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Research Article

Necessary and sufficient conditions for a bivariate mean of three parameters to be the Schur m-power convex

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Abstract

In this paper, the authors find necessary and sufficient conditions for a bivariate mean of three parameters to be the Schur m-power convex or the Schur m-power concave, by using techniques of the majorization theory.

Keywords: Schur-convexity; Schur m-power convexity; necessary and sufficient condition; majorization; bivariate mean.

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1. Introduction

In 2009, Kuang [1, p. 61] defined the mean

$$K_{p;\omega_1,\omega_2}(a,b)=\begin{cases}\left(\dfrac{\omega_1 A(a^p,b^p)+\omega_2 G(a^p,b^p)}{\omega_1+\omega_2}\right)^{1/p}, & p\neq 0\\ G(a,b), & p=0\end{cases} \tag{1}$$

for $(a,b)\in\mathbb{R}_+^2=(0,\infty)\times(0,\infty)$, where $p\in\mathbb{R}=(-\infty,\infty)$, $\omega_1,\omega_2\in\mathbb{R}_0=[0,\infty)$ with $\omega_1+\omega_2\neq 0$, $A(a,b)=\frac{a+b}{2}$, and $G(a,b)=\sqrt{ab}$. In [4], Wang and his two coauthors investigated the Schur m-power convexity of the mean $K_{p;\omega_1,\omega_2}(a,b)$ and obtained the following theorem.

Theorem 1.1 (see [4, Theorem 1.1]). *Let $p,m\in\mathbb{R}$ and $\omega_1,\omega_2\in\mathbb{R}_0$ with $m\neq 0$ and $\omega_1+\omega_2\neq 0$.*

1. *For $m>0$,*
   (a) *if $p\geq\max\{(1+\frac{\omega_2}{\omega_1})m,2m\}$, then the mean $K_{p;\omega_1,\omega_2}(a,b)$ is Schur m-power convex with respect to $(a,b)\in\mathbb{R}_+^2$;*
   (b) *if $m\leq p\leq\min\{(1+\frac{\omega_2}{\omega_1})m,2m\}$, then the mean $K_{p;\omega_1,\omega_2}(a,b)$ is Schur m-power concave with respect to $(a,b)\in\mathbb{R}_+^2$;*
   (c) *if $0\leq p<m$, then the mean $K_{p;\omega_1,\omega_2}(a,b)$ is Schur m-power concave with respect to $(a,b)\in\mathbb{R}_+^2$;*
   (d) *if $p<0$, then the mean $K_{p;\omega_1,\omega_2}(a,b)$ is Schur m-power concave with respect to $(a,b)\in\mathbb{R}_+^2$.*
2. *For $m<0$,*
   (a) *if $p\geq 0$, then the mean $K_{p;\omega_1,\omega_2}(a,b)$ is Schur m-power convex with respect to $(a,b)\in\mathbb{R}_+^2$;*
   (b) *if $m\leq p<0$, then the mean $K_{p;\omega_1,\omega_2}(a,b)$ is Schur m-power convex with respect to $(a,b)\in\mathbb{R}_+^2$;*
   (c) *if $2m\leq p<m$, $p=(1+\frac{\omega_2}{\omega_1})m$, and $0<\frac{\omega_2}{\omega_1}<1$, then the mean $K_{p;\omega_1,\omega_2}(a,b)$ is Schur m-power convex with respect to $(a,b)\in\mathbb{R}_+^2$;*
   (d) *if $p<2m$, $p=(1+\frac{\omega_2}{\omega_1})m$, and $\frac{\omega_2}{\omega_1}>1$, then the mean $K_{p;\omega_1,\omega_2}(a,b)$ is Schur m-power concave with respect to $(a,b)\in\mathbb{R}_+^2$.*

The main aim of the present paper is to find sufficient and necessary conditions for the mean $K_{p;\omega_1,\omega_2}(a,b)$ to be Schur m-power convex with respect to $(a,b)\in\mathbb{R}_+^2$ for $p\in\mathbb{R}$ and $(\omega_1,\omega_2)\in\Omega$ by using some results reported in the paper [5].

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2. Definitions and lemmas

In order to obtain our main results, we need the following definitions and lemmas.

Definition 2.1 (see [2,3]). *Let $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_n)$ belong to $\mathbb{R}^n$.*

1. *The n-tuple $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if*

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}, \quad 1 \le k \le n-1$$

and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i,$$

where

$$x_{[1]} \ge x_{[2]} \ge \dots \ge x_{[n]} \quad \text{and} \quad y_{[1]} \ge y_{[2]} \ge \dots \ge y_{[n]}$$

are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in descending order.

2. *A set $\mathcal{D} \subseteq \mathbb{R}^n$ is said to be convex if*

$$(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \dots, \alpha x_n + \beta y_n) \in \mathcal{D}$$

for any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{D}$, where $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$.

3. *A function $\varphi : \mathcal{D} \to \mathbb{R}$ is said to be Schur-convex (or Schur-concave, respectively) if the majorizing relation $\boldsymbol{x} \prec \boldsymbol{y}$ on $\mathcal{D}$ implies the inequality $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$ (or $\varphi(\boldsymbol{x}) \ge \varphi(\boldsymbol{y})$, respectively).*

Definition 2.2 (see [5]). *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be the function defined by*

$$f(x) = \begin{cases} \dfrac{x^m - 1}{m}, & m \ne 0; \\ \ln x, & m = 0. \end{cases}$$

A function $\varphi : \mathcal{D} \subseteq \mathbb{R}_+^n \to \mathbb{R}$ is said to be Schur m-power convex (or Schur m-power concave, respectively) on $\mathcal{D}$ if the majorizing relation

$$f(\boldsymbol{x}) = (f(x_1), f(x_2), \dots, f(x_n)) \prec f(\boldsymbol{y}) = (f(y_1), f(y_2), \dots, f(y_n))$$

on $\mathcal{D}$ implies the inequality $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$ (or $\varphi(\boldsymbol{x}) \ge \varphi(\boldsymbol{y})$, respectively).

In proofs of our main results, we use the following lemmas.

Lemma 2.1 (see [5]). *Let $\mathcal{D} \subset \mathbb{R}_+^n$ be a symmetric set with nonempty interior $\mathcal{D}^\circ$ and let $\varphi : \mathcal{D} \to \mathbb{R}_+$ be continuous on $\mathcal{D}$ and differentiable in $\mathcal{D}^\circ$. Then φ is Schur m-power convex on $\mathcal{D}$ if and only if φ is symmetric on $\mathcal{D}$ and the function*

$$\begin{cases} \dfrac{x_1^m - x_2^m}{m}\left(x_1^{1-m}\dfrac{\partial\varphi(\boldsymbol{x})}{\partial x_1} - x_2^{1-m}\dfrac{\partial\varphi(\boldsymbol{x})}{\partial x_2}\right), & m \ne 0 \\ (\ln x_1 - \ln x_2)\left(x_1\dfrac{\partial\varphi(\boldsymbol{x})}{\partial x_1} - x_2\dfrac{\partial\varphi(\boldsymbol{x})}{\partial x_2}\right), & m = 0 \end{cases}$$

is nonnegative for $\boldsymbol{x} \in \mathcal{D}^\circ$.

In the paper [5], Yang established necessary and sufficient conditions for the Daróczy mean

$$H_{p,\omega}(a,b) = \begin{cases} \left(\dfrac{a^p + \omega(ab)^{p/2} + b^p}{\omega + 2}\right)^{1/p}, & p \ne 0 \\ \sqrt{ab}\,, & p = 0 \end{cases} \tag{2}$$

to be Schur m-power convex, where $(a,b) \in \mathbb{R}_+^2$, $p \in \mathbb{R}$, and $\omega > -2$.

Lemma 2.2 (see [5, Theorem 7]). *For a fixed $p \in \mathbb{R}$, $m = 0$, and $\omega > -2$, the Daróczy mean $H_{p,\omega}(a,b)$ is Schur m-power convex (or Schur m-power concave, respectively) with respect to $(a,b) \in \mathbb{R}_+^2$ if and only if $p \ge 0$ (or $p \le 0$, respectively).*

Lemma 2.3 (see [5, Theorems 3 and 4]). *For a fixed $p \in \mathbb{R}$, $m > 0$, and $\omega > -2$, the Daróczy mean $H_{p,\omega}(a,b)$ is Schur m-power convex (or Schur m-power concave, respectively) with respect to $(a,b) \in \mathbb{R}_+^2$ if and only if $(p,\omega) \in V_1$ (or $(p,\omega) \in V_2$, respectively), where*

$$V_1 = \left\{(p,\omega) : -2 < \omega \le 0, p \ge \frac{\omega+2}{2}m\right\} \cup \left\{(p,\omega) : \omega > 0, p \ge \max\left\{\frac{\omega+2}{2}m, 2m\right\}\right\}$$

and

$$V_2 = \left\{(p,\omega) : -2 < \omega < 0, p < 0\right\} \cup \left\{(p,\omega) : \omega \ge 0, p \le \min\left\{\frac{\omega+2}{2}m, 2m\right\}\right\}.$$

Lemma 2.4 (see [5, Theorems 5 and 6]). *For a fixed $p \in \mathbb{R}$, $m < 0$, and $\omega > -2$, the Daróczy mean $H_{p,\omega}(a,b)$ is Schur m-power convex (or Schur m-power concave, respectively) with respect to $(a,b) \in \mathbb{R}_+^2$ if and only if $(p,\omega) \in E_1$ (or $(p,\omega) \in E_2$, respectively), where*

$$E_1 = \left\{(p,\omega) : -2 < \omega < 0, p > 0\right\} \cup \left\{(p,\omega) : \omega \ge 0, p \ge \max\left\{\frac{\omega+2}{2}m, 2m\right\}\right\}$$

and

$$E_2 = \left\{(p,\omega) : -2 < \omega \le 0, p \le \frac{\omega+2}{2}m\right\} \cup \left\{(p,\omega) : \omega > 0, p \le \min\left\{\frac{\omega+2}{2}m, 2m\right\}\right\}.$$

3. Main results and their proofs

The value range of the parameter $(\omega_1,\omega_2) \in \mathbb{R}_0^2$ with $\omega_1 + \omega_2 \ne 0$ in (1) can be extended to $\Omega = \Omega_1 \cup \Omega_2$, where

$$\Omega_1 = \left\{(\omega_1,\omega_2) : (\omega_1,\omega_2) \in \mathbb{R}^2, \omega_1\omega_2 \ge 0, \omega_1+\omega_2 \ne 0\right\}$$

and

$$\Omega_2 = \left\{(\omega_1,\omega_2) : (\omega_1,\omega_2) \in \mathbb{R}^2, \omega_1\omega_2 \le 0, |\omega_1| > |\omega_2|\right\}.$$

Remark 3.1. *Let $(\omega_1,\omega_2) \in \Omega$.*

1. *If $\omega_1 = 0$ and $\omega_2 \ne 0$, then $K_{p;0,\omega_2}(a,b) = G(a,b)$ for $(a,b) \in \mathbb{R}_+^2$ and $p \in \mathbb{R}$.*
2. *When $\omega_1 \ne 0$, we take $\omega = \frac{2\omega_2}{\omega_1}$. If $(\omega_1,\omega_2) \in \Omega_1$, then we have $\omega \ge 0$, and if $(\omega_1,\omega_2) \in \Omega_2$, then we obtain $-2 < \omega \le 0$. By the definitions in (1) and (2), we acquire $K_{p;\omega_1,\omega_2}(a,b) = H_{p,\omega}(a,b)$ for $(a,b) \in \mathbb{R}_+^2$ and $p \in \mathbb{R}$.*

Now, we are in a position to state and prove our main results.

Theorem 3.1. *Let $p,m \in \mathbb{R}$ and $\omega_2 \in \mathbb{R}$ with $\omega_2 \ne 0$. For every $p \in \mathbb{R}$, the symmetric function $K_{p;0,\omega_2}(a,b)$ is Schur m-power convex (or Schur m-power concave, respectively) with respect to $(a,b) \in \mathbb{R}_+^2$ if and only if $m \le 0$ (or $m \ge 0$, respectively).*

Proof. Since $K_{p;0,\omega_2}(a,b) = G(a,b)$ for $(a,b) \in \mathbb{R}_+^2$ and $p \in \mathbb{R}$, the desired conclusion follows from Lemma 2.1 immediately. □

Theorem 3.2. *Let $p \in \mathbb{R}$, $m = 0$, and $(\omega_1,\omega_2) \in \Omega$ with $\omega_1 \ne 0$. Then the symmetric function $K_{p;\omega_1,\omega_2}(a,b)$ is Schur m-power convex (or Schur m-power concave, respectively) with respect to $(a,b) \in \mathbb{R}_+^2$ if and only if $p \ge 0$ (or $p \le 0$, respectively).*

Proof. Since $\omega_1 \ne 0$, using the second item in Remark 3.1, we obtain

$$\frac{\omega+2}{2}m = \frac{\omega_1+\omega_2}{\omega_1}m$$

and

$$K_{p;\omega_1,\omega_2}(a,b) = H_{p,\omega}(a,b),$$

where $\omega = \frac{2\omega_2}{\omega_1}$. Combining this with Lemma 2.2 leads to the desired conclusion readily. □

Theorem 3.3. *Let $p \in \mathbb{R}$, $m > 0$, and $(\omega_1,\omega_2) \in \Omega$ with $\omega_1 \ne 0$. Then the symmetric function $K_{p;\omega_1,\omega_2}(a,b)$ is Schur m-power convex (or Schur m-power concave, respectively) with respect to $(a,b) \in \mathbb{R}_+^2$ if and only if $(p,\omega_1,\omega_2) \in S_1$ (or $(p,\omega_1,\omega_2) \in S_2$, respectively), where*

$$S_1 = \left\{(p,\omega_1,\omega_2) : \omega_1\omega_2 < 0, |\omega_1| > |\omega_2|, p \ge \frac{\omega_1+\omega_2}{\omega_1}m\right\} \cup \left\{(p,\omega_1,\omega_2) : \omega_1\omega_2 \ge 0, \omega_1 \ne 0, p \ge \max\left\{\frac{\omega_1+\omega_2}{\omega_1}m, 2m\right\}\right\}$$

and

$$S_2 = \left\{(p,\omega_1,\omega_2) : \omega_1\omega_2 < 0, |\omega_1| > |\omega_2|, p < 0\right\} \cup \left\{(p,\omega_1,\omega_2) : \omega_1\omega_2 \ge 0, \omega_1 \ne 0, p \le \min\left\{\frac{\omega_1+\omega_2}{\omega_1}m, 2m\right\}\right\}.$$

Proof. Since $\omega_1 \neq 0$, for $\omega = \frac{2\omega_2}{\omega_1}$, by virtue of the second item in Remark 3.1, we have $K_{p;\omega_1,\omega_2}(a,b) = H_{p,\omega}(a,b)$. Combining this with Lemma 2.3 results in the required conclusion directly. □

Theorem 3.4. *Let $p \in \mathbb{R}$, $m < 0$, and $(\omega_1, \omega_2) \in \Omega$ with $\omega_1 \neq 0$. Then the symmetric function $K_{p;\omega_1,\omega_2}(a,b)$ is Schur m-power convex (or Schur m-power concave, respectively) with respect to $(a,b) \in \mathbb{R}_+^2$ if and only if $(p,\omega_1,\omega_2) \in T_1$ (or $(p,\omega_1,\omega_2) \in T_2$, respectively), where*

$$T_1 = \{(p,\omega_1,\omega_2) : \omega_1\omega_2 < 0, |\omega_1| > |\omega_2|, p > 0\} \cup \left\{(p,\omega_1,\omega_2) : \omega_1\omega_2 \geq 0, \omega_1 \neq 0, p \geq \max\left\{\frac{\omega_1+\omega_2}{\omega_1}m, 2m\right\}\right\}$$

and

$$T_2 = \left\{(p,\omega_1,\omega_2) : \omega_1\omega_2 < 0, |\omega_1| > |\omega_2|, p \leq \frac{\omega_1+\omega_2}{\omega_1}m\right\} \cup \left\{(p,\omega_1,\omega_2) : \omega_1\omega_2 \geq 0, \omega_1 \neq 0, p \leq \min\left\{\frac{\omega_1+\omega_2}{\omega_1}m, 2m\right\}\right\}.$$

Proof. Since $\omega_1 \neq 0$, by combining the second item of Remark 3.1 with Lemma 2.4, we arrive at the desired conclusion. □

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