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Research Article

On the strength and independence number of graphs

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Abstract

A numbering f of a graph G of order n is a labeling that assigns distinct elements of the set $\{1, 2, \ldots, n\}$ to the vertices of G. The strength $\text{str}_f(G)$ of a numbering $f : V(G) \to \{1, 2, \ldots, n\}$ of G is defined by $\text{str}_f(G) = \max\{f(u) + f(v) \mid uv \in E(G)\}$, that is, $\text{str}_f(G)$ is the maximum edge label of G and the strength $\text{str}(G)$ of a graph G itself is the minimum of the set $\{\text{str}_f(G) \mid f \text{ is a numbering of } G\}$. In this paper, we present a necessary and sufficient condition for the strength of a graph G of order n to meet the constraints $\text{str}(G) = 2n - 2\beta(G) + 1$ and $\text{str}(G) = n + \delta(G) = 2n - 2\beta(G) + 1$, where $\beta(G)$ and $\delta(G)$ denote the independence number and the minimum degree of G, respectively. This answers open problems posed by Gao, Lau, and Shiu [*Symmetry* **13** (2021) #513]. Also, an earlier result leads us to determine a formula for the strength of graphs containing a particular class of graphs as a subgraph. We also extend what is known in the literature about k-stable properties.

Keywords: strength; independence number; k-stable; graph labeling; combinatorial optimization.

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1. Introduction

We refer to the book [3] for graph-theoretical notation and terminology not described in this paper. We use the notation $[a, b]$ for the interval of integers x such that $a \leq x \leq b$. For a graph G of order n, a *numbering* f of G is a labeling that assigns distinct elements of the set $[1, n]$ to the vertices of G, where each $uv \in E(G)$ is labeled $f(u) + f(v)$. The *strength* $\text{str}_f(G)$ *of a numbering* $f : V(G) \to [1, n]$ of G is defined by

$$\text{str}_f(G) = \max\{f(u) + f(v) \mid uv \in E(G)\},$$

that is, $\text{str}_f(G)$ is the maximum edge label of G and the *strength* $\text{str}(G)$ of a graph G itself is

$$\text{str}(G) = \min\{\text{str}_f(G) \mid f \text{ is a numbering of } G\}.$$

A numbering f of a graph G for which $\text{str}_f(G) = \text{str}(G)$ is called a *strength labeling* of G. Since empty graphs nK_1 do not have edges, this definition does not apply to such graphs. Consequently, we may define $\text{str}(nK_1) = +\infty$ for every positive integer n. This type of numberings was introduced in [8] as a generalization of the problem of finding whether a graph is super edge-magic or not (see [4] for the definition of a super edge-magic graph, and also consult either [1] or [5] for alternative and often more useful definitions of the same concept).

There are other related parameters that have been studied in the area of graph labelings. Excellent sources for more information on this topic are found in the extensive survey by Gallian [6], which also includes information on other kinds of graph labeling problems as well as their applications.

Several bounds for the strength of a graph have been found in terms of other parameters defined on graphs (see [7,8,12]). Among others, the following result established in [8] that provides a lower bound for the strength of a graph G in terms of its order and the minimum degree $\delta(G)$ is particularly useful.

Lemma 1.1. *For every graph G of order n with $\delta(G) \geq 1$,*

$$\text{str}(G) \geq n + \delta(G).$$

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It is worth to mention that the lower bound given in Lemma 1.1 is sharp in the sense that there are infinitely many graphs G for which $\mathrm{str}(G) = |V(G)| + \delta(G)$ (see [7–9, 12] for a detailed list of such graphs and other sharp bounds).

For every graph G of order n, it is clear that $3 \leq \mathrm{str}(G) \leq 2n-1$. In fact, it was shown in [10] that for every $k \in [1, n-1]$, there exists a graph G of order n satisfying $\delta(G) = k$ and $\mathrm{str}(G) = n + k$.

In the process of settling the problem (proposed in [8]) of finding sufficient conditions for a graph G of order n with $\delta(G) \geq 1$ to ensure that $\mathrm{str}(G) = n + \delta(G)$, an equivalent definition of the following class of graphs was defined in [11]. For integers $k \geq 2$, let F_k be the graph with $V(F_k) = \{v_i \,|i \in [1,k]\}$ and

$$E(F_k) = \{v_i v_j \,|i \in [1, \lfloor k/2 \rfloor] \text{ and } j \in [1+i, k+1-i]\}.$$

Let $\overline{G}$ denote the complement of a graph G. The following result found in [11] provides a necessary and sufficient condition for a graph G of order n to hold the inequality $\mathrm{str}(G) \leq 2n-k$, where $k \in [2, n-1]$.

Theorem 1.1. *Let G be a graph of order n. Then $\mathrm{str}(G) \leq 2n-k$ if and only if $\overline{G}$ contains F_k as a subgraph, where $k \in [2, n-1]$.*

The preceding result plays an important role in the study of the strength of graphs (see [13]). The following result was deduced from Lemma 1.1 and Theorem 1.1.

Theorem 1.2. *Let G be a graph of order n with $\delta(G) = n-k$, where $k \in [2, n-1]$. Then $\mathrm{str}(G) = n + \delta(G)$ if and only if $\overline{G}$ contains F_k as a subgraph.*

The following lemma taken from [7] provides a lower bound for the strength of a graph G in terms of its independence number $\beta(G)$.

Lemma 1.2. *For every graph G of order n,*

$$\mathrm{str}(G) \geq 2n - 2\beta(G) + 1.$$

It is known from [8] that $\mathrm{str}(C_{2n+1}) = 2n+3$ $(n \geq 1)$. It is also clear that $\beta(C_{2n+1}) = n$ $(n \geq 1)$. Using these facts, Gao, Lau and Shiu [7] pointed out that

$$\begin{aligned}\mathrm{str}(C_{2n+1}) &= 2n+3\\ &= |V(C_{2n+1})| + \delta(C_{2n+1})\\ &= 2|V(C_{2n+1})| - 2\beta(C_{2n+1}) + 1\end{aligned}$$

for all positive integers n. They also proposed the following two problems in [7].

Problem 1.1. *For a graph G of order n, find necessary and/or sufficient conditions for which $\mathrm{str}(G) = 2n - 2\beta(G) + 1$.*

Problem 1.2. *Characterize all graphs G of order n for which $\mathrm{str}(G) = n + \delta(G) = 2n - 2\beta(G) + 1$.*

In this paper, we provide an answer to Problem 1.1. This together with Theorem 1.2 gives us an answer to Problem 1.2 under certain conditions. An earlier result also leads us to determine formulas for the strength of F_n and graphs containing F_n as a subgraph. In addition, we extend what is known in the literature about k-stable properties.

2. Results involving the independence number

In this section, we present the proof of the following theorem. We also provide formulas for the strength of F_n and graphs containing F_n as a subgraph.

Theorem 2.1. *Let G be a graph of order n with $\beta(G) = k$, where $k \in [2, \lceil n/2 \rceil]$. Then $\mathrm{str}(G) = 2n - 2\beta(G) + 1$ if and only if $\overline{G}$ contains F_{2k-1} as a subgraph.*

Proof. First, suppose that $\mathrm{str}(G) = 2n - 2k + 1$, where $\beta(G) = k$ $(k \in [2, \lceil n/2 \rceil])$. Let $V(G) = \{v_i \,|i \in [1,n]\}$, and assume, without loss of generality, that there exists a strength labeling of G that assigns i to v_i $(i \in [1,n])$. Since $\mathrm{str}(G) = 2n-2k+1$, every two vertices v_i and v_j for which $i + j > 2n - 2k + 1$ are not adjacent in G. This means that every two vertices v_i and v_j for which $i + j > 2n - 2k + 1$ are adjacent in $\overline{G}$. Let $v_i = w_{n+1-i}$ $(i \in [1,n])$ so that $V(\overline{G}) = \{w_i \,|i \in [1,n]\}$. Then if w_{n+1-i} and w_{n+1-j} are adjacent in $\overline{G}$, it follows that

$$(n+1-i) + (n+1-j) = 2n + 2 - (i+j) < 2n + 2 - (2n - 2k + 1) = 2k + 1.$$

Thus, $\overline{G}$ contains F_{2k-1} as a subgraph.

Next, suppose that $\overline{G}$ contains F_{2k-1} as a subgraph, where $\beta(G) = k$ ($k \in [2, \lceil n/2 \rceil]$). Then it follows from Theorem 1.1 that

$$\text{str}(G) \leq 2n - (2k-1) = 2n - 2\beta(G) + 1.$$

It also follows from Lemma 1.2 that

$$\text{str}(G) \geq 2n - 2\beta(G) + 1$$

and therefore $\text{str}(G) = 2n - 2\beta(G) + 1$. □

The preceding theorem together with Theorem 1.2 establishes the following result.

Theorem 2.2. *Let G be a graph of order n with $\delta(G) = n - (2k-1)$ and $\beta(G) = k$, where $k \in [2, \lceil n/2 \rceil]$. Then*

$$\text{str}(G) = n + \delta(G) = 2n - 2\beta(G) + 1$$

if and only if $\overline{G}$ contains F_{2k-1} as a subgraph.

The conditions described in Theorem 2.2 are strictly necessary for a graph G to meet $\text{str}(G) = n+\delta(G) = 2n-2\beta(G)+1$. Indeed, as we have seen earlier, the cycle C_{2n+1} of odd order meets all the conditions.

Since $\overline{F}_{2k-1} = F_{2k-2} \cup K_1$, it follows that $F_{2k-2} \subseteq \overline{F}_{2k-1}$. This together with Theorem 1.1 and Lemma 1.1 gives us the following result.

Lemma 2.1. *For every integer $k \geq 2$,*

$$\text{str}(F_{2k-1}) = 2k.$$

The following result is obtained from Theorem 2.1 rather easily.

Lemma 2.2. *For every integer $k \geq 2$,*

$$\text{str}(F_{2k}) = 2k + 1.$$

Proof. Let $S = \{v_i \,|\, i \in [1, k]\}$. Then the vertices $v_1, v_2, \ldots, v_{k+1}$ are mutually adjacent in F_{2k}, producing K_{k+1}. Thus,

$$\beta(F_{2k}) \leq 2k - (k+1) + 1 = k.$$

On the other hand, $v_{k+1}, v_{k+2}, \ldots, v_{2k}$ are k independent vertices in F_{2k}. Thus, $\beta(F_{2k}) \geq k$. Consequently, $\beta(F_{2k}) = k$. It remains to observe that $\overline{F}_{2k} = F_{2k-1} \cup K_1$, which implies that $F_{2k-1} \subseteq \overline{F}_{2k}$. Therefore, the result follows from Theorem 2.1. □

It is clear that $\text{str}(F_2) = 2 + 1 = 3$. Combining this with Lemmas 2.1 and 2.2, we have the following result.

Theorem 2.3. *For every integer $n \geq 2$,*

$$\text{str}(F_n) = n + 1.$$

Lemma 2.1 suggests the possibility of determining a formula for the strength of graphs containing F_{2k-1} as a subgraph as the next result indicates.

Corollary 2.1. *Let G be a graph of order $2k-1$ such that $\beta(G) = k$ and $\overline{G}$ contains F_{2k-1} as a subgraph, where $k \geq 2$. Then $\text{str}(G) = 2k$.*

Analogously, we have the following result for graphs containing F_{2k} as a subgraph.

Corollary 2.2. *Let G be a graph of order $2k$ such that $\beta(G) = k$ and $\overline{G}$ contains F_{2k} as a subgraph, where $k \geq 2$. Then $\text{str}(G) = 2k + 1$.*

We conclude this section with the following formula for the strength of the complement of graphs containing F_n as a subgraph.

Theorem 2.4. *Let G be a graph of order n such that $\beta(G) = \lceil n/2 \rceil$ and $\overline{G}$ contains F_n as a subgraph, where $n \geq 3$. Then $\text{str}(G) = n + 1$.*

3. Relations with stability

Motivated by the following theorem of Ore [14], the concept of k-stable was introduced by Bondy and Chvátal [2].

Theorem 3.1. *If G is a graph of order $n \geq 3$ such that for all distinct non-adjacent vertices u and v,*

$$\deg u + \deg v \geq n,$$

then G is hamiltonian.

Let P be a property defined on all graphs of order n and let k be a nonnegative integer. Then P is said to be k*-stable* if whenever $G + uv$ has property P and $\deg_G u + \deg_G v \geq k$ then G itself has property P. Therefore, Theorem 3.1 can be stated as follows: The property of containing a hamiltonian cycle is n-stable.

In [2], Bondy and Chvátal provided a variety of graph theoretical properties for stability. The relations between stability and strength as well as their related parameters were established in [11]. In this section, we prove the stability of a graph theoretical property, which combines both the strength and the independence number of a graph.

The proof of the following theorem is similar to the proof provided by Bondy and Chvátal [2] when they established the property that $\beta(G) \leq k$ is $(2n-2k-1)$-stable for positive integers n and k with $k \leq n$.

Theorem 3.2. *Let n and k be integers with $k \in [2, \lceil n/2 \rceil]$. Then the property that*

$$\beta(G) = k \textit{ and } \mathrm{str}(G) = 2n - 2k + 1$$

is $(2n-2k-1)$-stable.

Proof. Let $G + uv$ be any graph of order n such that

$$\beta(G + uv) = k \text{ and } \mathrm{str}(G + uv) = 2n - 2k + 1,$$

and assume that

$$\deg_G u + \deg_G v \geq 2n - 2k - 1,$$

where $u, v \in V(G)$ and $uv \notin E(G)$. First, notice that $k = \beta(G + uv) \leq \beta(G)$, since $G \subseteq G + uv$. Next, we show that $\beta(G) \leq k$. For this purpose, suppose, to the contrary, that $\beta(G) > k$. Then there is a set S of $k-1$ vertices of G such that $u, v \notin S$ and $S \cup \{u, v\}$ is independent in G. However,

$$\deg_G u \leq n - 2 - |S| \text{ and } \deg_G u \leq n - 2 - |S|,$$

implying that

$$\deg_G u + \deg_G v \leq 2(n - 2 - |S|) = 2n - 2k - 2.$$

This contradicts our assumption that $\deg_G u + \deg_G v \geq 2n - 2k - 1$ and so $\beta(G) = k$ for $k \in [2, \lceil n/2 \rceil]$. It is now immediate from Theorem 2.1 that $F_{2k-1} \subseteq \overline{G + uv}$. However, since $G \subseteq G + uv$, it follows that $\overline{G + uv} \subseteq \overline{G}$ and $F_{2k-1} \subseteq \overline{G}$. Therefore, Theorem 2.1 implies that $\mathrm{str}(G) = 2n - 2k + 1$. □

The preceding theorem has the following consequence.

Theorem 3.3. *Let n and k be integers with $k \in [2, \lceil n/2 \rceil]$. Then the property that*

$$\delta(G) = n - (2k - 1), \ \beta(G) = k \textit{ and } \mathrm{str}(G) = 2n - 2k + 1$$

is $(2n-2k)$-stable.

Proof. In light of the proof of Theorem 3.2, it suffices to verify that if

$$\delta(G + uv) = n - 2k + 1 \text{ and } \deg_G u + \deg_G v \geq 2n - 2k$$

for all distinct non-adjacent vertices u and v in G, then $\delta(G) = n - 2k + 1$. Since $G \subseteq G + uv$, it follows that

$$\delta(G) \leq \delta(G + uv) = n - 2k + 1.$$

To show the reverse inequality, suppose that $\delta(G + uv) = n - 2k + 1$, but $\delta(G) < n - 2k + 1$. Then there exists a vertex u with $\deg_G u \leq n - 2k$. Since $\deg_G v \leq n - 1$ for all $v \in V(G)$, it follows that

$$\deg_G u + \deg_G v \leq (n - 2k) + (n - 1) = 2n - 2k - 1,$$

producing a contradiction. □

4. Conclusions

In the following lines, we summarize the results established in this work. We have investigated the relation existing between the strength and independence number of graphs. By means of this, we have found a characterization of the graphs G of order n with $\beta(G) = k$ ($k \in [2, \lceil n/2 \rceil]$) to satisfy $\mathrm{str}(G) = 2n - 2\beta(G) + 1$ in terms of their subgraph structure (see Theorem 2.1). As a consequence of this, we have provided a necessary and sufficient condition for a graph G to meet $\mathrm{str}(G) = |V(G)| + \delta(G) = 2|V(G)| - 2\beta(G) + 1$ under certain conditions (see Theorem 2.2). This and the aforementioned result answer Problems 1.1 and 1.2 posed by Gao, Lau and Shiu [7], and Theorem 2.1 produces formulas for the strength of F_n and the complement of graphs containing F_n as a subgraph (see Theorems 2.3 and 2.4, respectively). We have also extended what is known in the literature about k-stable properties (see Theorems 3.2 and 3.3).

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