Three determinants evaluated in circular products

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(Received: 6 June 2022. Received in revised form: 12 July 2022. Accepted: 13 July 2022. Published online: 14 July 2022.)

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Abstract

Three determinants for symmetric and skew-symmetric matrices are explicitly evaluated, in closed form, as circular products. One of them gives a solution to a problem proposed by Dzhumadil’daev [Amer. Math. Monthly 129 (2022) 486].

Keywords: determinant; Laplace expansion; symmetric matrix; skew-symmetric matrix; circular product.

2020 Mathematics Subject Classification: 15A15, 15B57.

1. Introduction and the main results

Evaluating determinants is one of the important topics in mathematics and physics. There are two classical results (cf. [2] and [3, §4.3]) about a skew-symmetric matrix \(\Xi\). When the order of \(\Xi\) is odd, then \(\det \Xi = 0\). When the order of \(\Xi\) is even, the determinant \(\det \Xi\) results in a square of a polynomial in the entries of \(\Xi\). In this paper, we explicitly evaluate three determinants, in closed form, as circular products. The main results are announced in advance as follows, whose proofs are given in the next section. The first one is about the symmetric matrix

\[
U_n = [u_{i,j}]_{1 \leq i,j \leq n}, \quad u_{i,j} = \begin{cases} x_i - x_j, & i \leq j; \\ x_j - x_i, & i > j. \end{cases}
\]

**Theorem 1.1** (Determinant identity for symmetric matrices).

\[
\det U_n = 2^{n-2} \prod_{k=1}^{n} (x_{k+1} - x_k), \quad \text{where} \quad x_{n+1} := x_1.
\]

The next two results are concerned with the following skew-symmetric matrices

\[
A_n = [a_{i,j}]_{1 \leq i,j \leq n}, \quad a_{i,j} = \begin{cases} (x_i - x_j)^2, & i \leq j; \\ -(x_i - x_j)^2, & i > j; \end{cases}
\]

\[
\Omega_n = [\omega_{i,j}]_{1 \leq i,j \leq n}, \quad \omega_{i,j} = \begin{cases} x_1^\lambda (x_i - x_j), & i \leq j; \\ x_j^\lambda (x_i - x_j), & i > j; \end{cases}
\]

where \(\lambda\) and \(\{x_k\}_{1 \leq k \leq n}\) are real numbers.

**Theorem 1.2** (Determinant identity for skew-symmetric matrices).

\[
\det A_{2n} = 4^{n-1} \prod_{k=1}^{2n} (x_k - x_{k+1})^2, \quad \text{where} \quad x_{2n+1} := x_1.
\]

It is remarked here that Theorem 1.2 resolves a monthly problem proposed recently by Dzhumadil’daev [1].

**Theorem 1.3** (Determinant identity for skew-symmetric matrices).

\[
\det \Omega_{2n} = x_1^{2\lambda} \prod_{k=1}^{n} (x_{2k} - x_{2k-1})^2 \prod_{k=1}^{n-1} (x_{2k}^\lambda - x_{2k+1}^\lambda)^2.
\]

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2. Proofs of Theorems 1.1, 1.2, 1.3

This section is divided into three subsections, dedicating to proofs of the three corresponding theorems anticipated in the previous section.

2.1. Proof of Theorem 1.1

By examining the difference between the \(i\)th row and the \((i+1)\)th row, we see that the resulting entry \(u'_{i,j}\) in the \((i,j)\) position equals

\[
u'_{i,j} = u_{i,j} - u_{i+1,j} = \begin{cases} 
    x_i - x_{i+1}, & i < j; \\
    x_{i+1} - x_i, & i \geq j.
\end{cases}
\]

Iterating this operation downwards from the first row to the penultimate row, and then extracting the common row factors, we get the following equality

\[
det U_n = \prod_{i=1}^{n-1} (x_i - x_{i+1}) \times det V_n,
\]

where the matrix \(V_n\) is given by

\[
V_n = [v_{i,j}]: \quad v_{i,j} = \begin{cases} 
    1, & i < j; \\
    -1, & i \geq j;
\end{cases}
\]

\[
x_j - x_n, \quad i = n.
\]

Next, for the matrix \(V_n\), we make the same row operations. Considering the difference between the \(i\)th row and the \((i+1)\)th row, where \(1 \leq i \leq n-2\), we can check without difficulty that the resulting entry \(v'_{i,j}\) in the \((i,j)\) position becomes

\[
v'_{i,j} = v_{i,j} - v_{i+1,j} = \begin{cases} 
    0, & j \neq i + 1; \\
    2, & j = i + 1.
\end{cases}
\]

Repeating this operation for \(i\) from 1 to \(n-2\), we derive another equality

\[
det V_n = det W_n,
\]

where the matrix \(W_n\) is given by

\[
W_n = [w_{i,j}]: \quad w_{i,j} = \begin{cases} 
    2, & i = j - 1; \\
    0, & i \neq j - 1;
\end{cases}
\]

\[
x_j - x_n, \quad i = n.
\]

Write this matrix explicitly

\[
W_n = \begin{bmatrix}
0 & 2 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 2 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 2 & 0 \\
-1 & -1 & -1 & -1 & \cdots & -1 & 1 \\
x_1 - x_n & x_2 - x_n & x_3 - x_n & \cdots & x_{n-1} - x_n & x_{n-1} - x_n & \cdots & x_1 - x_n \end{bmatrix}
\]

Expanding the determinant of \(W_n\) with respect to the first and the last columns, we find that

\[
det W_n = (-1)^n (x_n - x_1) 2^{n-2}
\]

which confirms the circular product formula stated in Theorem 1.1

\[
det U_n = 2^{n-2} \prod_{k=1}^{n} (x_{k+1} - x_k), \quad \text{where} \quad x_{n+1} := x_1.
\]
2.2. Proof of Theorem 1.2

Firstly, we reduce the matrix $A_{2n}$ by row and column operations. If we subtract the $(i+1)$th row from the $i$th row, then the resulting entry $a'_{i,j}$ in the $(i,j)$ position becomes

$$a'_{i,j} = a_{i,j} - a_{i+1,j} = \begin{cases} (x_i - x_{i+1})(x_i + x_{i+1} - 2x_j), & i < j - 1; \\ (x_i - x_{i+1})^2, & i = j & i = j - 1; \\ (x_i - x_{i+1})(2x_j - x_i - x_{i+1}), & i > j. \end{cases}$$

Repeating this operation downwards for all the rows except for the last one and then pulling out the common factors from the first row to the $(2n-1)$th row, we get the equality

$$\det A_{2n} = \prod_{i=1}^{2n-1} (x_i - x_{i+1}) \times \det B_{2n},$$

where the square matrix $B_{2n}$ is given by

$$B_{2n} = [b_{i,j}]: \quad b_{i,j} = \begin{cases} x_i + x_{i+1} - 2x_j, & 1 \leq i < j - 1; \\ x_i - x_{i+1}, & i = j & i = j - 1; \\ 2x_j - x_i - x_{i+1}, & j < i < 2n; \\ -(x_{2n} - x_j)^2, & i = 2n. \end{cases}$$

Analogously, for the matrix $B_{2n}$, we make the corresponding column operations. By subtracting the $(j+1)$th column from the $j$th column, the resulting entry $b'_{i,j}$ in the $(i,j)$ position becomes

$$b'_{i,j} = b_{i,j} - b_{i,j+1} = \begin{cases} 2(x_{j+1} - x_j), & i < j; \\ 0, & i = j; \\ 2(x_j - x_{j+1}), & i > j; \\ (x_{j+1} - x_j)(x_j + x_{j+1} - 2x_{2n}), & i = 2n. \end{cases}$$

Iterating this operation rightwards for all the columns except for the last one and then extracting the common column factors from the first column to the $(2n-1)$th column, we derive another equality

$$\det B_{2n} = \prod_{j=1}^{2n-1} (x_j - x_{j+1}) \times \det C_{2n},$$

where the matrix $C_{2n}$ returns to skew-symmetric one:

$$C_{2n} = [c_{i,j}]: \quad c_{i,j} = \begin{cases} -2, & i < j < 2n; \\ 0, & i = j; \\ 2, & j < i < 2n; \\ 2x_{2n} - x_j - x_{j+1}, & i = 2n; \\ x_i + x_{i+1} - 2x_{2n}, & j = 2n. \end{cases}$$

Finally, for each $k$ with $1 \leq k < 2n$, performing simultaneously the operations on the matrix $C_{2n}$ by subtracting the $(k+1)$th row and column from the $k$th row and column, respectively, we find the following reduced expression

$$\det C_{2n} = \det D_{2n},$$

where $D_{2n}$ is the double bordered skew-symmetric matrix

$$D_{2n} = [d_{i,j}]: \quad d_{i,j} = \begin{cases} 2i - 2j, & |i - j| = 1 & i, j < 2n; \\ 0, & |i - j| \neq 1 & i, j < 2n; \\ x_{j+2} - x_j, & i = 2n & j < 2n - 1; \\ x_{j+1} - x_j, & i = 2n & j = 2n - 1; \\ x_i - x_{i+2}, & j = 2n & i < 2n - 1; \\ x_i - x_{i+1}, & j = 2n & i = 2n - 1. \end{cases}$$
Now, we return to present an inductive proof for

\[ \begin{bmatrix} E_{2n-1} & -\vec{y} \\ \cdots & \cdots & \cdots \\ \vec{y} & 0 \end{bmatrix}, \text{ where } E_{2n-1} = \begin{bmatrix} e_{i,j} \end{bmatrix} : e_{i,j} = \begin{cases} \pm 2, & i - j = \pm 1; \\ 0, & i - j \neq \pm 1; \end{cases} \]

and

\[ \vec{y} = (y_1, y_2, \ldots, y_{2n-1}) : \begin{cases} y_k = x_{k+2} - x_k, & 1 \leq k < 2n - 1; \\ y_{2n-1} = x_2 - x_{2n-1}, & k = 2n - 1. \end{cases} \]

By means of the Laplace formula, expanding the determinant of \( D_{2n} \) along the last row and then the last column, we have the double sum expression

\[ \det D_{2n} = \sum_{1 \leq i, j < 2n} (-1)^{i+j} y_i y_j E_{2n-1}(i, j), \]

where \( E_{2n-1}(i, j) \) stands for the minor of \( E_{2n-1} \) after the \( i \)th row and the \( j \)th column having been removed. In general, we have the following remarkable formula

\[ E_{2n-1}(i, j) = 2^{2n-2} \times \begin{cases} 0, & i \times j \equiv 0 \pmod{2}; \\ 1, & i \times j \equiv 1 \pmod{2}. \end{cases} \]

According to this formula, we can rewrite, under the replacements \( i \rightarrow 2i - 1 \) and \( j \rightarrow 2j - 1 \), the former double sum for \( \det D_{2n} \) as

\[ \det D_{2n} = 4^{n-1} \sum_{1 \leq i, j \leq n} y_{2i-1} y_{2j-1} \]

\[ = 4^{n-1} \sum_{i=1}^{n} y_{2i-1} \sum_{j=1}^{n} y_{2j-1} \]

\[ = 4^{n-1} (x_1 - x_{2n})^2, \]

where the last step is justified by applying twice the telescoping method. Summing up, we have proved that

\[ \det A_{2n} = 4^{n-1} \prod_{k=1}^{2n} (x_k - x_{k+1})^2, \text{ where } x_{2n+1} := x_1. \]

\[ \square \]

**Induction principle** Now, we return to present an inductive proof for [★]. It is routine to verify that the formula [★] is true for \( E_3 \). Suppose that the same formula is valid for \( E_{2n-1} \). Then we have to validate it for \( E_{2n+1} \). In order to facilitate the intuitive reasoning below, we write explicitly the corresponding matrix

\[ n = 4 : \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \]

Now, we can prove the formula in [★] for \( E_{2n+1} \) case by case as follows:

- **\([i, j < 2n]\)** Expanding the determinant along the last row and then the last column, we see that \( E_{2n+1}(i, j) = E_{2n-1}(i, j) \).

- **\([i = 2n]\)** When \( j \neq 2n + 1 \), we have directly \( E_{2n+1}(2n, j) = 0 \) since all the entries in the last column result in zero. Instead, for \( j = 2n + 1 \), we have \( E_{2n+1}(2n, 2n + 1) = 0 \) because we can reduce the related matrix, by simple row and column operations, to a lower triangular matrix containing zero diagonal entries.

- **\([i = 2n + 1]\)** When \( j = 1 \), we have immediately \( E_{2n+1}(2n + 1, 1) = 1 \) since the matrix is lower triangular with all the diagonal entries equal to \(-1\). When \( j = 2 \), the minor \( E_{2n+1}(2n + 1, 2) = 0 \) since all the entries in the first row vanish. Finally for \( j > 2 \), by expanding the determinant along the first row and then the first column, we find that the invariant relation \( E_{2n+1}(2n + 1, j) = E_{2n-1}(2n - 1, j - 2) \).

In conclusion, we have verified that [★] is true for all the \( E_{2n+1} \) with \( n > 1 \).
2.3. Proof of Theorem 1.3

For the matrix $\Omega_{2n}$, by examining the difference of the $i$th row minus $(i-1)$th row and then iterating this operation upwards for all the rows except for the first one, we get the resulting matrix

$$\Omega'_{2n} = [\omega'_{i,j}]: \quad \omega'_{i,j} = \omega_{i,j} - \omega_{i-1,j} = \begin{cases} \lambda_i^2(x_i-x_j), & i = 1; \\ \lambda_i^2(x_i-x_j) - \lambda_{i-1}^2(x_{i-1}-x_j), & i \leq j; \\ \lambda_i^2(x_i-x_{i-1}), & i > j. \end{cases}$$

Similarly, by making the column operations leftwards, we transform the matrix $\Omega'_{2n}$ into another one

$$\Omega''_{2n} = [\omega''_{i,j}]: \quad \omega''_{i,j} = \omega_{i,j} - \omega'_{i,j-1} = \begin{cases} \lambda_i^2(x_{i-1}-x_j), & i = 1; \\ \lambda_i^2(x_i-x_{i-1}), & j = 1; \\ 0, & i = j; \\ (\lambda_i^2 - \lambda_{i-1}^2)(x_j-x_{j-1}), & i < j; \\ (\lambda_i^2 - \lambda_{i-1}^2)(x_i-x_{i-1}), & i > j. \end{cases}$$

Now, extracting the common factor $\lambda_i^2$ from both the first row and the first column, and then $\lambda_k - \lambda_{k-1}$ from the $k$th row and the $k$th column for $k$ from 2 to $2n$, we find the following determinant equality

$$\det \Omega_{2n} = \det \Omega'_{2n} = \det \Omega''_{2n} = x_1^2 \prod_{k=2}^{2n} (x_k - x_{k-1})^2 \det \Phi_{2n}(y_2, y_3, \ldots, y_{2n-1}).$$

The above skew-symmetric matrix $\Phi_{2n}$ can be expressed in blocks as

$$\Phi_{2n}(y_2, y_3, \ldots, y_{2n-1}) = [\phi_{i,j}] = \begin{bmatrix} 0 & -1 & -1 & \cdots & -1 \\ 1 & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1^2 & \cdots & \lambda_n^2 \\ \psi_{2n-1} \end{bmatrix} : \quad \phi_{i,j} = \begin{cases} -1, & i = 1; \\ 1, & j = 1; \\ 0, & i = j; \\ -y_i, & i < j; \\ y_j, & i > j; \end{cases}$$

where the submatrix $\psi_{2n-1}$ is defined by

$$\psi_{2n-m} = [\psi_{i,j}]_{m<i,j\leq 2n}: \quad \psi_{i,j} = \begin{cases} 0, & i = j; \\ -y_i, & i < j; \\ y_j, & i > j; \end{cases} \quad \text{with} \quad y_k = \frac{\lambda_k - \lambda_{k-1}}{x_k - x_{k-1}}.$$ 

Now, by subtracting $y_2$ times the first row and the first column, respectively, from the second row and the second column, we can further reduce the matrix $\Phi_{2n}$ to the following skew-symmetric matrix:

$$\Phi'_{2n} = [\phi'_{i,j}] = \begin{bmatrix} 0 & -1 & -1 & \cdots & -1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1^2 & \cdots & \lambda_n^2 \\ \psi_{2n-2} \end{bmatrix} : \quad \phi'_{i,j} = \begin{cases} 0, & 1 \leq i = j \leq 2n; \\ -1, & i = 1 \& j > 1; \\ 1, & j = 1 \& i > 1; \\ 0, & i = 2 \& j \geq 2; \\ 0, & j = 2 \& i \geq 2; \\ -y_i, & 2 < i < j \leq 2n; \\ y_j, & 2 < j < i \leq 2n; \end{cases}$$

where $\psi_{2n-2}$ is the skew-symmetric matrix explicitly given by

$$\psi_{2n-2} = [\psi_{i,j}]_{2<i,j\leq 2n}: \quad \psi_{i,j} = \begin{cases} 0, & i = j; \\ -y_{i+2}, & i < j; \\ y_{j+2}, & i > j. \end{cases}$$
Expanding the determinant of $\Phi^{'}_{2n}$ along the second row and then the second column, we find the determinant equality
\[
\det \Phi_{2n}(y_2, y_3, \cdots, y_{2n-1}) = \det \Phi^{'}_{2n} = \det \Psi_{2n-2}.
\]
By pulling out the common factor $y_3$ from the first row and the first column of $\Psi_{2n-2}$, we derive the recurrence relation below
\[
\det \Phi_{2n}(y_2, y_3, \cdots, y_{2n-1}) = y_3^2 \times \det \Phi^{'}_{2n-2}(y_4, y_5, \cdots, y_{2n-1}).
\]
Iterating this equation $(n - 1)$ times, we find the closed formula
\[
\det \Phi_{2n}(y_2, y_3, \cdots, y_{2n-1}) = \det \Phi_2 \prod_{k=1}^{n-1} y_{2k+1}^{2}.
\]
Since $\det \Phi_2 = 1$, we find finally that
\[
\det \Omega_{2n} = \det \Omega^{(1)}_{2n} = x_1^{2\lambda} \prod_{k=2}^{2n} (x_k - x_{k-1})^2 \det \Phi_{2n}(y_2, y_3, \cdots, y_{2n-1})
\]
\[
= x_1^{2\lambda} \prod_{k=2}^{2n} (x_k - x_{k-1})^2 \prod_{k=1}^{n-1} y_{2k+1}^{2}
\]
\[
= x_1^{2\lambda} \prod_{k=2}^{2n} (x_k - x_{k-1})^2 \prod_{k=1}^{n-1} \left(\frac{x_{2k} - x_{2k+1}}{x_{2k} - x_{2k+1}}\right)^2
\]
\[
= x_1^{2\lambda} \prod_{k=1}^{n} (x_{2k} - x_{2k-1})^2 \prod_{k=1}^{n-1} \left(\frac{x_{2k}}{x_{2k}} - \frac{x_{2k+1}}{x_{2k+1}}\right)^2.
\]

References