Contributions to Mathematics
www.shahindp.com/locate/cm

Contrib. Math. **6** (2022) 15–20
DOI: 10.47443/cm.2022.029

Research Article

Three determinants evaluated in circular products

Shuling Gao[1], Wenchang Chu[1,2,*]

[1]*School of Mathematics and Statistics, Zhoukou Normal University, Henan, China*
[2]*Department of Mathematics and Physics, University of Salento, Lecce, Italy*

(Received: 6 June 2022. Received in revised form: 12 July 2022. Accepted: 13 July 2022. Published online: 14 July 2022.)

Abstract

Three determinants for symmetric and skew-symmetric matrices are explicitly evaluated, in closed form, as circular products. One of them gives a solution to a problem proposed by Dzhumadil'daev [*Amer. Math. Monthly* **129** (2022) 486].

Keywords: determinant; Laplace expansion; symmetric matrix; skew-symmetric matrix; circular product.

2020 Mathematics Subject Classification: 15A15, 15B57.

1. Introduction and the main results

Evaluating determinants is one of the important topics in mathematics and physics. There are two classical results (cf. [2] and [3, §4.3]) about a skew-symmetric matrix Ξ. When the order of Ξ is odd, then $\det \Xi = 0$. When the order of Ξ is even, the determinant $\det \Xi$ results in a square of a polynomial in the entries of Ξ. In this paper, we explicitly evaluate three determinants, in closed form, as circular products. The main results are announced in advance as follows, whose proofs are given in the next section. The first one is about the symmetric matrix

$$U_n = \big[u_{i,j}\big]_{1\le i,j\le n}: \quad u_{i,j} = \begin{cases} x_i - x_j, & i \le j; \\ x_j - x_i, & i > j. \end{cases}$$

Theorem 1.1 (Determinant identity for symmetric matrices)**.**

$$\det U_n = 2^{n-2}\prod_{k=1}^{n}(x_{k+1}-x_k), \quad \textit{where} \quad x_{n+1} := x_1.$$

The next two results are concerned with the following skew-symmetric matrices

$$A_n = \big[a_{i,j}\big]_{1\le i,j\le n}: \quad a_{i,j} = \begin{cases} (x_i - x_j)^2, & i \le j; \\ -(x_i - x_j)^2, & i > j; \end{cases}$$

$$\Omega_n = \big[\omega_{i,j}\big]_{1\le i,j\le n}: \quad \omega_{i,j} = \begin{cases} x_i^{\lambda}(x_i - x_j), & i \le j; \\ x_j^{\lambda}(x_i - x_j), & i > j; \end{cases}$$

where λ and $\{x_k\}_{1\le k\le n}$ are real numbers.

Theorem 1.2 (Determinant identity for skew-symmetric matrices)**.**

$$\det A_{2n} = 4^{n-1}\prod_{k=1}^{2n}(x_k - x_{k+1})^2, \quad \textit{where} \quad x_{2n+1} := x_1.$$

It is remarked here that Theorem 1.2 resolves a monthly problem proposed recently by Dzhumadil'daev [1].

Theorem 1.3 (Determinant identity for skew-symmetric matrices)**.**

$$\det \Omega_{2n} = x_1^{2\lambda}\prod_{k=1}^{n}(x_{2k}-x_{2k-1})^2\prod_{k=1}^{n-1}(x_{2k}^{\lambda}-x_{2k+1}^{\lambda})^2.$$

*Corresponding author (chu.wenchang@unisalento.it).

2. Proofs of Theorems 1.1, 1.2, 1.3

This section is divided into three subsections, dedicating to proofs of the three corresponding theorems anticipated in the previous section.

2.1. Proof of Theorem 1.1

By examining the difference between the ith row and the $(i+1)$th row, we see that the resulting entry $u'_{i,j}$ in the (i,j) position equals

$$u'_{i,j} = u_{i,j} - u_{i+1,j} = \begin{cases} x_i - x_{i+1}, & i < j; \\ x_{i+1} - x_i, & i \geq j. \end{cases}$$

Iterating this operation downwards from the first row to the penultimate row, and then extracting the common row factors, we get the following equality

$$\det U_n = \prod_{i=1}^{n-1} (x_i - x_{i+1}) \times \det V_n,$$

where the matrix V_n is given by

$$V_n = \big[v_{i,j}\big] : \quad v_{i,j} = \begin{cases} 1, & i < j; \\ -1, & i \geq j; \\ x_j - x_n, & i = n. \end{cases}$$

Next, for the matrix V_n, we make the same row operations. Considering the difference between the ith row and the $(i+1)$th row, where $1 \leq i \leq n-2$, we can check without difficulty that the resulting entry $v'_{i,j}$ in the (i,j) position becomes

$$v'_{i,j} = v_{i,j} - v_{i+1,j} = \begin{cases} 0, & j \neq i+1; \\ 2, & j = i+1. \end{cases}$$

Repeating this operation for i from 1 to $n-2$, we derive another equality

$$\det V_n = \det W_n,$$

where the matrix W_n is given by

$$W_n = \big[w_{i,j}\big] : \quad w_{i,j} = \begin{cases} 2, & i = j-1; \\ 0, & i \neq j-1; \\ -1, & i = n-1 \;\&\; j < n; \\ 1, & i = n-1 \;\&\; j = n; \\ x_j - x_n, & i = n. \end{cases}$$

Write this matrix explicitly

$$W_n = \begin{bmatrix} 0 & 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & 0 \\ -1 & -1 & -1 & -1 & \cdots & -1 & 1 \\ x_1 - x_n & x_2 - x_n & x_3 - x_n & \cdots & x_{n-2} - x_n & x_{n-1} - x_n & 0 \end{bmatrix}.$$

Expanding the determinant of W_n with respect to the first and the last columns, we find that

$$\det W_n = (-1)^n (x_n - x_1) 2^{n-2}$$

which confirms the circular product formula stated in Theorem 1.1

$$\det U_n = 2^{n-2} \prod_{k=1}^{n} (x_{k+1} - x_k), \quad \text{where} \quad x_{n+1} := x_1. \qquad \square$$

2.2. Proof of Theorem 1.2

Firstly, we reduce the matrix A_{2n} by row and column operations. If we subtract the $(i+1)$th row from the ith row, then the resulting entry $a'_{i,j}$ in the (i,j) position becomes

$$a'_{i,j} = a_{i,j} - a_{i+1,j} = \begin{cases} (x_i - x_{i+1})(x_i + x_{i+1} - 2x_j), & i < j-1; \\ (x_i - x_{i+1})^2, & i = j \ \& \ i = j-1; \\ (x_i - x_{i+1})(2x_j - x_i - x_{i+1}), & i > j. \end{cases}$$

Repeating this operation downwards for all the rows except for the last one and then pulling out the common factors from the first row to the $(2n-1)$th row, we get the equality

$$\det A_{2n} = \prod_{i=1}^{2n-1} (x_i - x_{i+1}) \times \det B_{2n},$$

where the square matrix B_{2n} is given by

$$B_{2n} = \big[b_{i,j}\big]: \quad b_{i,j} = \begin{cases} x_i + x_{i+1} - 2x_j, & 1 \le i < j-1; \\ x_i - x_{i+1}, & i = j \ \& \ i = j-1; \\ 2x_j - x_i - x_{i+1}, & j < i < 2n; \\ -(x_{2n} - x_j)^2, & i = 2n. \end{cases}$$

Analogously, for the matrix B_{2n}, we make the corresponding column operations. By subtracting the $(j+1)$th column from the jth column, the resulting entry $b'_{i,j}$ in the (i,j) position becomes

$$b'_{i,j} = b_{i,j} - b_{i,j+1} = \begin{cases} 2(x_{j+1} - x_j), & i < j; \\ 0, & i = j; \\ 2(x_j - x_{j+1}), & i > j; \\ (x_{j+1} - x_j)(x_j + x_{j+1} - 2x_{2n}), & i = 2n. \end{cases}$$

Iterating this operation rightwards for all the columns except for the last one and then extracting the common column factors from the first column to the $(2n-1)$th column, we derive another equality

$$\det B_{2n} = \prod_{j=1}^{2n-1} (x_j - x_{j+1}) \times \det C_{2n},$$

where the matrix C_{2n} returns to skew-symmetric one:

$$C_{2n} = \big[c_{i,j}\big]: \quad c_{i,j} = \begin{cases} -2, & i < j < 2n; \\ 0, & i = j; \\ 2, & j < i < 2n; \\ 2x_{2n} - x_j - x_{j+1}, & i = 2n; \\ x_i + x_{i+1} - 2x_{2n}, & j = 2n. \end{cases}$$

Finally, for each k with $1 \le k < 2n$, performing simultaneously the operations on the matrix C_{2n} by subtracting the $(k+1)$th row and column from the kth row and column, respectively, we find the following reduced expression

$$\det C_{2n} = \det D_{2n},$$

where D_{2n} is the double bordered skew-symmetric matrix

$$D_{2n} = \big[d_{i,j}\big]: \quad d_{i,j} = \begin{cases} 2i - 2j, & |i-j| = 1 \ \& \ i,j < 2n; \\ 0, & |i-j| \ne 1 \ \& \ i,j < 2n; \\ x_{j+2} - x_j, & i = 2n \ \& \ j < 2n-1, \\ x_{j+1} - x_j, & i = 2n \ \& \ j = 2n-1; \\ x_i - x_{i+2}, & j = 2n \ \& \ i < 2n-1, \\ x_i - x_{i+1}, & j = 2n \ \& \ i = 2n-1. \end{cases}$$

Now, we write the matrix D_{2n} in blocks

$$D_{2n} = \begin{bmatrix} E_{2n-1} & \vdots & -\vec{y}^t \\ \cdots\cdots & \cdots & \cdots \\ \vec{y} & \vdots & 0 \end{bmatrix}, \quad \text{where } E_{2n-1} = \big[e_{i,j}\big] : e_{i,j} = \begin{cases} \pm 2, & i-j=\pm 1; \\ 0, & i-j \neq \pm 1; \end{cases}$$

and

$$\vec{y} = (y_1, y_2, \cdots, y_{2n-1}) : \begin{cases} y_k = x_{k+2} - x_k, & 1 \le k < 2n-1; \\ y_{2n-1} = x_{2n} - x_{2n-1}, & k = 2n-1. \end{cases}$$

By means of the Laplace formula, expanding the determinant of D_{2n} along the last row and then the last column, we have the double sum expression

$$\det D_{2n} = \sum_{1 \le i,j < 2n} (-1)^{i+j} y_i y_j E_{2n-1}(i,j),$$

where $E_{2n-1}(i,j)$ stands for the minor of E_{2n-1} after the ith row and the jth column having been removed. In general, we have the following remarkable formula

$$E_{2n-1}(i,j) = 2^{2n-2} \times \begin{cases} 0, & i \times j \equiv 0 \pmod 2; \\ 1, & i \times j \equiv 1 \pmod 2. \end{cases} \qquad \boxed{\bigstar}$$

According to this formula, we can rewrite, under the replacements $i \to 2\imath - 1$ and $j \to 2\jmath - 1$, the former double sum for $\det D_{2n}$ as

$$\begin{aligned} \det D_{2n} &= 4^{n-1} \sum_{1 \le \imath,\jmath \le n} y_{2\imath-1} y_{2\jmath-1} \\ &= 4^{n-1} \sum_{\imath=1}^{n} y_{2\imath-1} \sum_{\jmath=1}^{n} y_{2\jmath-1} \\ &= 4^{n-1} (x_1 - x_{2n})^2, \end{aligned}$$

where the last step is justified by applying twice the telescoping method. Summing up, we have proved that

$$\det A_{2n} = 4^{n-1} \prod_{k=1}^{2n} (x_k - x_{k+1})^2, \quad \text{where} \quad x_{2n+1} := x_1. \qquad \square$$

Induction principle Now, we return to present an inductive proof for $\boxed{\bigstar}$. It is routine to verify that the formula $\boxed{\bigstar}$ is true for E_3. Suppose that the same formula is valid for E_{2n-1}. Then we have to validate it for E_{2n+1}. In order to facilitate the intuitive reasoning below, we write explicitly the corresponding matrix

$$\boxed{n=4} : \; E_{2n+1} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Now, we can prove the formula in $\boxed{\bigstar}$ for E_{2n+1} case by case as follows:

- $\boxed{i,j < 2n}$ Expanding the determinant along the last row and then the last column, we see that $E_{2n+1}(i,j) = E_{2n-1}(i,j)$.
- $\boxed{i = 2n}$ When $j \neq 2n+1$, we have directly $E_{2n+1}(2n,j) = 0$ since all the entries in the last column result in zero. Instead, for $j = 2n+1$, we have $E_{2n+1}(2n, 2n+1) = 0$ because we can reduce the related matrix, by simple row and column operations, to a lower triangular matrix containing zero diagonal entries.
- $\boxed{i = 2n+1}$ When $j = 1$, we have immediately $E_{2n+1}(2n+1, 1) = 1$ since the matrix is lower triangular with all the diagonal entries equal to -1. When $j = 2$, the minor $E_{2n+1}(2n+1, 2) = 0$ since all the entries in the first row vanish. Finally for $j > 2$, by expanding the determinant along the first row and then the first column, we find that the invariant relation $E_{2n+1}(2n+1, j) = E_{2n-1}(2n-1, j-2)$.

In conclusion, we have verified that $\boxed{\bigstar}$ is true for all the E_{2n+1} with $n > 1$. $\square$

2.3. Proof of Theorem 1.3

For the matrix Ω_{2n}, by examining the difference of the ith row minus $(i-1)$th row and then iterating this operation upwards for all the rows except for the first one, we get the resulting matrix

$$\Omega'_{2n} = \left[\omega'_{i,j}\right]: \quad \omega'_{i,j} = \omega_{i,j} - \omega_{i-1,j} = \begin{cases} x_1^\lambda(x_1 - x_j), & i = 1; \\ x_i^\lambda(x_i - x_j) - x_{i-1}^\lambda(x_{i-1} - x_j), & i \le j; \\ x_j^\lambda(x_i - x_{i-1}), & i > j. \end{cases}$$

Similarly, by making the column operations leftwards, we transform the matrix Ω'_{2n} into another one

$$\Omega''_{2n} = \left[\omega''_{i,j}\right]: \quad \omega''_{i,j} = \omega'_{i,j} - \omega'_{i,j-1} = \begin{cases} x_1^\lambda(x_{j-1} - x_j), & i = 1; \\ x_1^\lambda(x_i - x_{i-1}), & j = 1; \\ 0, & i = j; \\ (x_i^\lambda - x_{i-1}^\lambda)(x_j - x_{j-1}), & i < j; \\ (x_j^\lambda - x_{j-1}^\lambda)(x_i - x_{i-1}), & i > j. \end{cases}$$

Now, extracting the common factor x_1^λ from both the first row and the first column, and then $x_k - x_{k-1}$ from the kth row and the kth column for k from 2 to $2n$, we find the following determinant equality

$$\det \Omega_{2n} = \det \Omega'_{2n} = \det \Omega''_{2n} = x_1^{2\lambda} \prod_{k=2}^{2n} (x_k - x_{k-1})^2 \det \Phi_{2n}(y_2, y_3, \cdots, y_{2n-1}).$$

The above skew-symmetric matrix Φ_{2n} can be expressed in blocks as

$$\Phi_{2n}(y_2, y_3, \cdots, y_{2n-1}) = \left[\phi_{i,j}\right] = \begin{bmatrix} 0 & -1 \ -1 \cdots -1 \\ 1 & \\ 1 & \Psi_{2n-1} \\ \vdots & \\ 1 & \end{bmatrix} : \quad \phi_{i,j} = \begin{cases} -1, & i = 1; \\ 1, & j = 1; \\ 0, & i = j; \\ -y_i, & i < j; \\ y_j, & i > j; \end{cases}$$

where the submatrix Ψ_{2n-1} is defined by

$$\Psi_{2n-m} = \left[\psi_{i,j}\right]_{m<i,j\le 2n}: \quad \psi_{i,j} = \begin{cases} 0, & i = j; \\ -y_i, & i < j; \\ y_j, & i > j; \end{cases} \quad \text{with} \quad y_k = \frac{x_k^\lambda - x_{k-1}^\lambda}{x_k - x_{k-1}}.$$

Now, by subtracting y_2 times the first row and the first column, respectively, from the second row and the second column, we can further reduce the matrix Φ_{2n} to the following skew-symmetric matrix:

$$\Phi'_{2n} = \left[\phi'_{i,j}\right] = \begin{bmatrix} 0 & -1 & -1 \ -1 \cdots -1 \\ 1 & 0 & 0 \ \ 0 \cdots \ 0 \\ 1 & 0 & \\ 1 & 0 & \Psi_{2n-2} \\ \vdots & \vdots & \\ 1 & 0 & \end{bmatrix} : \quad \phi'_{i,j} = \begin{cases} 0, & 1 \le i = j \le 2n; \\ -1, & i = 1 \ \& \ j > 1; \\ 1, & j = 1 \ \& \ i > 1; \\ 0, & i = 2 \ \& \ j \ge 2; \\ 0, & j = 2 \ \& \ i \ge 2; \\ -y_i, & 2 < i < j \le 2n; \\ y_j, & 2 < j < i \le 2n; \end{cases}$$

where Ψ_{2n-2} is the skew-symmetric matrix explicitly given by

$$\Psi_{2n-2} = \left[\psi_{i,j}\right]_{2<i,j\le 2n}: \quad \psi_{i,j} = \begin{cases} 0, & i = j; \\ -y_{i+2}, & i < j; \\ y_{j+2}, & i > j. \end{cases}$$

Expanding the determinant of Φ'_{2n} along the second row and then the second column, we find the determinant equality

$$\det \Phi_{2n}(y_2, y_3, \cdots, y_{2n-1}) = \det \Phi'_{2n} = \det \Psi_{2n-2}.$$

By pulling out the common factor y_3 from the first row and the first column of Ψ_{2n-2}, we derive the recurrence relation below

$$\det \Phi_{2n}(y_2, y_3, \cdots, y_{2n-1}) = y_3^2 \times \det \Phi_{2n-2}(y_4, y_5, \cdots, y_{2n-1}).$$

Iterating this equation $(n-1)$ times, we find the closed formula

$$\det \Phi_{2n}(y_2, y_3, \cdots, y_{2n-1}) = \det \Phi_2 \prod_{k=1}^{n-1} y_{2k+1}^2.$$

Since $\det \Phi_2 = 1$, we find finally that

$$\begin{aligned}
\det \Omega_{2n} = \det \Omega_{2n}^{\langle 1 \rangle} &= x_1^{2\lambda} \prod_{k=2}^{2n} (x_k - x_{k-1})^2 \det \Phi_{2n}(y_2, y_3, \cdots, y_{2n-1}) \\
&= x_1^{2\lambda} \prod_{k=2}^{2n} (x_k - x_{k-1})^2 \prod_{k=1}^{n-1} y_{2k+1}^2 \\
&= x_1^{2\lambda} \prod_{k=2}^{2n} (x_k - x_{k-1})^2 \prod_{k=1}^{n-1} \frac{(x_{2k}^{\lambda} - x_{2k+1}^{\lambda})^2}{(x_{2k} - x_{2k+1})^2} \\
&= x_1^{2\lambda} \prod_{k=1}^{n} (x_{2k} - x_{2k-1})^2 \prod_{k=1}^{n-1} (x_{2k}^{\lambda} - x_{2k+1}^{\lambda})^2. \qquad \square
\end{aligned}$$

References

[1] A. Dzhumadil'daev, Problem 12322, *Amer. Math. Monthly* **129** (2022) 486.
[2] W. Ledermann, A note on skew-symmetric determinants, *Proc. Edin. Math. Soc.* **36** (1993) 335–338.
[3] R. Vein, P. Dale, *Determinants and Their Applications in Mathematical Physics*, Springer Verlag, New York, 1999.