only dominated by the vertex u_1 labeled 8. Furthermore, every vetrex of G_8 is dominated by at least one vertex in S_8. Therefore, S_8 is a minimal irregular dominating set of G_8 and $G_8[S_8] \cong C_8$.

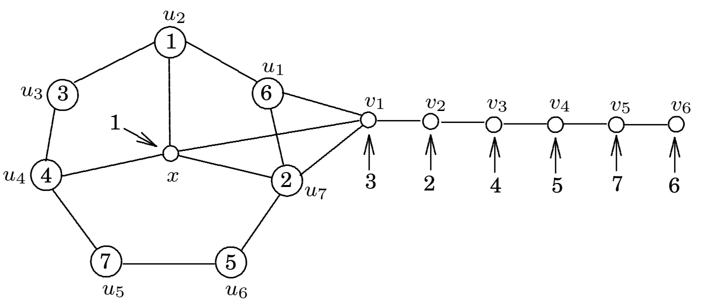

Figure 10: Showing that C_7 is an irregular domination graph.

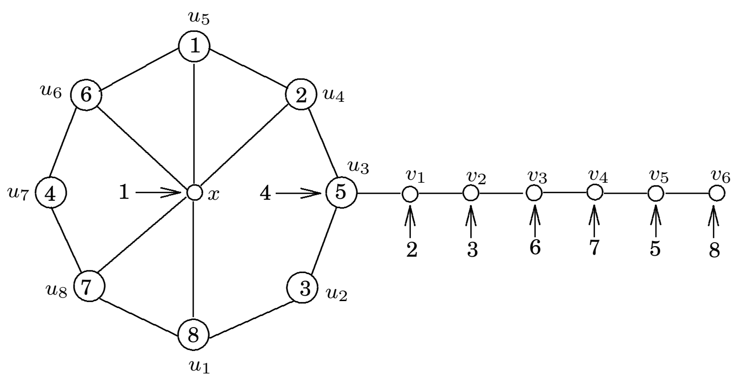

Figure 11: Showing that C_8 is an irregular domination graph.

Finally, suppose that $n \geq 9$. Let G_n be the graph obtained from G_8 by (1) replacing the edge u_8u_1 by the path $Q = (u_8, x_1, x_2, \dots, x_{n-8}, u_1)$ and joining x to each vertex of Q_1, (2) adding two new vertices w_1 and w_2 and five new edges u_5w_1, u_7w_1, w_1w_2, w_2x, w_2u_2, and (3) adding the path $(y_1, y_2, \dots, y_{n-8})$ at v_6 by joining v_6 to y_1. Then $\text{diam}(G_n) = d(u_7, y_{n-8}) = n+2$. Let $S_n = \{u_1, u_2, \dots, u_8, x_1, x_2, \dots, x_{n-8}\}$. Define a labeling $f_n : S_n \to [n+1]$ by $f_n(u_i) = f_8(u_i)$ for $1 \leq i \leq 8$ and $f(x_i) = 9+i$ for $1 \leq i \leq n-8$. First, every vertex of G_n is dominated by at least one vertex in S_n and so S_n is an irregular dominating set of G_n. Since G_8 is a distance-preserving subgraph of G_n, it follows that if $u_i \in S_n$, where $1 \leq i \leq 8$, then there is a vertex of $G_8 \subseteq G_n$ that is only dominated by u_i. Furthermore, for $1 \leq i \leq n-8$, the vertex y_i is only dominated by x_i labeled $9+i$. Therefore, S_n is a minimal irregular dominating set of G_n and $G_n[S_n] \cong C_n$. □

Corollary 3.2. *A cycle C_n of order $n \geq 3$ is an irregular domination graph if and only if $n \geq 7$.*

It was mentioned that no connected vertex-transitive graph has an irregular dominating set; consequently, no cycle has an irregular dominating set. By Corollary 3.2, however, almost all cycles are irregular domination graphs. That is, a graph may be an irregular domination graph even if it fails to have an irregular dominating set.

4. Irregular domination grids and prisms

We now turn our attention to two well-known classes of graphs constructed from paths and cycles. For two vertex-disjoint graphs G and H, the *Cartesian product* $G \,\square\, H$ of G and H has vertex set $V(G \,\square\, H) = V(G) \times V(H)$ and two distinct vertices (u, v) and (x, y) of $G \,\square\, H$ are adjacent if either (1) $u = x$ and $vy \in E(H)$ or (2) $v = y$ and $ux \in E(G)$. For an integer $n \geq 2$, the graph $P_n \,\square\, K_2$ is often referred to as a *ladder* and for each integer $n \geq 3$, the graph $C_n \,\square\, K_2$ is referred to as a *prism*.

We saw that a path P_n of order $n \geq 2$ is an irregular domination graph if and only if $n \notin \{2, 3\}$. While $P_2 \,\square\, K_2 = C_4$ is not an irregular dominating graph, $P_n \,\square\, K_2$ is an irregular domination graph for all $n \geq 3$.

Theorem 4.1. *For each integer $n \geq 3$, the ladder $P_n \,\square\, K_2$ is an irregular domination graph.*

Proof. For an integer $n \geq 3$, let $H_n = P_n \,\square\, K_2$ where $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ are two vertex-disjoint copies of an n-path in H_n and $u_i v_i \in E(H_n)$ for $1 \leq i \leq n$. Then $\operatorname{diam}(H_n) = d(u_1, v_n) = d(v_1, u_n) = n$. We show that there is a graph G_n having a minimal irregular dominating set S_n with corresponding labeling f_n such that $G_n[S_n] \cong H_n$.

For $n = 3, 4$, the graph G_n is shown in Figure 12. Let $S_n = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$ with the corresponding labeling f_n also shown in Figure 12.

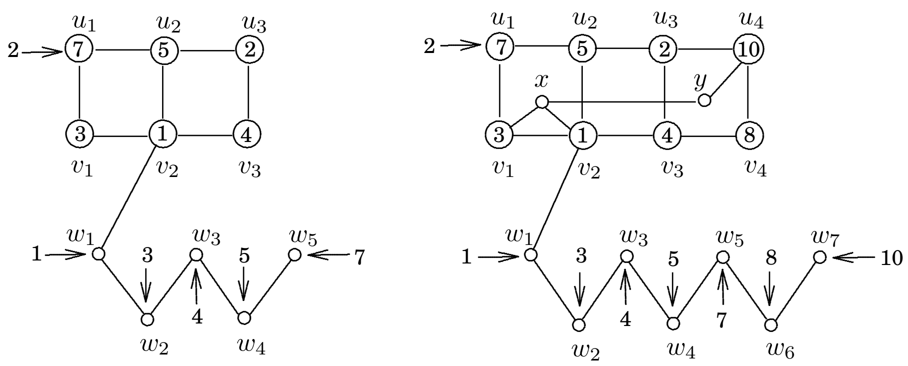

Figure 12: The graphs G_3 and G_4 in the proof of Theorem 4.1.

In G_3, the vertex w_1 is only dominated by the vertex labeled 1, the vertex u_1 is only dominated by the vertex labeled 2, the vertex w_2 is only dominated by the vertex labeled 3, the vertex w_3 is only dominated by the vertex labeled 4, the vertex w_4 is only dominated by the vertex u_6 labeled 5, and the vertex w_5 is only dominated by the vertex u_1 labeled 7. In G_4, the vertex w_1 is only dominated by the vertex labeled 1, the vertex u_1 is only dominated by the vertex labeled 2, the vertex w_2 is only dominated by the vertex labeled 3, the vertex w_3 is only dominated by the vertex labeled 4, the vertex w_4 is only dominated by the vertex u_6 labeled 5, the vertex w_5 is only dominated by the vertex u_1 labeled 7, the vertex w_6 is only dominated by the vertex v_4 labeled 8, and the vertex w_7 is only dominated by the vertex u_4 labeled 10. Furthermore, every vertex of G_n, $n = 3, 4$, is dominated by at least one vertex in S_n. Thus, S_n is a minimal irregular dominating set of G_n and $G_n[S_n] \cong H_n$ for $n = 3, 4$.

For $n \geq 5$, let G_n be the graph constructed from H_n and the path $P = (w_1, w_2, \ldots, w_{2n-1})$ of order $2n - 1$ by (1) adding two new vertices x and y and the four new edges xv_1, xv_2, xy, yu_4, (2) joining y to both u_i and v_i for $5 \leq i \leq n$, and (3) joining w_1 to v_2. Thus, $V(G_n) = V(H_n) \cup \{x, y\} \cup V(P)$ and $E(G_n) = E(H_n) \cup E(P) \cup \{v_2 w_1, xv_1, xv_2, xy, yu_4\} \cup \{yu_i, yv_i : 5 \leq i \leq n\}$. Then $\operatorname{diam}(G_n) = d(u_1, w_{2n-1}) = 2n+2$. Notice that G_4 is a distance-preserving subgraph of G_n. Let $S_n = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$. We define a labeling $f_n : S_n \to [2n + 2]$ by extending the labeling f_4 of G_4 shown in Figure 12; that is, we define $f_n(u_i) = f_4(u_i)$ and $f_n(v_i) = f_4(v_i)$ for $1 \leq i \leq 4$ (see Figure 12) and $f_n(u_i) = 2i + 2$ for $5 \leq i \leq n$ and $f_n(v_i) = 2i + 1$ for $5 \leq i \leq n$. Thus, the set of labels used by f_n is $[2n + 2] - \{6, 9\}$. The graph G_7 is shown in Figure 13 together with the corresponding labeling f_7 of G_7. Since G_4 is a distance-preserving subgraph of G_n, it follows that for each $\ell \in \{1, 2, 3, 4, 5, 7, 8, 10\}$, there is a vertex of $G_4 \subseteq G_n$ that is only dominated by the vertex labeled ℓ. Furthermore, if $\ell \in \{11, 12, \ldots, 2n+2\}$, then the vertex $w_{\ell-3}$ is only dominated by the vertex labeled ℓ. Also, every vertex of G_n is dominated by at least one vertex in S_n. In particular, if $5 \leq i \leq n$, then u_i and v_i are dominated by v_1 labeled 3. Therefore, S_n is a minimal irregular dominating set of G_n and $G_n[S_n] \cong P_n \,\square\, K_2$. $\square$

The graphs $P_m \,\square\, P_n$ for $m, n \geq 2$ are commonly referred to as *grids*. Thus, ladders form a subset of the grids. While it is an open problem to determine which of these graphs are irregular domination graphs, it is known that $P_m \,\square\, P_n$ is an irregular domination graph for each pair m, n of integers with $2 \leq m \leq n \leq 4$. That $P_4 \,\square\, P_4$ is an irregular domination graph is shown in Figure 14.

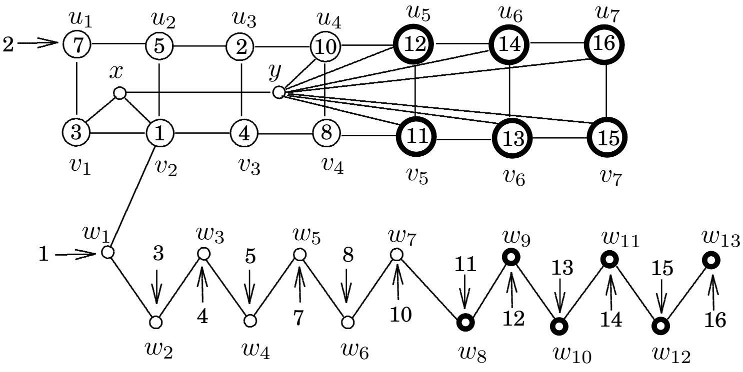

Figure 13: The graph G_7 in the proof of Theorem 4.1.

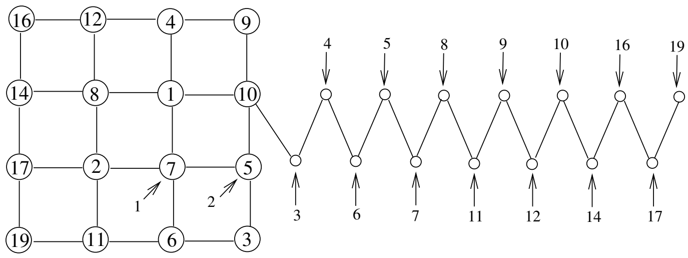

Figure 14: Showing that $P_4 \,\square\, P_4$ is an irregular domination graph.

We now turn our attention to prisms. First, we determine those prisms that are not irregular domination graphs.

Proposition 4.1. *For $n = 3, 4, 5$, the prism $C_n \,\square\, K_2$ is not an irregular domination graph.*

Proof. Since $\operatorname{diam}(C_3 \,\square\, K_2) = 2$, it follows that $C_3 \,\square\, K_2$ is not an irregular domination graph by Proposition 2.2. Since $C_4 \,\square\, K_2$ is a 3-regular graph of diameter 3 with the property that for each vertex x of H, there is exactly one vertex y such that $d(x, y) = 3$, it follows by Proposition 3.1 that $C_4 \,\square\, K_2$ is not an irregular domination graph. It remains to consider $C_5 \,\square\, K_2$. Let $H = C_5 \,\square\, K_2$ where $(u_1, u_2, \ldots, u_5, u_1)$ and $(v_1, v_2, \ldots, v_5, v_1)$ are two vertex-disjoint copies of a 5-cycle in H and $u_iv_i \in E(H)$ for $1 \leq i \leq 5$. Assume, to the contrary, that H is an irregular domination graph. Then there is a graph G with a minimal irregular dominating set S with corresponding irregular dominating labeling f such that $G[S] \cong H$. Since $\operatorname{diam}(H) = 3$, each vertex of S is dominated by a vertex of S labeled 1, 2, or 3. A vertex labeled 3 in S dominates at most two vertices of S, a vertex labeled 2 in S dominates at least four and at most six vertices of S, and a vertex labeled 1 in S dominates exactly three vertices of S. Since S has ten vertices, all three labels 1, 2, and 3 must be used. Furthermore, the vertex labeled 2 dominates at least five vertices of S. Suppose, without loss of generality, that $f(u_1) = 2$.

Case 1. *u_1 dominates exactly five vertices of S.* We may assume that u_1 dominates v_2, v_4, v_5, u_3, u_4 and u_1 does not dominate v_3. Therefore, there is a vertex $x \notin S$ such that x is a neighbor of both u_1 and v_4. Furthermore, there is no vertex $y \notin S$ such that y is a neighbor of both u_1 and v_3. In this case, every vertex of S is dominated by exactly one vertex labeled 1, 2, or 3. Since v_2 is the only unlabeled vertex whose three neighbors are not dominated, it follows that $f(v_2) = 1$. Since u_1 and u_5 are the only vertices not dominated by a vertex labeled 1 or 2, it follows that u_1 and u_5 must be dominated by a vertex labeled 3, which implies that $f(v_3) = 3$. Since $d_H(x, w) \leq 3$ for each $w \in V(H)$, it follows that x is not dominated by any labeled vertex, which is a contradiction.

Case 2. *u_1 dominates exactly six vertices of S.* Then u_1 dominates $v_2, v_3, v_4, v_5, u_3, u_4$. Thus, there is a vertex $x \notin S$ such that x is a neighbor of both u_1 and v_4 and there is a vertex $y \notin S$ such that y is a neighbor of both u_1 and v_3, where possibly $x = y$. Since the vertex labeled 1 must dominate at least two unlabeled vertices, we may assume that $f(v_2) = 1$ and so v_2 dominates u_2 and v_1. Therefore, the vertex z labeled 3 must dominate u_1 and u_5. However, no such vertex z has this property. This is a contradiction. Therefore, $C_5 \,\square\, K_2$ is not an irregular domination graph. □

Theorem 4.2. *For each integer $n \geq 6$, the prism $C_n \,\square\, K_2$ is an irregular domination graph.*

Proof. For $n \geq 6$, the diameter $C_n \,\square\, K_2$ is $\operatorname{diam}(C_n \,\square\, K_2) = \operatorname{diam}(C_n) + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$. Let $H_n = C_n \,\square\, K_2$ where $(u_1, u_2, \ldots, u_n, u_1)$ and $(v_1, v_2, \ldots, v_n, v_1)$ are two vertex-disjoint copies of an n-cycle in H_n and $u_iv_i \in E(H_n)$ for $1 \leq i \leq n$. We show for each integer $n \geq 6$ that there is a graph G_n having a minimal irregular dominating set S_n such that $G_n[S_n] \cong H_n$.

First, suppose that $n = 6$ and $\operatorname{diam}(H_6) = 4$. Let G_6 be the graph shown in Figure 15. Let $S_6 = \{u_1, u_2, \ldots, u_6, v_1, v_2, \ldots, v_6\}$ with the corresponding irregular dominating labeling f_6 as shown in Figure 15. Observe that the vertex y_1 is only dominated by the vertex u_1 labeled 1, the vertex u_1 is only dominated by the vertex v_6 labeled 2, for $i = 3, 4, 5$, the vertex y_{i-1} is only dominated by the vertex labeled i, for $i = 7, 8, \ldots, 12$, the vertex y_{i-2} is only dominated by the vertex

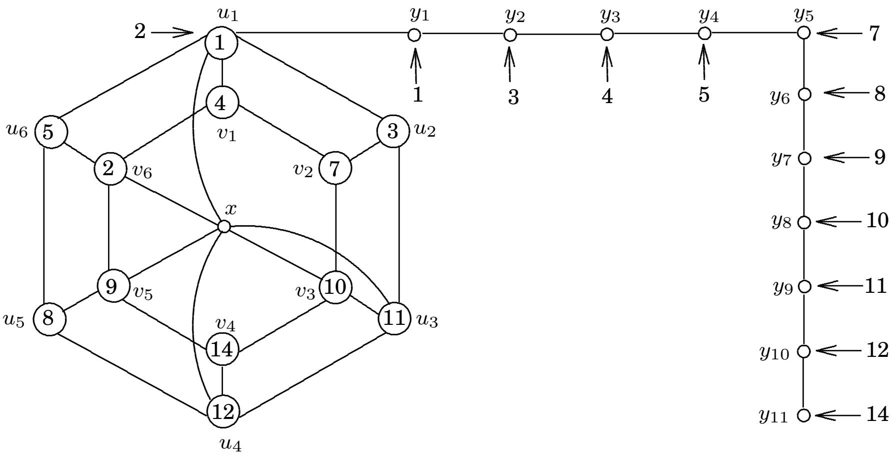

Figure 15: The graph G_6 the proof of Theorem 4.2.

labeled i, and the vertex y_{11} is only dominated by the vertex labeled 14. Furthermore, every vertex of G_6 is dominated by at least one vertex in S_6. Therefore, S_6 is a minimal irregular dominating set of G_6 and $G_6[S_6] \cong H_6 = C_6 \,\Box\, K_2$.

First, suppose that $n = 7$. Let F_7 be the graph constructed in the proof of Theorem 3.2 with $V(F_7) = \{u_1, u_2, \dots, u_7\} \cup \{x, y_1, y_2, \dots, y_6\}$, where $(u_1, u_2, \dots, u_7, u_1) = C_7$ and $(y_1, y_2, \dots, y_6) = P_6$, and $E(F_7) = E(C_7) \cup E(P_6) \cup \{xu_2,\ xu_4,\ xu_7,\ xy_1,\ u_1y_1,\ u_7y_1\}$. Then $T_7 = \{u_1, u_2, \dots, u_7\}$ is a minimal irregular dominating set of F_7 with corresponding irregular dominating labeling g_7 defined by $g_7(u_1) = 6$, $g_7(u_2) = 1$, $g_7(u_3) = 3$, $g_7(u_4) = 4$, $g_7(u_5) = 7$, $g_7(u_6) = 5$, and $g_7(u_7) = 2$ such that $F_7[T_7] = C_7 = (u_1, u_2, \dots, u_7, u_1)$. Let G_7 be the graph obtained from F_7 by (1) adding the 7-cycle $(v_1, v_2, \dots, v_7, v_1)$ and joining v_i to u_i for $1 \le i \le 7$, (2) joining x to v_i for $1 \le i \le 7$, and (3) adding the 7-path $(z_1, z_2, \dots, z_7)$ and joining z_1 to y_6. Then $\operatorname{diam}(G_7) = d(u_3, z_7) = 15$. Let $S_7 = \{u_1, u_2, \dots, u_7\} \cup \{v_1, v_2, \dots, v_7\}$. Define a labeling $f_7 : S_7 \to [14]$ by $f_7(u_i) = g_7(u_i)$ for $1 \le i \le 7$ and $f(v_i) = 7 + i$ for $1 \le i \le 7$. Observe that the vertex x is only dominated by the vertex u_2 labeled 1, the vertex y_2 is only dominated by the vertex u_7 labeled 2, the vertex y_1 is only dominated by the vertex u_3 labeled 3, the vertex y_3 is only dominated by the vertex u_4 labeled 4, the vertex y_4 is only dominated by the vertex u_6 labeled 5, the vertex y_6 is only dominated by the vertex u_1 labeled 6, the vertex y_5 is only dominated by the vertex u_5 labeled 7, and the vertex z_i is only dominated by the vertex v_i labeled $7 + i$ for $1 \le i \le 7$. Furthermore, every vertex of G_7 is dominated by at least one vertex in S_7. In particular, the vertex v_7 is dominated by u_3 labeled 3 and if $i \ne 7$, then v_i is dominated by u_7 labeled 2. Therefore, S_7 is a minimal irregular dominating set of G_7 and $G_7[S_7] \cong H_7 = C_7 \,\Box\, K_2$.

Next, suppose that $n = 8$. Let F_8 be the graph constructed in the proof of Theorem 3.2 with $V(F_8) = \{u_1, u_2, \dots, u_8\} \cup \{x, y_1, y_2, \dots, y_6\}$, where $(u_1, u_2, \dots, u_8, u_1) = C_8$ and $(y_1, y_2, \dots, y_6) = P_6$, and $E(F_8) = E(C_7) \cup E(P_6) \cup \{xu_i : i \in [8] - \{2, 3, 7\}\} \cup \{u_3y_1\}$. Then $T_8 = \{u_1, u_2, \dots, u_8\}$ is a minimal irregular dominating set of F_8 with corresponding irregular dominating labeling g_8 defined by $g_8(u_1) = 6$, $g_8(u_2) = 1$, $g_8(u_3) = 3$, $g_8(u_4) = 4$, $g_8(u_5) = 7$, $g_8(u_6) = 5$, $g_8(u_7) = 2$, and $g_8(u_8) = 2$ such that $F_8[T_8] = C_8 = (u_1, u_2, \dots, u_8, u_1)$. Let G_8 be the graph obtained from F_8 by (1) adding the 8-cycle $(v_1, v_2, \dots, v_8, v_1)$ and joining v_i to u_i for $1 \le i \le 8$, (2) joining x to v_i for $i \in [8] - \{4\}$ (and so $xv_4 \notin E(G_4)$), and (3) adding the 8-path $(z_1, z_2, \dots, z_8)$ and joining z_1 to y_6. Then $\operatorname{diam}(G_8) = d(u_7, z_8) = 18$. Let $S_8 = \{u_1, u_2, \dots, u_8\} \cup \{v_1, v_2, \dots, v_8\}$. Define a labeling $f_8 : S_8 \to [17]$ by $f_8(u_i) = g_8(u_i)$ for $1 \le i \le 8$, $f_8(v_1) = 14$, $f_8(v_2) = 10$, $f_8(v_3) = 11$, $f_8(v_4) = 9$, $f_8(v_5) = 12$, $f_8(v_6) = 17$, $f_8(v_7) = 16$, $f_8(v_8) = 15$. An argument similar to the one used for G_7 shows that S_8 is a minimal irregular dominating set of G_8 and $G_8[S_8] \cong H_8 = C_8 \,\Box\, K_2$.

Finally, suppose that $n \ge 9$. Let F_n be the graph (constructed in the proof of Theorem 3.2) with $V(F_n) = \{u_1, u_2, \dots, u_n\} \cup \{w_1, w_2, x, y_1, y_2, \dots, y_{n-2}\}$, where $C_n = (u_1, u_2, \dots, u_n, u_1)$, $P_{n-2} = (y_1, y_2, \dots, y_{n-2})$, and $E(F_n) = E(C_n) \cup E(P_{n-2}) \cup \{u_5w_1, u_7w_1, w_1w_2, w_2x, w_2u_2, u_3y_1\} \cup \{xu_i : i \in [n] - \{2, 3, 7\}\}$. Let G_n be the graph obtained from F_n by (1) adding the n-cycle $(v_1, v_2, \dots, v_n, v_1)$ and joining v_i to u_i for $1 \le i \le n$, (2) joining x to v_i for each $i \in [n] - \{4\}$ (and so $xv_4 \notin E(G_n)$), and (3) adding the n-path $(y_{n-1}, y_n, y_{n+1}, \dots, y_{2n-2})$ and joining y_{n-1} to y_{n-2}. Then $\operatorname{diam}(G_n) = d(u_7, z_n) = 2n + 1$. Let $S_n = \{u_1, u_2, \dots, u_n\} \cup \{v_1, v_2, \dots, v_n\}$. Define a labeling $f_n : S_n \to [2n + 1]$ by $f_n(u_1) = 8$, $f_n(u_2) = 3$, $f_n(u_3) = 5$, $f_n(u_4) = 2$, $f_n(u_5) = 1$, $f_n(u_6) = 6$, $f_n(u_7) = 4$, $f_n(u_8) = 7$, $f_n(u_9) = 10$, and $f_n(u_i) = 4 + i$ for $10 \le i \le n$, and $f_n(v_1) = n + 5$, $f_n(v_2) = 11$, $f_n(v_3) = 9$, $f_n(v_4) = 12$, and $f_n(v_i) = n + 1 + i$ for $5 \le i \le n$. Thus, the set of labels used by f_n is $[2n + 1] - \{13\}$. The graph G_{10} is shown in Figure 16 where the labeling f_{10} is also shown and the set of labels used by f_{10} is $[21] - \{13\}$.

An argument similar to the one used for G_7 shows that S_n is a minimal irregular dominating set of G_n and $G_n[S_n] \cong H_n = C_n \,\Box\, K_2$.

$\Box$

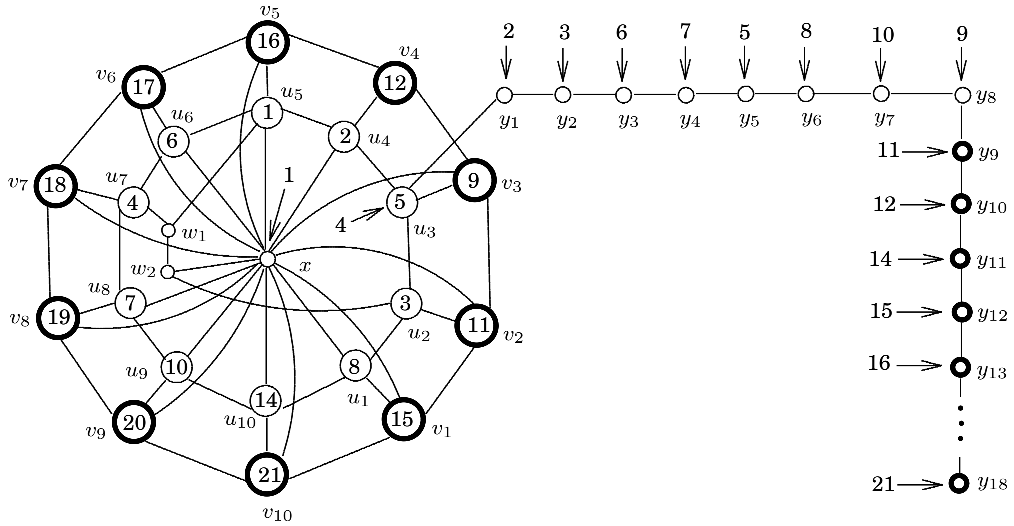

Figure 16: The graph G_{10} the proof of Theorem 4.2.

The following result is a consequence of Proposition 4.1 and Theorem 4.2.

Corollary 4.1. *The prism $C_n \,\square\, K_2$ is an irregular domination graph if and only of $n \geq 6$.*

5. Problems for further study

By Proposition 2.2, no connected graph of diameter at most 2 is an irregular domination graph. By Proposition 3.1, there is an infinite class of connected graphs of diameter 3, none of which are irregular domination graphs. On the other hand, there is also an infinite class of connected graphs of diameter 3 that are irregular domination graphs. For example, it can be shown that all trees of diameter 3 (double stars) are irregular domination graphs. These facts lead to the following problem.

Problem 5.1. *Can connected irregular domination graphs of diameter 3 be characterized?*

Based on those graphs that have been shown to be irregular domination graphs and the irregular dominating labelings of graphs that have verified, we conclude with the following problems.

Problem 5.2. *Is every connected graph of diameter at least 4 an irregular domination graph?*

Problem 5.3. *For every graph H that has been shown to be an irregular domination graph and each graph G with a minimal irregular dominating set S such that $G[S] \cong H$, the graph G is connected and the corresponding labeling assigns the label 1 to some vertex of S. Is this true in general?*

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References

[1] A. Ali, G. Chartrand, P. Zhang, *Irregularity in Graphs*, Springer, New York, 2021.
[2] A. Ali, G. Chartrand, P. Zhang, On irregular and antiregular domination in graphs, *Electron. J. Math.* **2** (2021) 26–36.
[3] C. Berge, Sur le couplage maximum d'un graphe, *C. R. Acad. Sci. Paris* **247** (1958) 258–259.
[4] P. Broe, G. Chartrand, P. Zhang, Irregular domination in trees, *Electron. J. Math.* **1** (2021) 89–100.
[5] P. Broe, G. Chartrand, P. Zhang, Irregular orbital domination in graphs, *Int. J. Comput. Math.: Comput. Syst. Theory* **7** (2022) 68–79.
[6] G. Chartrand, M. A. Henning, K. Schultz, On orbital domination numbers of graphs, *J. Combin. Math. Combin. Comput.* **37** (2001) 3–26.
[7] G. Chartrand, P. Zhang, A chessboard problem and irregular domination, *Bull. Inst. Combin Appl.*, To appear.
[8] E. J. Cockayne, R. M. Dawes, S. T. Hedetniemi, Total domination in graphs, *Networks* **10** (1977) 211–219.
[9] E. J. Cockayne, S. T. Hedetniemi, Towards a theory of domination in graphs, *Networks* **7** (1977) 247–261.
[10] T. W. Haynes, S. T. Hedetniemi, M. A. Henning, *Domination in Graphs: Core Concepts*, Springer, New York, 2022.
[11] M. A. Henning, A. Yeo, *Total Domination in Graphs*, Springer, New York, 2013.
[12] O. Ore, *Theory of Graphs*, Amer. Math. Soc. Colloq. Pub., Providence, 1962.