Contributions to Mathematics
www.shahindp.com/locate/cm

Contrib. Math. **6** (2022) 30–38
DOI: 10.47443/cm.2022.038

Research Article

Oscillation results of third-order nonlinear dynamic equations with damping on time scales

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(Received: 7 July 2022. Received in revised form: 5 August 2022. Accepted: 13 August 2022. Published online: 24 August 2022.)

Abstract

In this paper, we study the oscillatory criteria of solutions for third-order dynamic equations with damping and obtain some sufficient conditions by using the generalized Riccati transformation. We extend and improve some well-known existing results. We also provide an example for illustrating our main result.

Keywords: nonlinear dynamic equations; oscillation solutions; damping equations; time scales.

2020 Mathematics Subject Classification: 34K11, 39A10, 39A99.

1. Introduction

The study of oscillatory behavior of nonlinear damped differential equations has become a well-researched area because of the fact that such equations appear in many real-life problems; for example, see [5, 8, 10–12, 14, 17, 18, 21–23, 25–30] and the references cited therein.

Hilger [7] introduced the time scales theory which unified the representation of continuous analysis and discrete analysis; see also [2, 3]. The past decade has witnessed the tremendous development of time scale theory in many fields. This theory has received a large amount of attention and studies. In general, we cannot obtain analytical solutions of higher order dynamic equations, so the oscillation and asymptotic behavior of solutions is what we often focus on. Very recently, there have been many researches regarding the oscillation criteria for solutions of dynamic equations on any time scales such as [4, 6, 9, 13–15, 15, 16, 18, 20–22, 24, 25, 28] and the references therein.

Nowadays, several researchers are interested in the following nonlinear differential equation (see [19]):

$$(r(t)z'(t))' + p(t)z'(t) + q(t)f(z(t)) = 0.$$

Aktaş et al. [1] gave some new oscillation results for the following difference equation:

$$\triangle(c_n\triangle(d_n\triangle z_n)) + p_n\triangle z_{n+1} + q_n f(z_{n-\sigma}) = 0, \quad n \geq n_0,$$

where $n_0 \in \mathbb{N}$ is a fixed integer and σ is a nonnegative integer.

By a solution for dynamic equations on time scales, we mean a nontrivial real valued function satisfying the dynamic equation on any time scales $\mathbb{T}$. If a solution of the dynamic equation is neither eventually negative nor eventually positive, it is called oscillatory. Or else, the solution is called non-oscillatory. If all of the solutions of dynamic equation are oscillatory, then we say that the dynamic equation is oscillatory.

Şenel [21] studied the oscillation behavior of the following second-order dynamic equations:

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} + p(t)(z^{\Delta}(t))^{\gamma} + f(t, z(t)) = 0$$

on a time scale $\mathbb{T}$. Hassan [6] obtained some oscillation results for the equation:

$$(a(t)(r(t)[(z^{\Delta}(t))^{\Delta}]^{\gamma})^{\Delta} + f(t, z(\tau(t))) = 0.$$

Afterwards, Erbe et al. [4] gave some new results for this problem. In [16], Qiu explored the following third-order damped dynamic equation:

$$(r_1(t)(r_2(t)(z^{\Delta}(t))^{\gamma})^{\Delta})^{\Delta} + f(t, z(t), z^{\sigma}(t), z(g(t)), z^{\Delta}(t)) = 0$$

on any time scale $\mathbb{T}$ such that $\inf \mathbb{T} = t_0$ and $\sup \mathbb{T} = \infty$.

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The aim of the present study is to give an oscillation behavior for the following third-order nonlinear damped dynamic equation with delay term on any time scale $\mathbb{T}$:

$$(r_1(t)(r_2(t)(x^{\Delta}(t))^{\gamma})^{\Delta})^{\Delta} + p(t)(r_2(x^{\Delta}(t))^{\gamma})^{\Delta} + f(t, x(t), x^{\sigma}(t), x(g(t)), x^{\Delta}(t)) = 0. \quad (1)$$

Throughout this study, we assume that the following assumptions hold:

(C_1). $r_1(t), r_2(t) \in C(\mathbb{T}, (0,\infty))$ such that

$$\int_{t_0}^{\infty} \frac{e_{-\frac{p}{r_1}}(s,t_0)}{r_1(s)} \Delta s = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{(r_2(s))^{\frac{1}{\gamma}}} \Delta s = \infty; \quad (2)$$

(C_2). $g(t) \in C(\mathbb{T}, \mathbb{T})$ and

$$g(t) \geq \begin{cases} \sigma(t), & 0 < \gamma < 1 \\ t, & \gamma \geq 1, \end{cases}$$

for all $t \in \mathbb{T}$;

(C_3). $f(t,u,v,w,r) \in C(\mathbb{T} \times \mathbb{R}^4, \mathbb{T})$ and there exists a function $q(t) \in C_{rd}(\mathbb{T}, (0,\infty))$ such that

$$f(t,u,v,w,r) sign(u) \geq q(t)\Big(\mid v \mid^2 + \mid w \mid^2 \Big),$$

for u, v and w with a same sign;

(C_4). The inequality

$$\int_{t_0}^{\infty} q(t) \Delta t < \infty,$$

satisfies whenever $\gamma \in (0,1)$;

(C_5). $p(t) \in C(\mathbb{T}, \mathbb{R})$ and $1 - \mu(t)\dfrac{p(t)}{r_1(t)} > 0$;

(C_6). γ is a quotient of odd positive integers.

The remaining part of this paper is arranged as follows. In Section 2, we present some lemmas that are required to prove the main result. In Section 3, we give the main oscillation result of the problem (1) and provide an example for illustrating it.

2. Preliminaries

In this section, we give some lemmas for establishing the oscillation result for the problem (1).

Lemma 2.1. *Suppose that the conditions* (C_1)–(C_6) *(given in the previous section) hold. Also, suppose that there exists a sufficiently large* $t_1 \in [t_0,\infty)_{\mathbb{T}}$ *such that* $x(t)$ *is a solution of* (1) *satisfying* $x(t) > 0$ *for* $t \in [t_1,\infty)_{\mathbb{T}}$. *Then there exists a* $T \in [t_1,\infty)_{\mathbb{T}}$ *such that*

$$\left(\frac{r_1(t)(r_2(t)(x^{\Delta}(t))^{\gamma})^{\Delta}}{e_{-\frac{p}{r_1}}(t,t_0)}\right)^{\Delta} < 0, \quad (r_2(t)(x^{\Delta}(t))^{\gamma})^{\Delta} > 0,$$

and

$$x^{\Delta}(t) > 0 \ \text{ or } \ x^{\Delta}(t) < 0,$$

for $t \in [T,\infty)_{\mathbb{T}}$.

Proof. Let $t_1 \in [t_0,\infty)_{\mathbb{T}}$ and $x(t)$ be a solution of (1) satisfying $x(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. Then, we have $x(\sigma(t))$ and $x(g(t)) > 0$. Using (1) and (C_3), we have

$$\begin{aligned}
\left(\frac{r_1(t)(r_2(t)(x^{\Delta}(t))^{\gamma})^{\Delta}}{e_{-\frac{p}{r_1}}(t,t_0)}\right)^{\Delta} &= \frac{(r_1(t)(r_2(t)(x^{\Delta}(t))^{\gamma})^{\Delta})^{\Delta}}{e_{-\frac{p}{r_1}}(\sigma(t),t_0)} - \frac{(e_{-\frac{p}{r_1}}(t,t_0))^{\Delta}(r_1(t)(r_2(t)(x^{\Delta}(t))^{\gamma})^{\Delta})}{e_{-\frac{p}{r_1}}(t,t_0)e_{-\frac{p}{r_1}}(\sigma(t),t_0)} \\
&= \frac{(r_1(t)(r_2(t)(x^{\Delta}(t))^{\gamma})^{\Delta})^{\Delta} + p(t)(r_2(t)(x^{\Delta}(t))^{\gamma})^{\Delta}}{e_{-\frac{p}{r_1}}(\sigma(t),t_0)} \\
&= \frac{-f(t,x(t),x^{\sigma}(t),x(g(t)),x^{\Delta}(t))}{e_{-\frac{p}{r_1}}(\sigma(t),t_0)}
\end{aligned}$$

$$\begin{aligned} &\leq \frac{-q(t)(x^\gamma(\sigma(t))+x^\gamma(g(t)))}{e_{-\frac{p}{r_1}}(\sigma(t),t_0)} \\ &< 0, \end{aligned}$$

for all $t\in[t_1,\infty)_{\mathbb{T}}$. Thus,

$$\frac{r_1(t)(r_2(t)(x^\Delta(t))^\gamma)^\Delta}{e_{-\frac{p}{r_1}}(t,t_0)}$$

is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$. Since $r_1(t)>0$ and

$$e_{-\frac{p}{r_1}}(t,t_0)>0,$$

we can say that $r_2(t)(x^\Delta(t))^\gamma)^\Delta$ is eventually of one sign. We want to show that the inequality

$$(r_2(t)(x^\Delta(t))^\gamma)^\Delta>0$$

is satisfied for all $t\in[t_1,\infty)_{\mathbb{T}}$; contrarily, suppose that it is not true. Then, there exists a $t_2\in[t_1,\infty)_{\mathbb{T}}$ such that $(r_2(t)(x^\Delta(t))^\gamma)^\Delta<0$ for $t\in[t_2,\infty)_{\mathbb{T}}$. Then

$$\begin{aligned} r_2(t)(x^\Delta(t))^\gamma-r_2(t_2)(x^\Delta(t_2))^\gamma &= \int_{t_2}^t \frac{e_{-\frac{p}{r_1}}(s,t_0)r_1(s)(r_2(s)(x^\Delta(s))^\gamma)^\Delta}{e_{-\frac{p}{r_1}}(s,t_0)r_1(s)}\Delta s \\ &\leq \frac{r_1(t_2)(r_2(t_2)(x^\Delta(t_2))^\gamma)^\Delta}{e_{-\frac{p}{r_1}}(t_2,t_0)}\int_{t_2}^t\frac{e_{-\frac{p}{r_1}}(s,t_0)}{r_1(s)}\Delta s. \end{aligned}$$

By using the fact that

$$\frac{r_1(t)(r_2(t)(x^\Delta(t))^\gamma)^\Delta}{e_{-\frac{p}{r_1}}(t,t_0)}$$

is strictly decreasing on the interval $[t_1,\infty)_{\mathbb{T}}$ and the condition (C_1), we have $\lim_{t\to\infty} r_2(t)(x^\Delta(t))^\gamma=-\infty$ and thus there exists a sufficiently large $t_3\in[t_2,\infty)_{\mathbb{T}}$ such that $r_2(t)(x^\Delta(t))^\gamma<0$. By the assumption $(r_2(t)(x^\Delta(t))^\gamma)^\Delta<0$ for $t\in[t_2,\infty)_{\mathbb{T}}$, we know that $r_2(t)(x^\Delta(t))^\gamma$ is strictly decreasing on $[t_3,\infty)_{\mathbb{T}}$. Thus, we have

$$r_2(t)(x^\Delta(t))^\gamma\leq r_2(t_3)(x^\Delta(t_3))^\gamma<0$$

and hence

$$x^\Delta(t)\leq r_2^{\frac{1}{\gamma}}(t_3)x^\Delta(t_3)\frac{1}{r_2^{\frac{1}{\gamma}}(t)}. \tag{3}$$

Integrating (3) from t_3 to $t\in[\sigma(t_3),\infty)_{\mathbb{T}}$, we get

$$x(t)-x(t_3)\leq r_2^{\frac{1}{\gamma}}(t_3)x^\Delta(t_3)\int_{t_3}^t\frac{\Delta s}{r_2^{\frac{1}{\gamma}}(s)}.$$

By (C_1), we obtain $\lim_{t\to\infty}x(t)=-\infty$, which is a contradiction. Thus, we have $(r_2(t)(x^\Delta(t))^\gamma)^\Delta>0$ on $[t_1,\infty)_{\mathbb{T}}$. So, we can say that $r_2(t)(x^\Delta(t))^\gamma$ is strictly increasing on $[t_1,\infty)_{\mathbb{T}}$. Because of this, $r_2(t)(x^\Delta(t))^\gamma$ is either eventually negative or eventually positive. Then there exists a $T\in[t_1,\infty)_{\mathbb{T}}$ such that $x^\Delta(t)>0$ or $x^\Delta(t)<0$ on $[T,\infty)_{\mathbb{T}}$.

□

Lemma 2.2. *Suppose that the conditions (C_1)–(C_6) hold. Also, assume that*

$$2^{\frac{1}{\gamma}}\alpha\int_{t_4}^t\left(\frac{1}{(r_2(\xi))^{\frac{1}{\gamma}}}\int_\xi^\infty\left(\frac{e_{-\frac{p}{r_1}}(\tau,t_0)}{r_1(\tau)}\int_\tau^\infty\frac{q(s)}{e_{\frac{-p}{r_1}}(\sigma(s),t_0)}\Delta s\right)\Delta\tau\right)\Delta\xi=\infty. \tag{4}$$

Then either $\lim_{t\to\infty}x(t)=0$ or there exists a sufficiently large t_4 such that $x^\Delta(t)>0$ on $[t_4,\infty)_{\mathbb{T}}$.

Proof. Using Lemma 2.1, we conclude that $x^\Delta(t)$ is eventually of one sign. Thus there exists a sufficiently large t_4 such that at least one of the inequalities $x^\Delta(t)<0$ and $x^\Delta(t)>0$ is satisfied on the interval $[t_4,\infty)_{\mathbb{T}}$. We suppose that $x^\Delta(t)<0$, by $x(t)$ is a positive solution of (1) on $[t_0,\infty)_{\mathbb{T}}$, we get $\lim_{t\to\infty}x(t)=\alpha\geq 0$ and $\lim_{t\to\infty}r_2(t)(x^\Delta(t))^\gamma=\beta\leq 0$. We assert that $\alpha=0$. Otherwise, suppose $\alpha>0$. So there exists $t_5>t_4$ such that $x(t)\geq\alpha$ on $[t_5,\infty)_{\mathbb{T}}$. If we integrate

$$\frac{r_1(t)(r_2(t)(x^\Delta(t))^\gamma)^\Delta}{e_{-\frac{p}{r_1}}(t,t_0)}$$

from t to ∞, we have

$$\begin{aligned}-\frac{r_1(t)(r_2(t)(x^\Delta(t))^\gamma)^\Delta}{e_{-\frac{p}{r_1}}(t,t_0)} &= -\lim_{t\to\infty}\frac{r_1(t)(r_2(t)(x^\Delta(t))^\gamma)^\Delta}{e_{-\frac{p}{r_1}}(t,t_0)}+\int_t^\infty\frac{-f(s,x(s),x^\sigma(s),x(g(s)),x^\Delta(s))}{e_{-\frac{p}{r_1}}(\sigma(s),t_0)}\Delta s\\ &\le -\int_t^\infty\frac{2q(s)x^\gamma(s)}{e_{-\frac{p}{r_1}}(\sigma(s),t_0)}\Delta s\\ &\le -2\alpha^\gamma\int_t^\infty\frac{q(s)}{e_{-\frac{p}{r_1}}(\sigma(s),t_0)}\Delta s.\end{aligned}$$

Thus, we get

$$-(r_2(t)(x^\Delta(t))^\gamma)^\Delta \le -2\alpha^\gamma\left[\frac{e_{-\frac{p}{r_1}}(t,t_0)}{r_1(t)}\int_t^\infty\frac{q(s)}{e_{-\frac{p}{r_1}}(\sigma(s),t_0)}\Delta s\right]. \tag{5}$$

By integrating (5) from t to ∞, we get

$$\begin{aligned}r_2(t)(x^\Delta(t))^\gamma &= \lim_{t\to\infty}r_2(t)(x^\Delta(t))^\gamma-2\alpha^\gamma\int_t^\infty\left(\frac{e_{-\frac{p}{r_1}}(\tau,t_0)}{r_1(\tau)}\int_\tau^\infty\frac{q(s)}{e_{-\frac{p}{r_1}}(\sigma(s),t_0)}\Delta s\right)\Delta\tau\\ &= \beta-2\alpha^\gamma\int_t^\infty\left(\frac{e_{-\frac{p}{r_1}}(\tau,t_0)}{r_1(\tau)}\int_\tau^\infty\frac{q(s)}{e_{-\frac{p}{r_1}}(\sigma(s),t_0)}\Delta s\right)\Delta\tau\\ &\le -2\alpha^\gamma\int_t^\infty\left(\frac{e_{-\frac{p}{r_1}}(\tau,t_0)}{r_1(\tau)}\int_\tau^\infty\frac{q(s)}{e_{-\frac{p}{r_1}}(\sigma(s),t_0)}\Delta s\right)\Delta\tau\end{aligned}$$

and hence

$$x^\Delta(t) \le -\frac{2^{\frac{1}{\gamma}}\alpha}{(r_2(t))^{\frac{1}{\gamma}}}\left(\int_t^\infty\frac{e_{-\frac{p}{r_1}}(\tau,t_0)}{r_1(\tau)}\int_\tau^\infty\frac{q(s)}{e_{-\frac{p}{r_1}}(\sigma(s),t_0)}\Delta s\right)\Delta\tau. \tag{6}$$

Again, by integrating both sides of (6) from t_5 to t, we have

$$x(t)-x(t_5)\le -2^{\frac{1}{\gamma}}\alpha\int_{t_5}^t\left(\frac{1}{(r_2(\xi))^{\frac{1}{\gamma}}}\int_\xi^\infty\left(\frac{e_{-\frac{p}{r_1}}(\tau,t_0)}{r_1(\tau)}\int_\tau^\infty\frac{q(s)}{e_{-\frac{p}{r_1}}(\sigma(s),t_0)}\Delta s\right)\Delta\tau\right)\Delta\xi. \tag{7}$$

From (4) and (7), it follows that $\lim_{t\to\infty}x(t)=-\infty$, that gives a contradiction. Thus, we have $\alpha=0$.

□

Lemma 2.3. *For $0<\gamma<1$, suppose that the conditions (C_1)–(C_6) hold and let $x(t)$ be a solution of the (1) with $x(t)>0$, $x^\Delta(t)>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$ such that $t_1\ge t_0$. Then, it holds that*

$$\left(\frac{x^\Delta(t)}{x^\sigma(t)}\right)^{1-\gamma}\ge\alpha(t)=\left(\frac{2e_{-\frac{p}{r_1}}(t,t_0)}{e_{-\frac{p}{r_1}}(\sigma(t),t_0)r_2(t)}\delta(t,t_1)\int_t^\infty q(s)\Delta s\right)^{\frac{1-\gamma}{\gamma}},$$

for $t\in[t_1,\infty)_{\mathbb{T}}$, where

$$\delta(t,t_1)=\int_{t_1}^t\frac{e_{-\frac{p}{r_1}}(s,t_0)}{r_1(s)}\Delta s.$$

Proof. Since $x(t)$ is a solution for (1) and $x(t)>0$ on the interval $t_1,\infty)_{\mathbb{T}}$ with $t_1\in[t_0,\infty)_{\mathbb{T}}$, by using Lemma 2.1, we have

$$(r_2(t)(x^\Delta(t))^\gamma)^\Delta>0,\quad t\in[t_1,\infty)_{\mathbb{T}}. \tag{8}$$

By (1), we get

$$\begin{aligned}\left(\frac{r_1(t)(r_2(t)(x^\Delta(t))^\gamma)^\Delta}{e_{-\frac{p}{r_1}}(t,t_0)}\right)^\Delta &= \frac{(r_1(t)(r_2(t)(x^\Delta(t))^\gamma)^\Delta)^\Delta}{e_{-\frac{p}{r_1}}(\sigma(t),t_0)}-\frac{(e_{-\frac{p}{r_1}}(t,t_0))^\Delta(r_1(t)(r_2(t)(x^\Delta(t))^\gamma)^\Delta)}{e_{-\frac{p}{r_1}}(t,t_0)e_{-\frac{p}{r_1}}(\sigma(t),t_0)}\\ &= \frac{-f(t,x(t),x^\sigma(t),x(g(t)),x^\Delta(t))}{e_{-\frac{p}{r_1}}(\sigma(t),t_0)}\\ &\le \frac{-q(t)(x^\gamma(\sigma(t))+x^\gamma(g(t)))}{e_{-\frac{p}{r_1}}(\sigma(t),t_0)}\end{aligned}$$

$$\leq \frac{-2q(t)x^{\gamma}(\sigma(t))}{e_{-\frac{p}{r_1}}(\sigma(t),t_0)}. \tag{9}$$

We integrate (9) from $t \in [t_1,\infty)_{\mathbb{T}}$ to ∞ and use (8) to obtain

$$\begin{aligned}
\frac{r_1(t)(r_2(t)(x^{\Delta}(t))^{\gamma})^{\Delta}}{e_{-\frac{p}{r_1}}(t,t_0)} &\geq \int_{\infty}^{t}\left(\frac{r_1(s)(r_2(s)(x^{\Delta}(s))^{\gamma})^{\Delta}}{e_{-\frac{p}{r_1}}(s,t_0)}\right)^{\Delta}\Delta s \\
&= -\int_{t}^{\infty}\left(\frac{r_1(s)(r_2(s)(x^{\Delta}(s))^{\gamma})^{\Delta}}{e_{-\frac{p}{r_1}}(s,t_0)}\right)^{\Delta}\Delta s \\
&\geq \int_{t}^{\infty}\frac{2q(s)x^{\gamma}(\sigma(s))}{e_{-\frac{p}{r_1}}(\sigma(s),t_0)}\Delta s \\
&\geq \frac{2x^{\gamma}(\sigma(t))}{e_{-\frac{p}{r_1}}(\sigma(t),t_0)}\int_{t}^{\infty} q(s)\Delta s.
\end{aligned}$$

Since $\dfrac{r_1(t)(r_2(t)(x^{\Delta}(t))^{\gamma})^{\Delta}}{e_{-\frac{p}{r_1}}(t,t_0)}$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned}
\frac{r_2(t)(x^{\Delta}(t))^{\gamma}}{e_{-\frac{p}{r_1}}(t,t_0)} &= \frac{r_2(t_1)(x^{\Delta}(t_1))^{\gamma}}{e_{-\frac{p}{r_1}}(t_1,t_0)} + \int_{t_1}^{t}\frac{r_1(s)(r_2(s)(x^{\Delta}(s))^{\gamma})^{\Delta}e_{-\frac{p}{r_1}}(s,t_0)}{r_1(s)e_{-\frac{p}{r_1}}(s,t_0)}\Delta s \\
&\geq \frac{(r_1(t)(r_2(t)(x^{\Delta}(t))^{\gamma})^{\Delta})}{e_{-\frac{p}{r_1}}(\sigma(t),t_0)}\int_{t_1}^{t}\frac{e_{-\frac{p}{r_1}}(s,t_0)}{r_1(s)}\Delta s \\
&\geq \frac{2x^{\gamma}(\sigma(t))}{e_{-\frac{p}{r_1}}(\sigma(t),t_0)}\delta(t)\int_{t}^{\infty} q(s)\Delta s,
\end{aligned}$$

for $t \in [\sigma(t_1),\infty)_{\mathbb{T}}$. Thus, for $0<\gamma<1$, we get

$$\left(\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\right)^{\gamma} \geq \frac{2e_{\frac{-p}{r_1}}(t,t_0)}{e_{\frac{-p}{r_1}}(\sigma(t),t_0)r_2(t)}\delta(t)\int_{t}^{\infty} q(s)\Delta s$$

and hence

$$\left(\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\right)^{1-\gamma} \geq \alpha(t) = \left(\frac{2e_{\frac{-p}{r_1}}(t,t_0)}{e_{\frac{-p}{r_1}}(\sigma(t),t_0)r_2(t)}\delta(t)\int_{t}^{\infty} q(s)\Delta s\right)^{\frac{1-\gamma}{\gamma}}.$$

□

3. Main result

In this section, we give a new oscillation criteria for the third-order nonlinear dynamic equation (1) with damping term by using the inequality technique and the generalized Riccati transformation.

Theorem 3.1. *Suppose that the conditions (C_1)–(C_6) hold. Also, suppose that there exist functions $A(t) \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, $B(t) \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and a sufficiently large $t_1 \in [t_0,\infty)_{\mathbb{T}}$ that x(t) is a solution of (1) satisfying $x(t)>0$ and $x^{\Delta}(t)>0$ for $t \in [t_1,\infty)_{\mathbb{T}}$ and*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(\frac{2A(s)q(s)}{e_{\frac{-p}{r_1}}(\sigma(s),t_0)} - B^{\Delta}(s) - \phi(s)\right)\Delta s = \infty, \tag{10}$$

where

$$\phi(t) \geq \begin{cases} \dfrac{(A^{\Delta}(t))^2 r_2(t)}{4\gamma A(t)\alpha\delta e_{-\frac{p}{r_1}}(t,t_0)}, & 0<\gamma<1 \\[2ex] \dfrac{r_2(t)}{\delta e_{-\frac{p}{r_1}}(t,t_0)A^{\gamma}}\left(\dfrac{A^{\Delta}(t)}{\gamma+1}\right)^{\gamma+1}, & \gamma \geq 1. \end{cases}$$

Then, every solution of (1) is either oscillatory on $[t_1,\infty)_{\mathbb{T}}$ or satisfies $\lim_{t\to\infty} x(t)$ exists.

Proof. Suppose that (1) is not oscillatory. Without loss of generality, we suppose that there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$ on $t_1, \infty)_{\mathbb{T}}$. By Lemma 2.1, for $t \in [t_1, \infty)_{\mathbb{T}}$, either $x^{\Delta}(t) < 0$ or $x^{\Delta}(t) > 0$ holds. Suppose that $x^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Defining the generalized Riccati function

$$u(t) = A(t)\frac{r_1(t)(r_2(t)(x^{\Delta}(t))^{\gamma})^{\Delta}}{e_{\frac{-p}{r_1}(t,t_0)}x^{\gamma}(t)} + B(t),$$

we have

$$\begin{aligned} u^{\Delta}(t) &= \left(\frac{A(t)}{x^{\gamma}(t)}\right)\left(\frac{r_1(t)(r_2(t)(x^{\Delta}(t))^{\gamma})^{\Delta}}{e_{-\frac{p}{r_1}(t,t_0)}}\right)^{\Delta} + \left(\frac{A(t)}{x^{\gamma}(t)}\right)^{\Delta}\left(\frac{r_1(t)(r_2(t)(x^{\Delta}(t))^{\gamma})^{\Delta}}{e_{-\frac{p}{r_1}(t,t_0)}}\right)^{\sigma} + B^{\Delta}(t) \\ &= \left(\frac{A(t)}{x^{\gamma}(t)}\right)\frac{-f(t,x(t),x^{\sigma}(t),x(g(t)),x^{\Delta}(t))}{e_{-\frac{p}{r_1}(\sigma(t),t_0)}} + \frac{A^{\Delta}(t)x^{\gamma}(t) - A(t)(x^{\gamma})^{\Delta}(t)}{x^{\gamma}(\sigma(t))x^{\gamma}(t)}\left(\frac{r_1(t)(r_2(t)(x^{\Delta}(t))^{\gamma})^{\Delta}}{e_{-\frac{p}{r_1}(t,t_0)}}\right)^{\sigma} + B^{\Delta}(t). \end{aligned}$$

Accepting the fact that

$$f(t,x(t),x^{\sigma}(t),x(g(t)),x^{\Delta}(t)) \geq q(t)(x^{\gamma}(\sigma(t)) + x^{\gamma}(g(t))) \geq 2q(t)x^{\gamma}(t),$$

we obtain

$$u^{\Delta}(t) \leq \frac{-2A(t)q(t)}{e_{-\frac{p}{r_1}(\sigma(t),t_0)}} + B^{\Delta}(t) + A^{\Delta}(t)\left(\frac{u(t)-B(t)}{A(t)}\right)^{\sigma} - A(t)\left(\frac{(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)}\right)\left(\frac{u(t)-B(t)}{A(t)}\right)^{\sigma}. \tag{11}$$

When $0 < \gamma < 1$, by using the Pötzsche chain rule, we get

$$(x^{\gamma})^{\Delta} = \gamma\int_0^1 (x + h\mu x^{\Delta})^{\gamma-1}dh \geq \gamma(x^{\sigma})^{\gamma-1}x^{\Delta}$$

and thus

$$\frac{(x^{\gamma})^{\Delta}}{x^{\gamma}} \geq \frac{\gamma(x^{\sigma})^{\gamma-1}x^{\Delta}}{x^{\gamma}} = \gamma\left(\frac{x^{\Delta}}{x^{\sigma}}\right)\left(\frac{x^{\sigma}}{x}\right)^{\gamma}.$$

By Lemmas 2.1 and 2.3, we obtain

$$\begin{aligned} \frac{x^{\Delta}(t)}{x^{\sigma}(t)} &= \frac{e_{-\frac{p}{r_1}(t,t_0)}}{r_2(t)}\frac{r_2(t)(x^{\Delta}(t))^{\gamma}}{e_{-\frac{p}{r_1}(t,t_0)}(x(\sigma(t))^{\gamma}}\left(\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\right)^{1-\gamma} \\ &\geq \frac{\alpha(t)\delta(t)e_{-\frac{p}{r_1}(t,t_0)}}{r_2(t)}\frac{r_1(t)(r_2(t)(x^{\Delta}(t))^{\gamma})^{\Delta}}{e_{-\frac{p}{r_1}(\sigma(t),t_0)}(x(\sigma(t)))^{\gamma}} \\ &\geq \frac{\alpha(t)\delta(t)e_{-\frac{p}{r_1}(t,t_0)}}{r_2(t)}\left(\frac{r_1(t)(r_2(t)(x^{\Delta}(t))^{\gamma})^{\Delta}}{e_{-\frac{p}{r_1}(t,t_0)}x^{\gamma}(t)}\right)^{\sigma} \\ &= \frac{\alpha(t)\delta(t)e_{-\frac{p}{r_1}(t,t_0)}}{r_2(t)}\left(\frac{u(t)-B(t)}{A(t)}\right)^{\sigma}, \end{aligned}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. Also, we know that $\frac{x^{\sigma}}{x} \geq 1$. So, (11) becomes

$$\begin{aligned} u^{\Delta}(t) &\leq \frac{-2A(t)q(t)}{e_{-\frac{p}{r_1}(\sigma(t),t_0)}} + B^{\Delta}(t) + A^{\Delta}(t)\left(\frac{u(t)-B(t)}{A(t)}\right)^{\sigma} - \frac{\gamma A(t)\alpha(t)\delta(t)e_{-\frac{p}{r_1}(t,t_0)}}{r_2(t)}\left[\left(\frac{u(t)-B(t)}{A(t)}\right)^{\sigma}\right]^2 \\ &= \frac{-2A(t)q(t)}{e_{-\frac{p}{r_1}(\sigma(t),t_0)}} + B^{\Delta}(t) + \frac{(A^{\Delta}(t))^2 r_2(t)}{4\gamma A(t)\alpha(t)\delta(t)e_{-\frac{p}{r_1}(t,t_0)}} \\ &\quad -\left[\frac{A^{\Delta}(t)}{2}\sqrt{\frac{r_2(t)}{\gamma A(t)\alpha(t)\delta(t)e_{-\frac{p}{r_1}(t,t_0)}}} - \sqrt{\frac{\gamma A(t)\alpha(t)\delta(t)e_{-\frac{p}{r_1}(t,t_0)}}{r_2(t)}}\left(\frac{u(t)-B(t)}{A(t)}\right)^{\sigma}\right]^2 \\ &\leq \frac{-2A(t)q(t)}{e_{-\frac{p}{r_1}(\sigma(t),t_0)}} + B^{\Delta}(t) + \frac{(A^{\Delta}(t))^2 r_2(t)}{4\gamma A(t)\alpha(t)\delta(t)e_{-\frac{p}{r_1}(t,t_0)}}. \end{aligned} \tag{12}$$

When $\gamma \geq 1$, we have

$$(x^\gamma)^\Delta = \gamma \int_0^1 (x + h\mu x^\Delta)^{\gamma-1} \geq \gamma x^{\gamma-1} x^\Delta$$

and it follows that

$$\frac{(x^\gamma)^\Delta}{x^\gamma} \geq \frac{\gamma x^{\gamma-1} x^\Delta}{x^\gamma} = \gamma \frac{x^\Delta}{x}.$$

Also, we obtain

$$\begin{aligned} \left(\frac{x^\Delta(t)}{x(t)}\right)^\gamma &= \frac{e_{-\frac{p}{r_1}(t,t_0)}}{r_2(t)} \frac{r_2(t)(x^\Delta(t))^\gamma}{e_{-\frac{p}{r_1}(t,t_0)} x^\gamma(t)} \\ &\geq \frac{\delta(t) e_{-\frac{p}{r_1}(t,t_0)}}{r_2(t)} \frac{r_1(t)(r_2(t)(x^\Delta(t))^\gamma)^\Delta}{e_{-\frac{p}{r_1}(\sigma(t),t_0)} x^\gamma(t)} \\ &\geq \frac{\delta(t) e_{-\frac{p}{r_1}(t,t_0)}}{r_2(t)} \left(\frac{r_1(t)(r_2(t)(x^\Delta(t))^\gamma)^\Delta}{e_{-\frac{p}{r_1}(t,t_0)} x^\gamma(t)}\right)^\sigma \\ &= \frac{\delta(t) e_{-\frac{p}{r_1}(t,t_0)}}{r_2(t)} \left(\frac{u(t) - B(t)}{A(t)}\right)^\sigma \end{aligned}$$

and so

$$\frac{x^\Delta(t)}{x(t)} \geq \left(\frac{\delta(t) e_{-\frac{p}{r_1}(t,t_0)}}{r_2(t)}\right)^{\frac{1}{\gamma}} \left[\left(\frac{u(t) - B(t)}{A(t)}\right)^\sigma\right]^{\frac{1}{\gamma}}.$$

Thus, we get

$$u^\Delta(t) \leq \frac{-2A(t)q(t)}{e_{-\frac{p}{r_1}(\sigma(t),t_0)}} + B^\Delta(t) + A^\Delta(t)\left(\frac{u(t) - B(t)}{A(t)}\right)^\sigma - \gamma A(t)\left(\frac{\delta(t) e_{-\frac{p}{r_1}(t,t_0)}}{r_2(t)}\right)^{\frac{1}{\gamma}} \left[\left(\frac{u(t) - B(t)}{A(t)}\right)^\sigma\right]^{\frac{1+\gamma}{\gamma}}.$$

Since $Q^\gamma - P^\gamma \geq \gamma P^{\gamma-1}(Q - P)$, where Q and P are nonnegative constants and $\gamma \geq 1$, we get

$$\lambda P^{\lambda-1} Q - Q^\lambda \leq (\lambda - 1) P^\lambda,$$

with $\lambda = \frac{\gamma+1}{\gamma}$. If

$$Q^\lambda = Q^{\frac{(\gamma+1)}{\gamma}} = \gamma A(t)\left(\frac{\delta(t) e_{-\frac{p}{r_1}(t,t_0)}}{r_2(t)}\right)^{\frac{1}{\gamma}} \left[\left(\frac{u(t) - B(t)}{A(t)}\right)^\sigma\right]^{\frac{1+\gamma}{\gamma}}$$

and

$$P^{\lambda-1} = P^{\frac{1}{\gamma}} = \frac{\gamma}{\gamma+1}\left(\frac{r_2(t)}{\delta(t) e_{-\frac{p}{r_1}(t,t_0)}}\right)^{\frac{1}{\gamma+1}} \frac{A^\Delta(t)}{(\gamma A(t))^{\frac{\gamma}{\gamma+1}}}$$

then we have

$$u^\Delta(t) \leq \frac{-2A(t)q(t)}{e_{-\frac{p}{r_1}(\sigma(t),t_0)}} + B^\Delta(t) + \frac{r_2(t)}{\delta(t) e_{-\frac{p}{r_1}(t,t_0)} A^\gamma(t)} \left(\frac{A^\Delta(t)}{\gamma+1}\right)^{\gamma+1}. \tag{13}$$

From (13) and (12), it follows that

$$u^\Delta(t) \leq \frac{-2A(t)q(t)}{e_{-\frac{p}{r_1}(\sigma(t),t_0)}} + B^\Delta(t) + \phi(t),$$

where

$$\phi(t) \geq \begin{cases} \dfrac{(A^\Delta(t))^2 r_2(t)}{4\gamma A(t)\alpha(t)\delta(t) e_{-\frac{p}{r_1}(t,t_0)}}, & 0 < \gamma < 1 \\ \dfrac{r_2(t)}{\delta(t) e_{-\frac{p}{r_1}(t,t_0)} A^\gamma(t)} \left(\dfrac{A^\Delta(t)}{\gamma+1}\right)^{\gamma+1}, & \gamma \geq 1. \end{cases}$$

Integrating the last inequality from t_1 to t, we obtain

$$u(t) - u(t_1) \leq \int_{t_1}^t \left(\frac{-2A(s)q(s)}{e_{-\frac{p}{r_1}(\sigma(s),t_0)}} + B^\Delta(s) + \phi(s)\right)\Delta s$$

and so

$$\int_{t_1}^{t}\left(\frac{2A(s)q(s)}{e_{-\frac{p}{r_1}}(\sigma(s),t_0)}-B^{\Delta}(s)-\phi(s)\right)\Delta s \le u(t_1)-u(t)\le u(t_1)<\infty,$$

which contradicts (10). Thus, $x^{\Delta}(t)<0$ on $[t_1,\infty)_{\mathbb{T}}$, and we see that $\lim_{t\to\infty} x(t)$ exists.

□

Example 3.1. *Take* $\mathbb{T}=\mathbb{N}_0$ *and consider the equation*

$$\left(t^2\left(\frac{1}{t^2+1}(x^{\Delta}(t))^{\gamma}\right)^{\Delta}\right)^{\Delta}-t\left(\frac{1}{t^2+1}(x^{\Delta}(t))^{\gamma}\right)^{\Delta}+\frac{1+(x^{\Delta}(t))^2}{t}x^{\gamma}(t+1)+\frac{t+1}{t^2}x^{\gamma}(2t)=0 \tag{14}$$

on $[1,\infty)_{\mathbb{T}}$ *with* $\gamma\ge 1$. *For this equation, we have*

$$r_1(t)=t^2,\ \ r_2(t)=\frac{1}{t^2+1},\ \ p(t)=-t,\ \ \textit{and}\ \ q(t)=\frac{1}{t}.$$

Since $e_{-\frac{p}{r_1}}(t,t_0)=t$, *we get*

$$\int_1^{\infty}\frac{e_{-\frac{p}{r_1}}(s,t_0)}{r_1(s)}\Delta s=\int_1^{\infty}\frac{s}{s^2}\Delta s=\int_1^{\infty}\frac{\Delta s}{s}=\infty,\quad \int_1^{\infty}\frac{\Delta s}{(r_2(s))^{\frac{1}{\gamma}}}=\int_1^{\infty}(s^2+1)^{\frac{1}{\gamma}}\Delta s=\infty$$

and

$$\delta(t,t_1)=\int_{t_1}^{t}\frac{s}{s^2}\Delta s=\int_{t_1}^{t}\frac{\Delta s}{s}>\int_{t_1}^{t}\frac{\Delta s}{s(s+1)}=\frac{t+1-t_1}{t_1(t+1)}.$$

For $(A(s),B(s))=(s,1)$, *it holds that*

$$\int_{t_1}^{\infty}\frac{2A(s)q(s)}{e_{-\frac{p}{r_1}}(\sigma(s),t_0)}\Delta s=\int_{t_1}^{\infty}\frac{2s}{s(s+1)}\Delta s=\int_{t_1}^{\infty}\frac{2}{(s+1)}\Delta s=\infty$$

and

$$\begin{aligned}\int_{t_1}^{\infty}\frac{r_2(s)}{\delta(s,t_1)e_{-\frac{p}{r_1}}(s,t_0)A^{\gamma}(s)}\left(\frac{A^{\Delta}(s)}{\gamma+1}\right)^{\gamma+1}\Delta s &< \int_{t_1}^{\infty}\frac{1}{s^2+1}\frac{t_1(s+1)}{(s+1-t_1)ss^{\gamma}}\left(\frac{1}{\gamma+1}\right)^{\gamma+1}\Delta s\\ &< t_1\left(\frac{1}{\gamma+1}\right)^{\gamma+1}\int_{t_1}^{\infty}\frac{\Delta s}{s^{\gamma+1}}\\ &< \infty.\end{aligned}$$

Consequently, we get

$$\int_{t_1}^{\infty}\left(\frac{2A(s)q(s)}{e_{-\frac{p}{r_1}}(\sigma(s),t_0)}-B^{\Delta}(s)-\phi(s)\right)\Delta s=\infty.$$

Therefore, by Theorem 3.1, every solution of (14) *is oscillatory on* $[1,\infty)_{\mathbb{T}}$.

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