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Research Article

X-ranks for embedded varieties and extensions of fields

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Abstract

Let $X \subset \mathbb{P}^r$ be a projective embedded variety defined over a field K. Results relating maximum and generic X-rank of points of $\mathbb{P}^r(K)$ and $\mathbb{P}^r(L)$ are given, where L is a field containing K. Some of these results are algebraically closed for K and L. In other results (e.g. on the cactus rank), L is a finite extension of K.

Keywords: cactus rank; Veronese variety; X-rank.

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1. Introduction

In this paper we fix an extension of fields, say $K \subset L$, a projective variety X defined over K and an embedding of X into a projective space $\mathbb{P}^r$ defined over K. Thus $\mathbb{P}^r(K) \subset \mathbb{P}^r(L)$. For each $a \in \mathbb{P}^r(K)$ there are several different notions of ranks with respect to $X(K)$ and $X(L)$. In Section 3 we consider the case in which K is not algebraically closed and L is a finite extension of K (see Theorem 3.1), and in the rest of the paper we consider the case in which both K and L are algebraically closed.

Fix algebraically closed fields $K \subset L$. Take $F \in \{K, L\}$. Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety defined over K. We also assume that the embedding of X in $\mathbb{P}^r$ is defined over K and that X is non-degenerate, i.e. $X(K)$ spans $\mathbb{P}^k(K)$. Thus $X(L)$ spans $\mathbb{P}^r(L)$. Set $n := \dim X$. For any scheme or algebraic subset $Z \subset \mathbb{P}^r(K)$ (respectively, $Z \subset \mathbb{P}^r(L)$) defined over K (respectively, over L) let $\langle Z\rangle_K \subseteq \mathbb{P}^r(K)$ (respectively, $\langle Z\rangle_L \subseteq \mathbb{P}^r(L)$) denote the linear span of Z over K (respectively, over L). Note that $\langle\langle Z\rangle_K\rangle_L = \langle Z\rangle_L$ for any $Z \subseteq \mathbb{P}^r(K)$. For all positive integers t let $S(X(F), t)$ denote the set of all subsets of $X(F)$ with cardinality t. The set $S(X(F), t)$ is an irreducible quasi-projective variety of dimension nt. For any $o \in \mathbb{P}^r(F)$ the $X(F)$-rank $r_{X(F)}(o)$ of o is the minimal cardinality of a subset of $X(F)$ containing o in its linear span. For any positive integer t let $S(X(F), o, t)$ denote the set of all $S \in S(X(F), t)$ such that $o \in \langle S\rangle_F$ and $o \notin \langle S'\rangle_F$ for any $S' \subsetneq S$. Each set $S(X(F), o, t)$ is constructible by a theorem of Chevalley (see [8, Ex. II.3.18]). Note that $r_{X(F)}(o)$ is the minimal integer t such that $S \in S(X(F), t) \neq \emptyset$. Now assume $o \in \mathbb{P}^r(K)$. Since K is algebraically closed, it is easy to check (and well-known) that $r_{X(L)}(o) = r_{X(K)}(o)$ and that $S(X(L), o, t)$ is the constructible L-set associated to $S(X(K), o, t)$ (see Remark 2.1 for more details). In particular $S(X(L), o, t)$ and $S(X(K), o, t)$ have the same number of irreducible components and the bijection between their irreducible components preserves the dimension of the components. In particular $S(X(L), o, t) = S(X(K), o, t)$ if and only if either $S(X(K), o, t) = \emptyset$ or $S(X(K), o, t)$ is finite. For any positive integer t let $\sigma_t(X(F)) \subseteq \mathbb{P}^r(F)$ denote the closure in $\mathbb{P}^n(F)$ of the union of all $\langle S\rangle_F$, $S \in S(X(F), t)$. Each $\sigma_t(X(F))$ is irreducible and $\sigma_t(X(L))$ is the L-variety associated to the K-variety $\sigma_t(X(K))$. The first integer a such that $\sigma_a(X(F)) = \mathbb{P}^r(F)$ is the same for $F = K$ and $F = L$. It is often call the generic $X(K)$-rank (respectively, generic $X(L)$), because it is the $X(K)$-rank (respectively, $X(L)$-rank) of a non-empty open subset of $\mathbb{P}^r(K)$ (respectively, $\mathbb{P}^r(L)$). For any positive integer t let $R(X(F), t)$ denote the set of all $o \in \mathbb{P}^r(F)$ such that $r_{X(F))}(o) = t$. Each $R(X(F), t)$ is constructible (Lemma 2.1). See Remark 2.1 for the definition and construction of the L-associated set of any constructible subset of $\mathbb{P}^r(K)$.

It is easy to prove the following result (its proof is given after Lemma 2.1).

Theorem 1.1. *For each positive integer t the constructible L-set $R(X(L), t)$ is the L-set associated to $R(X(K), t)$.*

The maximum among all $X(F)$-rank is the largest integer a such that $R(X(F), a) \neq \emptyset$. Thus Theorem 1.1 has the following corollary.

Corollary 1.1. *The maxima of the $X(K)$-ranks and of the $X(L)$-ranks are the same.*

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Take any $o \in \mathbb{P}^r(F)$. The *open* $X(F)$*-rank* $or_{X(F)}(o)$ of o is the minimal integer $t > 0$ such that for all closed sets $T \subsetneq X(F)$ there is $S \in S(X(F), t)$ such that $o \in \langle S \rangle_F$ and $S \cap T = \emptyset$ (see [1, 9]). Obviously $or_{X(F)}(o) \ge r_{X(F)}(o)$, but very often the strict inequality holds. For instance, $or_{X(F)}(o) > 1$ for all $o \in \mathbb{P}^r(F)$. Since $X(K)$ is Zariski dense in $X(L)$, $or_{X(L)}(o) \le or_{X(K)}(o)$ for all $o \in \mathbb{P}^r(K)$. Let $O(X(F), t)$ denote the set of all $o \in \mathbb{P}^r(F)$ such that $or_{X(F)}(t)$. We also prove the following results.

Theorem 1.2. *We have* $or_{X(L)}(o) = or_{X(K)}(o)$ *for all* $o \in \mathbb{P}^r(K)$.

Theorem 1.3. *The following properties are true:*

1. *The generic open* $X(F)$*-rank is the same for* $F = K$ *and* $F = L$.
2. *The maximum open* $X(F)$*-rank is the same for* $F = K$ *and* $F = L$.
3. *Each set* $O(X(F), t)$ *is constructible and* $O(X(L), t)$ *is the* L*-constructible set associated to* $O(X(K), t)$.

2. Proofs of Theorems 1.1, 1.2, and 1.3

Remark 2.1. *Let* $Y(K)$ *be a projective variety defined over* K *and let* $E \subseteq Y(K)$ *be a constructible subset. We define the* L*-constructible set* $E(L) \subseteq Y(L)$ *associated to* E *in the following way. If* E *is a finite set, then take* $E(L) := E$. *Thus we may assume* $\dim E > 0$ *and use induction on the integer* $\dim E$. *Let* $\overline{E}$ *be the closure of* E *in* $Y(K)$. *Let* $\overline{E} = A_1 \cup \cdots \cup A_x$ *be the irreducible components of* $\overline{E}$. *Set* $E_i := A_i \cap E$. *Each set* E_i *is constructible. Note that each* E_i *contains a non-empty open subset* U_i *of* A_i. *Thus the set* $E_i \setminus U_i$ *is a constructible set of dimension* $< \dim E$. *By the inductive assumption we have defined the constructible sets* $(E_i \setminus U_i)(L)$. *Set* $E_i(L) := U_i(L) \cup (E_i \setminus U_i)(L)$ *and* $E(L) := E_1(L) \cup \cdots \cup E_x(L)$. *It is easy to check that the definition of* $E(L)$ *does not depend on the choice of* $Y(K)$, *we only need a* K*-variety containing* $\overline{E}$. *Note that there is a bijection between the irreducible component of* $\overline{E}(L)$ *and* $\overline{E}$ *(respectively,* $\overline{E}(L) \setminus E(L)$ *and* $\overline{E} \setminus E$*) which preserves the dimension.*

Observation 2.1. *Note that* $E(L) = E$ *if and only if* E *is finite. In all other cases* $E(L)$ *(respectively,* E*) has the cardinality of* L *(respectively,* K*) and hence* $E(L) \setminus E(K)$ *is infinite and its Zariski closure contains all non-isolated points of* $E(L)$.

Observation 2.1 is applied to $R(X(K), t)$ *and* $R(X(L), t)$ *by Theorem 1.1 and to all constructible sets used in the proofs of the results stated in the introduction. By [6, Theorem 3.1] each* $R(X(F), t)$ *has positive dimension, except at most when* t *is the maximal* $X(F)$*-rank.*

Lemma 2.1. *Each set* $R(X(F), t)$ *is constructible.*

Proof. Since $R(X(F), 1) = X(F)$, we may assume $t > 1$ and use induction on the integer t. Since $R(X(F), t) \cap R(X(F), x) = \emptyset$ for all $x < t$, it is sufficient to prove that $A := \cup_{1 \le x \le t} R(X(F), x)$ is constructible. The set E is the image of $S(X(F), t)$ by the evaluation map. □

Proof of Theorem 1.1. Lemma 2.1 says that $R(X(K), t)$ and $R(X(L), t)$ are constructible. Since $X(L)$ is the L-set of $X(K)$, we may use induction on t to prove the theorem. It is sufficient to mimic the proof of Lemma 2.1. □

Proof of Theorem 1.2. For any positive integer t set

$$X(F)(t) := \cup_{S \in S(X(F), t)} S \subseteq X(F).$$

Since $S(X(F), t)$ is constructible, a theorem of Chevalley gives that $X(F)(t)$ is constructible and that for each constructible set $\Sigma \subseteq S(X(F), t)$ the set $\mathrm{ev}(\Sigma) := \cup_{S \in S(X(F), t)} S \subseteq X(F)$ is constructible (see [8, Ex. II.3.18, II.3.19]). Fix any $o \in \mathbb{P}^r(F)$.

Observation: The open $X(F)$-rank $or_{X(F)}(o)$ of o is the first positive integer t such that $\mathrm{ev}(S(X(F), o, t))$ is Zariski dense in $X(F)$.

Now assume $o \in \mathbb{P}^r(K)$. Since K is algebraically closed, the Observation gives $or_{X(L)}(o) = or_{X(K)}(o)$. □

Proof of Theorem 1.3. It is sufficient to prove Part (3). The observation in the proof of Theorem 1.2 and a theorem of Chevalley (see [8, Ex. II.3.18, II.3.19]) first gives that each $O(X(F), t)$ is constructible and then that $O(X(L), t)$ is the L-constructible set associated to $O(X(K), t)$. □

3. When K is not algebraically closed

Let K be a field which is not algebraically closed. We fix an inclusion $K \subset \overline{K}$. Let $X \subset \mathbb{P}^r$ be an embedding (defined over K) of the integral projective variety X defined over K. We assume that $X(\overline{K})$ is non-degenerate, but we do not assume that $X(K)$ spans $\mathbb{P}^r(K)$ (we allow the case $X(K) = \emptyset$). For each $a \in \overline{K}$ let $\deg(a)$ be the degree of the minimum polynomial of a over K, i.e. the dimension of the K-vector space $K(a)$. We fix a system of homogeneous coordinates $x_0, \dots, x_r$ of $\mathbb{P}^r(K)$. For each $a = (a_0 : \cdots : a_r) \in \mathbb{P}^r(\overline{K})$ with, say, $a_i \neq 0$ the degree $\deg_1(a)$ of a is the maximum of all integers $\deg(a_j/a_i)$, $0 \le j \le r$, and let $\deg_2(a)$ be the the degree of the extension $K(a_0/a_1, \dots, a_r/a_i)$ of K. The integer $\deg_3(a)$ is the degree of the normal closure of $K(a_0/a_1, \dots, a_r/a_i)$ as an extension of K. The integers $\deg_1(a)$, $\deg_2(a)$ and $\deg_3(a)$ are well-defined, i.e. they do not depend upon the choice of the index i such that $a_i \neq 0$. If K is a finite field $\mathbb{F}_q$, then $\deg_2(a) = \deg_3(a)$ for all a, because all finite extensions of $\mathbb{F}_q$ are Galois extensions. However, even for a finite field we may have $\deg_1(a) < \deg_2(a)$ if $r \ge 2$ (Example 3.1). If K is real closed ([4]), then $\overline{K} = K(i)$ and any a has $\deg_1(a) = \deg_2(a) = \deg_3(a) \in \{1, 2\}$. For $K = \mathbb{Q}$ and any $r \ge 2$ there are easy examples with $\deg_1(a) < \deg_2(a) < \deg_3(a)$. For any finite set $S \subset \mathbb{P}^r(\overline{K})$, $S \neq \emptyset$ let $\deg_1(S)$ be the maximum of all $\deg_1(a)$, $a \in S$. Let $\deg_2(S)$ (respectively, $\deg_3(S)$) be the degree of the extension (respectively, normal extension) of K generated by the ratios of the homogeneous coordinates of all $a \in S$.

Take $o \in \mathbb{P}^r(\overline{K})$ and fix $i \in \{1, 2, 3\}$. Set $t := r_{X(\overline{K})}(o)$. Let $DR_i(X, K, o)$ denote the minimum of all $\deg_i(S)$ for some $S \in S(X(\overline{K}, t)$. We say that $a = (a_0 : \cdots : a_r) \in \mathbb{P}^r(\overline{K})$ is *separable over* K if all ratios a_j/a_i with $a_i \neq 0$ are separable over K. Obviously if $a_i \neq 0$ it is sufficient to test all a_j/a_i. If K is perfect, then every $a \in \mathbb{P}^r(\overline{K})$ is separable over K. The field K is perfect if either K is a finite field or $\mathrm{char}(K) = 0$.

Example 3.1. *Take $r = 2$, $K = \mathbb{F}_q$ and $a = (1 : u : v)$ with $u \in \mathbb{F}_{q^3} \setminus \mathbb{F}_q$ and $v \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. We have $\deg_1(a) = 3$ and $\deg_2(a) = 6$.*

The fact that all finite extensions of a finite field are Galois extensions has the following byproduct.

Proposition 3.1. *Take $K = \mathbb{F}_q$. Fix $o \in \mathbb{P}^r(\overline{K})$ and set $t := r_{X(\overline{K})}(o)$. Assume $\#S(X(\overline{K}), o, t) = 1$. Then*

$$DR_2(X, K, o) \le t \deg_2(o).$$

Proof. Set $x := \deg_2(o)$ and $y := DR_2(X, K, o)$. Write $\{S\} = S(X(\overline{K}), o, t)$. Consider X over $\mathbb{F}_{q^x}$. Since $o \in \mathbb{P}^r(\mathbb{F}_{q^x})$ and S is the unique element of $S(X(\overline{K}), t)$ computing the $X(\overline{K})$-rank of o, S is invariant for the Galois group of the extension $\mathbb{F}_{q^y}/\mathbb{F}_{q^x}$. Thus $y \le (\#S)x$. We have $t = \#S$. □

For any field $E \supseteq K$ let $\rho(X(E))$ denote the maximal integer t such that any subset of $X(F)$ with cardinality t is linearly independent. Of course, if $E \subset E'$, then $X(E) \subseteq E'$ and hence $\rho(X(E')) \le \rho(X(E))$. If E is algebraically closed, it is easy to check that $\rho(X(E')) = \rho(X(E))$ for any field $E' \supset E$.

Remark 3.1. *Fix $o \in \mathbb{P}^r(\overline{K})$ and assume $2r_{X(\overline{K})} \le \rho(X(\overline{K}))$. Then we have*

$$\#S\left(X(\overline{K}), o, r_{X(\overline{K})}\right) = 1.$$

Remark 3.2. *Let $\nu_d : \mathbb{P}^n \to \mathbb{P}^r$, $r = \binom{n+d}{n} - 1$, be the d-Veronese embedding of $\mathbb{P}^n$, i.e. the embedding induced The cohomology of a projective space easily gives that $\rho(\nu_d(\mathbb{P}^n)(E)) = d + 1$ for any field E. In particular we may apply Remark 3.1 to any $o \in \mathbb{P}^r(\overline{K})$ such that*

$$r_{X(\overline{K})}(o) \le \left\lfloor \frac{d+1}{2} \right\rfloor.$$

The proof of Proposition 3.1 gives the following result.

Proposition 3.2. *Fix a separable $o \in \mathbb{P}^r(\overline{K})$ and assume $\#S(X(\overline{K}), o, r_{X(\overline{K})}(o)) = 1$. Then $DR_3(X, K, o) \le r_{X(\overline{K})}(o) \deg_3(o)$.*

By Remark 3.2, Proposition 3.2 may be applied to the d-Veronese embedding of any projective space, but just in a very restricted range of ranks.

Other notions of ranks for homogeneous polynomials are the *slice rank* and the *Schmidt rank* (often called *strength*). The recent preprint [10] by Lempert and Ziegler proves stronger versions of all our attempts related to this notion over a non-algebraically closed field with characteristic 0.

Remark 3.3. *Take any field K such that $\mathrm{char}(K) = 0$ and let $X \subset \mathbb{P}^r$, $r = \binom{n+d}{n} - 1$, be the image of the the d-Veronese embedding of $\mathbb{P}^n$. Fix $a \in X(K)$ and $o \in \mathbb{P}^r(\overline{K})$.*

If we do not search for a small degree extension of K on which it is defined all points (or the set) defining the $X(\overline{K})$-rank, then we may get far better bounds. We recall that the cactus $X(\overline{K})$-rank of $a \in \mathbb{P}^r(\overline{K})$ is the minimal degree of a zero-dimensional scheme $Z \subset X(\overline{K})$ whose linear span contains a (see [2,3,5,7]). Fix a finite extension L of K such that a is defined over K. We call *cactus L-rank* the minimal degree of a zero-dimensional scheme $Z \subset X(\overline{K})$ defined over L and whose linear span contains a. We call *strong cactus L-rank* the minimal degree of a zero-dimensional scheme $Z \subset X(\overline{K})$ such that all connected components of Z are defined over L and the linear span of Z contains a. Obviously every connected Z defined over L may be use to test the strong cactus rank.

Theorem 3.1. *Assume* $\mathrm{char}(K) = 0$*. Let L be any finite extension of K. Fix an integer $d \geq 3$. Let $X(\overline{K}) \subset \mathbb{P}^r$, $r = \binom{n+d}{n} - 1$, be the image of the d-Veronese embedding of $\mathbb{P}^n$. If $d = 2k+1$ is odd, set $N := 2\binom{n+k}{n}$. If $d = 2k+2$ is even, set*

$$N := \binom{n+k}{n} + \binom{n+k+1}{n}.$$

Then every $a \in \mathbb{P}^r(L)$ has strong cactus L-rank $\leq N$.

Proof. Fix $b \in X(L)$. The proof of [3, Theorem 3] gives the existence of a zero-dimensional scheme $Z \subset X(\overline{K})$ defined over L, spanning a and with $Z_{\mathrm{red}} = \{b\}$. Since Z is connected, it gives an upper bound for the strict cactus L-rank of a. □

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