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Research Article

On the existence and uniqueness of solutions for a class of nonlinear degenerate elliptic problems via Browder-Minty theorem*

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Abstract

The purpose of this paper is to investigate the existence and uniqueness of weak solutions for a class of nonlinear degenerate elliptic problems of the form:

$$-\text{div}\,[\nu_1\, a(y,\nabla\varphi) + \nu_2\, b(y,\varphi,\nabla\varphi)] + \nu_3\, g(y,\varphi) = \phi(y),$$

where ν_1, ν_2, and ν_3 are A_p-weight functions and the operators a, b and g are Caratéodory functions that satisfy some certain conditions, and $\phi \in L^{p'}(\Omega, \nu_1^{1-p'})$. The approach used for attaining the mentioned purpose is based on the Browder-Minty theorem and the theory of weighted Sobolev spaces.

Keywords: nonlinear degenerate elliptic problems; Browder-Minty theorem; weighted Sobolev spaces; weak solution.

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1. Introduction

The goal of this paper is to show that there is a unique weak solution in $W_0^{1,p}(\Omega, \nu_1)$ (p is not necessarily equal to 2) for the Dirichlet problem associated with the nonlinear degenerate elliptic equation of the form:

$$\begin{cases} -\text{div}\Big[\nu_1 a(y,\nabla\varphi) + \nu_2 b(y,\varphi,\nabla\varphi)\Big] + \nu_3 g(y,\varphi) = \phi(y) & \text{in } \Omega, \\ \varphi(y) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded open set in $\mathbb{R}^N$; ν_1, ν_2, and ν_3 are A_p-weight functions, and the functions $b : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$, $a : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ and $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ are Caratéodory functions that satisfy some assumptions with $\phi \in L^{p'}(\Omega, \nu_1^{1-p'})$.

The problems of the type (1) have already been studied for the case $\nu_1 \equiv \nu_2 \equiv \nu_3 \equiv 1$; the existence results have been reported in [4] (see also [7]) when $a(y,\nabla\varphi) = 0$. Also, the degenerate case with different conditions have been investigated in many papers; for example, see [1–3, 9, 14–23]. Moreover, Cavalheiro established the existence of solution for (1) in [5] when $a(y,\nabla\varphi) = 0$ and in [6] when $g(y,\varphi) = 0$.

The remaining part of this paper consists of five sections. Definitions and some preliminary results are presented in the next section. The assumptions on a, b, and g, as well as the notion of weak solutions for (1) are outlined in Section 3. Section 4 is concerned with the main result and its proof. An example is presented in Section 5.

2. Preliminaries

In this section, we recall some definitions and basic properties of weighted Lebesgue and Sobolev spaces. Detailed expositions on these concepts can be found in [10, 24].

Let ν be a weight function on $\mathbb{R}^N$ such that ν is measurable and strictly positive a.e. in $\mathbb{R}^N$. The space $L^p(\Omega,\nu)$ is defined as

$$L^p(\Omega,\nu) := \left\{ f : \Omega \longrightarrow \mathbb{R} \ \text{ such that } \ ||f||_{L^p(\Omega,\nu)} = \left(\int_\Omega |f(y)|^p \nu(y)dy\right)^{\frac{1}{p}} < \infty \right\}.$$

We now establish conditions on ν that ensure $L^p(\Omega,\nu) \subset L^1_{loc}(\Omega)$.

*This paper is dedicated to the memory of Professor Adil Abbassi.

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Proposition 2.1 (see [12, 13]). *Let $1 \le p < \infty$ and $B \subset \Omega$ be a ball. If $\nu^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega)$ for $p > 1$ and*

$$ess \sup_{y \in B} \frac{1}{\nu(y)} < +\infty \quad for\ p = 1,$$

then $L^p(\Omega, \nu) \subset L^1_{loc}(\Omega)$.

The class of A_p-weight is a particularly well-understood class of weights. In harmonic analysis, these classes have a variety of applications (see [24]).

Definition 2.1. *For $1 \le p < \infty$, one has $\nu \in A_p$-weight, if there exists $\theta = \theta(p, \nu)$ so that*

$$\left(\frac{1}{|B|}\int_B \nu(y)dy\right)\left(\frac{1}{|B|}\int_B (\nu(y))^{\frac{-1}{p-1}}\, dy\right)^{p-1} \leqslant \theta \ \ for\ p > 1,\ and\ \left(\frac{1}{|B|}\int_B \nu(y)dy\right) ess \sup_{x\in B} \frac{1}{\nu(y)} \leqslant \theta \ \ for\ p = 1.$$

The A_p constant of ν is the infimum over all such constants θ. The set of all A_p-weights is denoted by A_p. Additional information on A_p-weights can be found in [12, 25].

Example 2.1. *In this example, we have two parts.*

1. *$\nu \in A_p \Leftrightarrow a \le \nu(z) \le b$ for a.e. $z \in \mathbb{R}^N$ with $a, b > 0$.*
2. *For $z \in \mathbb{R}^N$, we have $\nu(z) := |Z|^\lambda \in A_p \Leftrightarrow -N < \lambda < N(p-1)$.*

Definition 2.2. *Let $\Omega \subset \mathbb{R}^n$ and $\nu \in A_p$. The space $W^{1,p}(\Omega, \nu)$ is defined as*

$$W^{1,p}(\Omega, \nu) := \Big\{u \in L^p(\Omega, \nu)\ and\ D_i u \in L^p(\Omega, \nu),\ i = 1, \dots, N\Big\}.$$

The norm of u in $W^{1,p}(\Omega, \omega)$ is given by

$$||\varphi||_{W^{1,p}(\Omega,\nu)} := \left(\int_\Omega |\varphi(y)|^p \nu(y)dy + \sum_{i=1}^N \int_\Omega |D_i\varphi(y)|^p \nu(y)dy\right)^{\frac{1}{p}}.$$

In addition, we define $W^{1,p}_0(\Omega, \nu)$ as the closure of $\mathcal{C}^\infty_0(\Omega)$ in $W^{1,p}(\Omega, \nu)$.

Proposition 2.2 (see [12, 13]). *The spaces $(W^{1,p}(\Omega, \nu), ||\cdot||_{W^{1,p}(\Omega,\nu)})$ and $(W^{1,p}_0(\Omega, \nu), ||\cdot||_{W^{1,p}(\Omega,\nu)})$ are separable and reflexive Banach spaces. The dual of $W^{1,p}_0(\Omega, \nu)$ is given by*

$$W_0^{-1,p'}(\Omega, \nu^{1-p'}) = \left\{u_0 - \sum_{i=1}^N D_i u_i\ :\ \frac{u_i}{\nu} \in L^{p'}(\Omega, \nu),\ i = 0, \dots, n\right\}.$$

Theorem 2.1 (see [11]). *Let $\nu \in A_p$ and $\Omega \subset \mathbb{R}^N$. If $v_i \longrightarrow v$ in $L^p(\Omega, \nu)$, then there exists a subsequence (v_{i_l}) and $\psi \in L^p(\Omega, \nu)$ such that*

(i) $u_{i_l}(z) \longrightarrow v(z)$, $i_l \longrightarrow \infty$.

(ii) $|v_{i_l}(z)| \leqslant \psi(z)$.

Theorem 2.2 (see [8]). *If $\nu \in A_p$ and $\Omega \subset \mathbb{R}^N$, then there exist B_Ω, ε, $\kappa > 0$ with $1 \leqslant \kappa \leqslant \frac{N}{N-1} + \varepsilon$ such that*

$$||v||_{L^{\kappa p}(\Omega,\nu)} \leqslant B_\Omega ||\nabla v||_{L^p(\Omega,\nu)}.$$

The Browder-Minty theorem is stated as follows

Theorem 2.3 (Browder-Minty theorem, see [26]). *Let $L : \mathcal{W} \longrightarrow \mathcal{W}^*$ where $\mathcal{W}$ is a reflexive, real, and separable Banach space. The following assertions hold:*

1. *If L is coercive, hemicontinuous and monotone operator on $\mathcal{W}$, the problem $Lv = T$, $T \in W^*$ admits a solution in $\mathcal{W}$.*
2. *If L is coercive, hemicontinuous and strictly monotone on $\mathcal{W}$, the problem $Lv = T$, $T \in W^*$ admits a unique solution in $\mathcal{W}$.*

3. Hypotheses and the concept of weak solution

Hypotheses

We now present some hypotheses on the problem (1). Suppose that $\Omega \subset \mathbb{R}^N (N \geq 2)$, ν_1, ν_2 and ν_3 are A_p-weights, $a_m : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}$, $b_m : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ $(m = 1, \dots, N)$, with $a(y,\delta) = \big(a_1(y,\delta), \dots, a_N(y,\delta)\big)$ and $b(y,\mu,\delta) = \big(b_1(y,\mu,\delta), \dots, b_N(y,\mu,\delta)\big)$ and $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ such that

(H1) a_m, b_m and g are Caratéodory functions;

(H2) there are h_1, h_2, h_3t, $h_4 \in L^\infty(\Omega)$ and $f_1 \in L^{p'}(\Omega,\nu_1)$ $\Big(\text{with } \frac{1}{p} + \frac{1}{p'} = 1\Big)$, $f_2 \in L^{q'}(\Omega,\nu_2)$ and $f_3 \in L^{s'}(\Omega,\nu_3)$ such that

$$|a(y,\delta)| \leq f_1(y) + h_1(y)|\delta|^{p-1},$$

$$|b(y,\mu,\delta)| \leq f_2(y) + h_2(y)|\mu|^{q-1} + h_3(y)|\delta|^{q-1},$$

$$|g(y,\mu)| \leq f_3(y) + h_4(y)|\mu|^{s-1},$$

where $(\mu,\delta) \in \mathbb{R} \times \mathbb{R}^n$;

(H3) there exits $\lambda > 0$ such that

$$\Big\langle a(y,\delta) - a(y,\delta'), \delta - \delta' \Big\rangle \geqslant \lambda|\delta - \delta'|^p,$$

$$\Big\langle b(y,\mu,\delta) - b(y,\mu',\delta'), \delta - \delta' \Big\rangle \geqslant 0,$$

$$\Big(g(y,\mu) - g(y,\mu')\Big)\Big(\mu - \mu'\Big) \geqslant 0,$$

where $\mu, \mu' \in \mathbb{R}$ and $\delta, \delta' \in \mathbb{R}^n$ with $\mu \neq \mu'$ and $\delta \neq \delta'$;

(H4) there exist κ_1, κ_2, $\kappa_3 > 0$ such that $\langle a(y,\delta),\delta\rangle \geqslant \kappa_1|\delta|^p$, $\langle b(y,\mu,\delta),\delta\rangle \geqslant \kappa_2|\delta|^q + \kappa_3|\mu|^q$, $g(y,\mu)\mu \geqslant 0$.

The concept of weak solution

The definition of a weak solution of (1) is stated as follows.

Definition 3.1. *A function $\varphi \in W_0^{1,p}(\Omega,\nu_1)$ is a weak solution of (1) if for any $v \in W_0^{1,p}(\Omega,\nu_1)$ it holds that*

$$\int_\Omega \langle a(y,\nabla\varphi), \nabla v\rangle \nu_1 dy + \int_\Omega \langle b(y,\varphi,\nabla\varphi), \nabla v\rangle \nu_2 dy + \int_\Omega g(y,\varphi) v \nu_3 dy = \int_\Omega \phi v dy.$$

Remark 3.1. *For all ν_1, ν_2, $\nu_3 \in A_p$ the following statements hold.*

(i) *If $1 < q < p < \infty$ and $\frac{\nu_2}{\nu_1} \in L^{k_1}(\Omega,\nu_1)$ where $k_1 = \frac{p}{p-q}$, then $||\varphi||_{L^q(\Omega,\nu_2)} \leqslant \vartheta_{p,q}||\varphi||_{L^p(\Omega,\nu_1)}$ with $\vartheta_{p,q} = ||\frac{\nu_2}{\nu_1}||^{1/q}_{L^{k_1}(\Omega,\nu_1)}$.*

(ii) *If $1 < s < p < \infty$ and $\frac{\nu_3}{\nu_1} \in L^{k_2}(\Omega,\nu_1)$ where $k_2 = \frac{p}{p-s}$, then $||\varphi||_{L^s(\Omega,\nu_3)} \leqslant \vartheta_{p,s}||\varphi||_{L^p(\Omega,\nu_1)}$ with $\vartheta_{p,s} = ||\frac{\nu_3}{\nu_1}||^{1/s}_{L^{k_2}(\Omega,\nu_1)}$.*

4. Main general result

The next theorem presents the paper's main result.

Theorem 4.1. *If the conditions **(H1)–(H4)** hold, then the problem (1) admits a unique solution in $W_0^{1,p}(\Omega,\nu_1)$.*

Proof. We reduce the problem (1) to a new one, governed by the operator problem $\Psi\varphi = \Upsilon$, and we apply Theorem 2.3. We define

$$\Phi : W_0^{1,p}(\Omega,\nu_1) \times W_0^{1,p}(\Omega,\nu_1) \longrightarrow \mathbb{R}$$

and

$$\Upsilon : W_0^{1,p}(\Omega,\nu_1) \longrightarrow \mathbb{R},$$

where Φ and Υ are specified in the following paragraphs.
Hence

$$\varphi \in W_0^{1,p}(\Omega,\nu_1) \text{ is a weak solution of (1)} \quad \Leftrightarrow \quad \Phi(\varphi,v) = \Upsilon(v) \text{ for all } v \in W_0^{1,p}(\Omega,\nu_1).$$

The theorem is proved in four steps.

Step 1.
We utilize some tools and the condition **(H2)** to show the existence of the operator Ψ and that the problem (1) is identical to the operator equation $\Psi\varphi = \Upsilon$. By employing the Hölder's inequality and Theorem 2.2, we get

$$\begin{aligned} |\Upsilon(\varphi)| &\leq \int_\Omega \frac{|\phi|}{\nu_1} |\varphi|\nu_1 \, dy \\ &\leq ||\phi/\nu_1||_{L^{p'}(\Omega,\nu_1)}||\varphi||_{L^p(\Omega,\nu_1)} \\ &\leq C_\Omega||\phi/\nu_1||_{L^{p'}(\Omega,\nu_1)}||\varphi||_{W_0^{1,p}(\Omega,\nu_1)}. \end{aligned}$$

Since $\phi \in L^{p'}(\Omega, \nu_1^{1-p'})$, then $\Upsilon \in W_0^{-1,p'}(\Omega, \nu_1^{1-p'})$.
The operator Φ can be written as

$$\Phi(\varphi, v) = \Phi_1(\varphi, v) + \Phi_2(\varphi, v) + \Phi_3(\varphi, v),$$

where

$$\begin{aligned} &\Phi_1 : W_0^{1,p}(\Omega,\nu_1) \times W_0^{1,p}(\Omega,\nu_1) \longrightarrow \mathbb{R} \\ &\Phi_1(\varphi, v) = \int_\Omega \langle a(y, \nabla\varphi), \nabla v\rangle \nu_1 dy, \\ &\Phi_2 : W_0^{1,p}(\Omega,\nu_1) \times W_0^{1,p}(\Omega,\nu_1) \longrightarrow \mathbb{R} \\ &\Phi_2(\varphi, v) = \int_\Omega \langle b(y, \varphi, \nabla\varphi), \nabla v\rangle \nu_2 dy, \\ &\Phi_3 : W_0^{1,p}(\Omega,\nu_1) \times W_0^{1,p}(\Omega,\nu_1) \longrightarrow \mathbb{R} \\ &\Phi_3(\varphi, v) = \int_\Omega g(y, \varphi) v \nu_3 dy. \end{aligned}$$

Then, we have

$$|\Phi(\varphi, v)| \leq |\Phi_1(\varphi, v)| + |\Phi_2(\varphi, v)| + |\Phi_3(\varphi, v)|. \tag{2}$$

Also, by utilizing Hölder inequality, Remark 3.1(i), **(H2)** and Theorem 2.2, we have

$$\begin{aligned} |\Phi_1(\varphi, v)| &\leq \int_\Omega |a(y, \nabla\varphi)||\nabla v|\nu_1 dy \\ &\leq \int_\Omega \left(f_1 + h_1|\nabla\varphi|^{p-1}\right)|\nabla v|\nu_1 dy \\ &\leq ||f_1||_{L^{p'}(\Omega,\nu_1)}||\nabla v||_{L^p(\Omega,\nu_1)} + ||h_1||_{L^\infty(\Omega)}||\nabla\varphi||^{p-1}_{L^p(\Omega,\nu_1)}||\nabla v||_{L^p(\Omega,\nu_1)} \\ &\leq \left(||f_1||_{L^{p'}(\Omega,\nu_1)} + ||h_1||_{L^\infty(\Omega)}||\varphi||^{p-1}_{W_0^{1,p}(\Omega,\nu_1)}\right)||v||_{W_0^{1,p}(\Omega,\nu_1)}, \end{aligned}$$

and

$$\begin{aligned} |\Phi_2(\varphi, v)| &\leq \int_\Omega |b(y, \varphi, \nabla\varphi)||\nabla v|\nu_2 dy \\ &\leq \int_\Omega \left(f_2 + h_2|\varphi|^{q-1} + h_3|\nabla\varphi|^{q-1}\right)|\nabla v|\nu_2 dy \\ &\leq ||f_2||_{L^{q'}(\Omega,\nu_2)}||\nabla v||_{L^q(\Omega,\nu_2)} + ||h_2||_{L^\infty(\Omega)}||\varphi||^{q-1}_{L^q(\Omega,\nu_2)}||\nabla v||_{L^q(\Omega,\nu_2)} + \ ||h_3||_{L^\infty(\Omega)}||\nabla\varphi||^{q-1}_{L^q(\Omega,\nu_2)}||\nabla v||_{L^q(\Omega,\nu_2)} \\ &\leq \left[\vartheta_{p,q}||f_2||_{L^{q'}(\Omega,\nu_2)} + \vartheta^q_{p,q}\left(C^{q-1}_\Omega||h_2||_{L^\infty(\Omega)} + \ ||h_3||_{L^\infty(\Omega)}\right)||\varphi||^{q-1}_{W_0^{1,p}(\Omega,\nu_1)}\right]||v||_{W_0^{1,p}(\Omega,\nu_1)}. \end{aligned}$$

Similarly, by using Hölder inequality, Theorem 2.2, **(H2)** and Remark 3.1, we get

$$\begin{aligned} |\Phi_3(\varphi, v)| &\leq \int_\Omega |g(y, \varphi)||v|\nu_3 dy \\ &\leq \left[C_\Omega\vartheta_{p,s}||f_3||_{L^{s'}(\Omega,\nu_3)} + \vartheta^s_{p,s}C^s_\Omega||h_4||_{L^\infty(\Omega)}||\varphi||^{s-1}_{W_0^{1,p}(\Omega,\nu_1)}\right] ||v||_{W_0^{1,p}(\Omega,\nu_1)}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} |\Phi(\varphi, v)| \leq\ & \Big[||f_1||_{L^{p'}(\Omega,\nu_1)} + ||h_1||_{L^\infty(\Omega)}||\varphi||^{p-1}_{W_0^{1,p}(\Omega,\nu_1)} + C_\Omega\vartheta_{p,s}||f_3||_{L^{s'}(\Omega,\nu_3)} + \vartheta_{p,q}||f_2||_{L^{q'}(\Omega,\nu_2)} \\ &+ \vartheta^s_{p,s}C^s_\Omega||h_4||_{L^\infty(\Omega)}||\varphi||^{s-1}_{W_0^{1,p}(\Omega,\nu_1)} + \vartheta^q_{p,q}\left(C^{q-1}_\Omega||h_2||_{L^\infty(\Omega)} + ||h_3||_{L^\infty(\Omega)}\right)||\varphi||^{q-1}_{W_0^{1,p}(\Omega,\nu_1)}\Big]||v||_{W_0^{1,p}(\Omega,\nu_1)}. \end{aligned}$$

Thus, $\Phi(\varphi, .)$ is linear and continuous for every $\varphi \in W_0^{1,p}(\Omega, \nu_1)$. As a result, there is a linear and continuous operator on $W_0^{1,p}(\Omega, \nu_1)$ labeled by Ψ that provides $\langle \Psi\varphi, v\rangle = \Phi(\varphi, v)$ for all $\varphi, v \in W_0^{1,p}(\Omega, \nu_1)$. We also have

$$\begin{aligned} \|\Psi\varphi\|_* \quad \leq \quad & ||f_1||_{L^{p'}(\Omega,\nu_1)} + ||h_1||_{L^\infty(\Omega)}||\varphi||^{p-1}_{W_0^{1,p}(\Omega,\nu_1)} + C_\Omega \vartheta_{p,s}||f_3||_{L^{s'}(\Omega,\nu_3)} + \vartheta_{p,q}||f_2||_{L^{q'}(\Omega,\nu_2)} \\ & + \vartheta^s_{p,s} C^s_\Omega ||h_4||_{L^\infty(\Omega)}||\varphi||^{s-1}_{W_0^{1,p}(\Omega,\nu_1)} + \vartheta^q_{p,q}\left(C^{q-1}_\Omega ||h_2||_{L^\infty(\Omega)} + ||h_3||_{L^\infty(\Omega)}\right)||\varphi||^{q-1}_{W_0^{1,p}(\Omega,\nu_1)}, \end{aligned}$$

where

$$\|\Psi\varphi\|_* := \sup\left\{ |\langle \Psi\varphi, v\rangle| = |\Phi(\varphi, v)| : v \in W_0^{1,p}(\Omega, \nu_1), \|v\|_{W_0^{1,p}(\Omega,\nu_1)} = 1\right\}$$

is the norm in $W_0^{-1,p'}(\Omega, \nu_1^{1-p'})$. Therefore, we get the operator

$$\begin{aligned} \Psi : W_0^{1,p}(\Omega, \nu_1) &\longrightarrow W_0^{-1,p'}(\Omega, \nu_1^{1-p'}) \\ \varphi &\longmapsto \Psi\varphi. \end{aligned}$$

Therefore, the problem (1) is equivalent to the operator equation

$$\Psi\varphi = \Upsilon, \quad \varphi \in W_0^{1,p}(\Omega, \nu_1).$$

Step 2.

In this step, we demonstrate that Ψ is strictly monotonic. For all $\varphi_1, \varphi_2 \in W_0^{1,p}(\Omega, \nu_1)$ with $\varphi_1 \neq \varphi_2$, we have

$$\begin{aligned} \langle \Psi\varphi_1 - \Psi\varphi_2, \varphi_1 - \varphi_2\rangle &= \Phi(\varphi_1, \varphi_1 - \varphi_2) - \Phi(\varphi_2, \varphi_1 - \varphi_2) \\ &= \int_\Omega \langle a(y, \nabla\varphi_1), \nabla(\varphi_1 - \varphi_2)\rangle \nu_1 dy - \int_\Omega \langle a(y, \nabla\varphi_2), \nabla(\varphi_1 - \varphi_2)\rangle \nu_1 dy \\ &\quad + \int_\Omega \langle b(y, \varphi_1, \nabla\varphi_1), \nabla(\varphi_1 - \varphi_2)\rangle \nu_2 dy - \int_\Omega \langle b(y, \varphi_2, \nabla\varphi_2), \nabla(\varphi_1 - \varphi_2)\rangle \nu_2 dy \\ &\quad + \int_\Omega g(y, \varphi_1)(\varphi_1 - \varphi_2)\nu_3 dy - \int_\Omega g(y, \varphi_2)(\varphi_1 - \varphi_2)\nu_3 dy \\ &= \int_\Omega \langle a(y, \nabla\varphi_1) - a(y, \nabla\varphi_2), \nabla(\varphi_1 - \varphi_2)\rangle \nu_1 dy + \int_\Omega \Big(g(y, \varphi_1) - g(y, \varphi_2)\Big)\Big(\varphi_1 - \varphi_2\Big)\nu_3 dy \\ &\quad + \int_\Omega \langle b(y, \varphi_1, \nabla\varphi_1) - b(y, \varphi_2, \nabla\varphi_2), \nabla(\varphi_1 - \varphi_2)\rangle \nu_2 dy. \end{aligned}$$

By usung **(H3)**, we obtain

$$\langle \Psi\varphi_1 - \Psi\varphi_2, \varphi_1 - \varphi_2\rangle \quad \geq \quad \int_\Omega \lambda|\nabla(\varphi_1 - \varphi_2)|^p \nu_1 dy \quad \geq \quad \lambda\|\nabla(\varphi_1 - \varphi_2)\|^p_{L^p(\Omega,\nu_1)},$$

and by Theorem 2.2, we conclude that

$$\langle \Psi\varphi_1 - \Psi\varphi_2, \varphi_1 - \varphi_2\rangle \quad \geq \quad \frac{\lambda}{(C^p_\Omega + 1)}\|\varphi_1 - \varphi_2\|^p_{W_0^{1,p}(\Omega,\nu_1)},$$

which implies that Ψ is strictly monotone.

Step 3.

This step establishes the coerciveness of the operator Ψ. For all $\varphi \in W_0^{1,p}(\Omega, \nu_1)$, we get

$$\begin{aligned} \langle \Psi\varphi, \varphi\rangle &= \Phi(\varphi, \varphi) \\ &= \int_\Omega \langle a(y, \nabla\varphi), \nabla\varphi\rangle \nu_1 dy + \int_\Omega \langle b(y, \varphi, \nabla\varphi), \nabla\varphi\rangle \nu_2 dy + \int_\Omega g(y, \varphi)u\, \nu_3 dy. \end{aligned}$$

From Theorem 2.2 and **(H4)**, it follows that

$$\begin{aligned} \langle \Psi\varphi, \varphi\rangle &\geq \kappa_1 \int_\Omega |\nabla\varphi|^p \nu_1 dy + \kappa_2 \int_\Omega |\nabla\varphi|^q \nu_2 dy + \kappa_3 \int_\Omega |\varphi|^q \nu_2 dy \\ &\geq \kappa_1 \int_\Omega |\nabla\varphi|^p \nu_1 dy + min(\kappa_2, \kappa_3)\left[\int_\Omega |\nabla\varphi|^q \nu_2 dy + \int_\Omega |\varphi|^q \nu_2 dy\right] \\ &= \kappa_1\|\nabla\varphi\|^p_{L^p(\Omega,\nu_1)} + min(\kappa_2, \kappa_3)\|\varphi\|^q_{W_0^{1,q}(\Omega,\nu_2)} \\ &\geq \kappa_1\|\nabla\varphi\|^p_{L^p(\Omega,\nu_1)} \\ &\geq \frac{\kappa_1}{(C^p_\Omega + 1)}\|\varphi\|^p_{W_0^{1,p}(\Omega,\nu_1)}. \end{aligned}$$

Hence, we obtain

$$\frac{\langle \Psi\varphi, \varphi\rangle}{\|\varphi\|_{W_0^{1,p}(\Omega,\nu_1)}} \geq \frac{\kappa_1}{(C_\Omega^p+1)}\|\varphi\|^{p-1}_{W_0^{1,p}(\Omega,\nu_1)}.$$

Therefore, as $p>1$, we conclude that

$$\frac{\langle \Psi\varphi, \varphi\rangle}{\|\varphi\|_{W_0^{1,p}(\Omega,\nu_1)}} \longrightarrow +\infty \text{ as } \|\varphi\|_{W_0^{1,p}(\Omega,\nu_1)} \longrightarrow +\infty,$$

which means that Ψ is coercive.

Step 4.

In this step, we show that Ψ is continuous. To do this, consider $\varphi_k \longrightarrow \varphi$ in $W_0^{1,p}(\Omega,\nu_1)$ as $k\longrightarrow\infty$. Then $\varphi_k\longrightarrow\varphi$ in $L^p(\Omega,\nu_1)$, $\nabla\varphi_k \longrightarrow \nabla\varphi$ in $(L^p(\Omega,\nu_1))^n$. Therefore, according to Theorem 2.1, there exist (φ_{k_i}), $\psi_1\in L^p(\Omega,\nu_1)$ and $\psi_2\in L^p(\Omega,\nu_1)$ in such a way that

$$\begin{aligned} &\varphi_{k_i}(y)\longrightarrow\varphi(y), && \text{in } \Omega\\ &|\varphi_{k_i}(y)|\leq\psi_1(y), && \text{in } \Omega\\ &\nabla\varphi_{k_i}(y)\longrightarrow\nabla\varphi(y), && \text{in } \Omega\\ &|\nabla\varphi_{k_i}(y)|\leq\psi_2(y), && \text{in } \Omega. \end{aligned} \tag{3}$$

We are going to establish that $\Psi\varphi_k\longrightarrow\Psi\varphi$ in $W_0^{-1,p'}(\Omega,\nu_1^{1-p'})$. It is proved in three steps.

Step 4.1.

Let us define the operator $B_j: W_0^{1,p}(\Omega,\nu_1)\longrightarrow L^{p'}(\Omega,\nu_1)$ by $(B_j\varphi)(y)=a_j(y,\nabla\varphi(y))$. We now show that $B_j\varphi_k\longrightarrow B_j\varphi$ in $L^{p'}(\Omega,\nu_1)$.

(i) For all $\varphi\in W_0^{1,p}(\Omega,\nu_1)$, by Theorem 2.2 and **(H2)**, we have

$$\begin{aligned} \|B_j\varphi\|^{p'}_{L^{p'}(\Omega,\nu_1)} &= \int_\Omega |B_j\varphi(y)|^{p'}\nu_1 dy = \int_\Omega |a_j(y,\nabla\varphi)|^{p'}\nu_1 dy\\ &\leq \int_\Omega \left(f_1+h_1|\nabla\varphi|^{p-1}\right)^{p'}\nu_1 dy\\ &\leq C_p\int_\Omega\left(f_1^{p'}+h_1^{p'}|\nabla\varphi|^p\right)\nu_1 dy\\ &\leq C_p\left[\|f_1\|^{p'}_{L^{p'}(\Omega,\nu_1)}+\|h_1\|^{p'}_{L^\infty(\Omega)}\|\nabla\varphi\|^p_{L^p(\Omega,\nu_1)}\right]\\ &\leq C_p\left[\|f_1\|^{p'}_{L^{p'}(\Omega,\nu_1)}+\|h_1\|^{p'}_{L^\infty(\Omega)}\|u\|^p_{W_0^{1,p}(\Omega,\nu_1)}\right]. \end{aligned}$$

(ii) By **(H2)** and (3), we obtain

$$\begin{aligned} \|B_j\varphi_{k_i}-B_j\varphi\|^{p'}_{L^{p'}(\Omega,\nu_1)} &= \int_\Omega |B_j\varphi_{k_i}(y)-B_j\varphi(y)|^{p'}\nu_1 dy\\ &\leq \int_\Omega\Big(|a_j(y,\nabla\varphi_{k_i})|+|a_j(y,\nabla\varphi)|\Big)^{p'}\nu_1 dy\\ &\leq C_p\int_\Omega\left(|a_j(y,\nabla\varphi_{k_i})|^{p'}+|a_j(y,\nabla\varphi)|^{p'}\right)\nu_1 dy\\ &\leq C_p\int_\Omega\left[\left(f_1+h_1|\nabla\varphi_{k_i}|^{p-1}\right)^{p'}+\left(f_1+h_1|\nabla\varphi|^{p-1}\right)^{p'}\right]\nu_1 dy\\ &\leq C_p\int_\Omega\left[\left(f_1+h_1\psi_2^{p-1}\right)^{p'}+\left(f_1+h_1\psi_2^{p-1}\right)^{p'}\right]\nu_1 dy\\ &\leq 2C_pC_p'\int_\Omega\left(f_1^{p'}+h_1^{p'}\psi_2^p\right)\nu_1 dy\\ &\leq 2C_pC_p'\left[\|f_1\|^{p'}_{L^{p'}(\Omega,\nu_1)}+\|h_1\|^{p'}_{L^\infty(\Omega)}\|\psi_2\|^p_{L^p(\Omega,\nu_1)}\right]. \end{aligned}$$

As $k \longrightarrow \infty$, by using **(H1)**, we get

$$B_j\varphi_{k_i}(y) = a_j(y, \nabla\varphi_{k_i}(y)) \longrightarrow a_j(y, \nabla\varphi(y)) = B_j\varphi(y), \text{ for almost all } x \in \Omega.$$

Consequently, by Lebesgue's theorem, we have

$$\|B_j\varphi_{k_i} - B_j\varphi\|_{L^{p'}(\Omega,\nu_1)} \longrightarrow 0 \Leftrightarrow B_j\varphi_{k_i} \longrightarrow B_j\varphi \quad \text{in} \quad L^{p'}(\Omega, \nu_1).$$

Finally, considering the principle of convergence in Banach spaces, we conclude

$$B_j\varphi_k \longrightarrow B_j\varphi \quad \text{in} \quad L^{p'}(\Omega, \nu_1). \tag{4}$$

Step 4.2.
Define $G_j : W_0^{1,p}(\Omega, \nu_1) \longrightarrow L^{q'}(\Omega, \nu_2)$ by $(G_j\varphi)(y) = b_j(y, \varphi(y), \nabla\varphi(y))$. We have

$$G_j\varphi_k \longrightarrow G_j\varphi \quad \text{in} \quad L^{q'}(\Omega, \nu_2). \tag{5}$$

(i) For all $\varphi \in W_0^{1,p}(\Omega, \nu_1)$, by Remark 3.1(i), **(H2)** and Theorem 2.2, we get

$$\begin{aligned}
\|G_j\varphi\|^{q'}_{L^{q'}(\Omega,\nu_2)} &= \int_\Omega |b_j(y, \varphi, \nabla\varphi)|^{q'} \nu_2 dy \\
&\leq \int_\Omega \left(f_2 + h_2|\varphi|^{q-1} + h_3|\nabla\varphi|^{q-1}\right)^{q'} \nu_2 dy \\
&\leq C_q \int_\Omega \left[f_2^{q'} + h_2^{q'}|\varphi|^q + h_3^{q'}|\nabla\varphi|^q\right]\nu_2 dy \\
&\leq C_q \left[\|f_2\|^{q'}_{L^{q'}(\Omega,\nu_2)} + \|h_2\|^{q'}_{L^\infty(\Omega)}\|\varphi\|^q_{L^q(\Omega,\nu_2)} + \|h_3\|^{q'}_{L^\infty(\Omega)}\|\nabla\varphi\|^q_{L^q(\Omega,\nu_2)}\right] \\
&\leq C_q \left[\|f_2\|^{q'}_{L^{q'}(\Omega,\nu_2)} + C^q_{p,q}\left(C^q_\Omega\|h_2\|^{q'}_{L^\infty(\Omega)} + \|h_3\|^{q'}_{L^\infty(\Omega)}\right)\|u\|^q_{W_0^{1,p}(\Omega,\nu_1)}\right].
\end{aligned}$$

(ii) By using Remark 3.1(i), **(H2)**, and the similar reasoning as employed in Step 4.1(ii), we get

$$G_j\varphi_k \longrightarrow G_j\varphi \quad \text{in} \quad L^{q'}(\Omega, \nu_2). \tag{6}$$

Step 4.3.
We define the operator $H : W_0^{1,p}(\Omega, \nu_1) \longrightarrow L^{s'}(\Omega, \nu_3)$ by $(H\varphi)(y) = g(y, \varphi(y))$. In this step, we show that $H\varphi_k \longrightarrow H\varphi$ in $L^{s'}(\Omega, \nu_3)$.

(i) For all $\varphi \in W_0^{1,p}(\Omega, \nu_1)$, by using Remark 3.1(ii) and **(H2)**, we get

$$\begin{aligned}
\|H\varphi\|^{s'}_{L^{s'}(\Omega,\nu_3)} &= \int_\Omega |g(y, \varphi)|^{s'} \nu_3 dy \\
&\leq \int_\Omega \left(f_3 + h_4|\varphi|^{s-1}\right)^{s'} \nu_3 dy \\
&\leq C_s \int_\Omega \left(f_3^{s'} + h_4^{s'}|\varphi|^s\right) \nu_3 dy \\
&\leq C_s \left[\|f_3\|^{s'}_{L^{s'}(\Omega,\nu_3)} + \|h_4\|^{p'}_{L^\infty(\Omega)}\|\varphi\|^s_{L^s(\Omega,\nu_3)}\right] \\
&\leq C_s \left[\|f_3\|^{s'}_{L^{s'}(\Omega,\nu_3)} + C^s_{p,s}\|h_4\|^{p'}_{L^\infty(\Omega)}\|\varphi\|^s_{L^p(\Omega,\nu_1)}\right] \\
&\leq C_s \left[\|f_3\|_{L^{s'}(\Omega,\nu_1)} + C^s_{p,s}C^s_\Omega\|h_4\|^{s'}_{L^\infty(\Omega)}\|\varphi\|^s_{W_0^{1,p}(\Omega,\nu_1)}\right].
\end{aligned}$$

(ii) From Remark 3.1(ii) and **(H2)**, it follows that

$$\begin{aligned}\|H\varphi_{k_i} - H\varphi\|^{s'}_{L^{s'}(\Omega,\nu_3)} &= \int_\Omega |H\varphi_{k_i}(y) - H\varphi(y)|^{p'}\nu_3 dy\\ &\le \int_\Omega \Big(|g(y,\varphi_{k_i})| + |g(y,\varphi)|\Big)^{s'}\nu_3 dy\\ &\le C_s\int_\Omega \Big(|g(y,\varphi_{k_i})|^{s'} + |g(y,\varphi)|^{s'}\Big)\nu_3 dy\\ &\le C_s\int_\Omega \Big[\Big(f_3 + h_4|\varphi_{k_i}|^{s-1}\Big)^{s'} + \Big(f_3 + h_4|\varphi|^{s-1}\Big)^{s'}\Big]\nu_3 dy\\ &\le C_s\int_\Omega \Big[\Big(f_3 + h_4|\psi_1|^{s-1}\Big)^{s'} + \Big(f_3 + h_4\psi_1^{s-1}\Big)^{s'}\Big]\nu_3 dy\\ &\le 2C_sC'_s\Big[\|f_3\|^{s'}_{L^{s'}(\Omega,\nu_3)} + \|h_4\|^{s'}_{L^\infty(\Omega)}\|\psi_1\|^{s}_{L^s(\Omega,\nu_3)}\Big]\\ &\le 2C_sC'_s\Big[\|f_3\|^{s'}_{L^{s'}(\Omega,\nu_3)} + \vartheta^s_{p,s}\|h_4\|^{s'}_{L^\infty(\Omega)}\|\psi_1\|^{s}_{L^p(\Omega,\nu_1)}\Big].\end{aligned}$$

As $k \longrightarrow \infty$, by using **(H1)**, we obtain

$$H\varphi_{k_i}(y) = g(y,\varphi_{k_i}(y)) \longrightarrow g(y,u(y)) = H\varphi(y), \quad \text{a.e. } x \in \Omega.$$

Consequently, by means of Lebesgue's theorem, we have

$$\|H\varphi_{k_i} - H\varphi\|_{L^{s'}(\Omega,\nu_3)} \longrightarrow 0,$$

that is,

$$H\varphi_{k_i} \longrightarrow H\varphi \quad \text{in} \quad L^{s'}(\Omega,\nu_3).$$

Finally, considering the principle of convergence in Banach spaces, we conclude that

$$H\varphi_k \longrightarrow H\varphi \quad \text{in} \quad L^{s'}(\Omega,\nu_3). \tag{7}$$

At last, by considering $v \in W^{1,p}_0(\Omega,\nu_1)$ and with the help of Theorem 2.2, Hölder inequality, and Remark 3.1, we arrive at

$$\begin{aligned}|\Phi_1(\varphi_k,v) - \Phi_1(\varphi,v)| &= \left|\int_\Omega \langle a(y,\nabla\varphi_k) - a(y,\nabla\varphi), \nabla v\rangle \nu_1 dy\right|\\ &\le \sum_{j=1}^n \int_\Omega |a_j(y,\nabla\varphi_k) - a_j(y,\nabla\varphi)||D_j v|\nu_1 dy\\ &= \sum_{j=1}^n \int_\Omega |B_j\varphi_k - B_j\varphi||D_j v|\nu_1 dy\\ &\le \sum_{j=1}^n \|B_j\varphi_k - B_j\varphi\|_{L^{p'}(\Omega,\nu_1)}\|D_j v\|_{L^p(\Omega,\nu_1)}\\ &\le \left(\sum_{j=1}^n \|B_j\varphi_k - B_j\varphi\|_{L^{p'}(\Omega,\nu_1)}\right)\|v\|_{W^{1,p}_0(\Omega,\nu_1)},\end{aligned}$$

$$\begin{aligned}|\Phi_2(\varphi_k,v)-\Phi_2(\varphi,v)| &= \left|\int_\Omega \langle b(y,\varphi_k,\nabla\varphi_k)-b(y,\varphi,\nabla\varphi),\nabla v\rangle\nu_2 dy\right| \\ &\le \sum_{j=1}^n\int_\Omega |b_j(y,\varphi_k,\nabla\varphi_k)-b_j(y,\varphi,\nabla\varphi)||D_jv|\nu_2 dy \\ &= \sum_{j=1}^n\int_\Omega |G_j\varphi_k-G_j\varphi||D_jv|\nu_2 dy \\ &\le \left(\sum_{j=1}^n\|G_j\varphi_k-G_j\varphi\|_{L^{q'}(\Omega,\nu_2)}\right)\|\nabla v\|_{L^q(\Omega,\nu_2)} \\ &\le \vartheta_{p,q}\left(\sum_{j=1}^n\|G_j\varphi_k-G_j\varphi\|_{L^{q'}(\Omega,\nu_2)}\right)\|\nabla v\|_{L^p(\Omega,\nu_1)} \\ &\le \vartheta_{p,q}\left(\sum_{j=1}^n\|G_j\varphi_k-G_j\varphi\|_{L^{q'}(\Omega,\nu_2)}\right)\|v\|_{W_0^{1,p}(\Omega,\nu_1)}, \\ |\Phi_3(\varphi_k,v)-\Phi_3(\varphi,v)| &\le \int_\Omega |g(y,\varphi_k)-g(y,\varphi)||v|\nu_3 dy \\ &= \int_\Omega |H\varphi_k-H\varphi||v|\nu_3 dy \\ &\le \|H\varphi_k-H\varphi\|_{L^{s'}(\Omega,\nu_3)}\|v\|_{L^s(\Omega,\nu_3)} \\ &\le \vartheta_{p,s}\|H\varphi_k-H\varphi\|_{L^{s'}(\Omega,\nu_3)}\|v\|_{L^p(\Omega,\nu_1)} \\ &\le \vartheta_{p,s}C_\Omega\|H\varphi_k-H\varphi\|_{L^{s'}(\Omega,\nu_3)}\|v\|_{W_0^{1,p}(\Omega,\nu_1)}.\end{aligned}$$

Hence, for all $v\in W_0^{1,p}(\Omega,\nu_1)$, we have

$$\begin{aligned}|\Phi(\varphi_k,v)-\Phi(\varphi,v)| &\le |\Phi_1(\varphi_k,v)-\Phi_1(\varphi,v)|+|\Phi_2(\varphi_k,v)-\Phi_2(\varphi,v)|+|\Phi_3(\varphi_k,v)-\Phi_3(\varphi,v)| \\ &\le \Bigg[\sum_{j=1}^n\Big(\|B_j\varphi_k-B_j\varphi\|_{L^{p'}(\Omega,\nu_1)}+\vartheta_{p,q}\|G_j\varphi_k-G_j\varphi\|_{L^{q'}(\Omega,\nu_2)}\Big)+\vartheta_{p,s}C_\Omega\|H\varphi_k-H\varphi\|_{L^{s'}(\Omega,\nu_3)}\Bigg]\|v\|_{W_0^{1,p}(\Omega,\nu_1)},\end{aligned}$$

and consequently, we get

$$\|\Psi\varphi_k-\Psi\varphi\|_* \le \sum_{j=1}^n\Big(\|B_j\varphi_k-B_j\varphi\|_{L^{p'}(\Omega,\nu_1)}+\vartheta_{p,q}\|G_j\varphi_k-G_j\varphi\|_{L^{q'}(\Omega,\nu_2)}\Big)+\vartheta_{p,s}C_\Omega\|H\varphi_k-H\varphi\|_{L^{s'}(\Omega,\nu_3)}.$$

Combining (4), (6) and (7), we deduce that

$$\|\Psi\varphi_k-\Psi\varphi\|_*\longrightarrow 0 \text{ as } m\longrightarrow\infty,$$

that is, $\Psi\varphi_k\longrightarrow\Psi\varphi$ in $W_0^{-1,p'}(\Omega,\nu_1^{1-p'})$, which implies that Ψ is continuous.
We have now proved that Ψ is strictly monotone, coercive and hemicontinuous, and $\Upsilon\in W_0^{-1,p'}(\Omega,\nu_1^{1-p'})$. Thus, we have verified all the conditions of Theorem 2.3. As a result, from Theorem 2.3, it follows that the operator equation $\Psi\varphi=\Upsilon$ admits the unique weak solution $\varphi\in W_0^{1,p}(\Omega,\nu_1)$ and it also follows that u is the unique weak solution for (1). This completes the proof of Theorem 4.1. □

5. Example

Set $\Omega=\{(y,z)\in\mathbb{R}^2 : x^2+y^2<1\}$, and let $\nu_1(y,z)=\big(y^2+z^2\big)^{-1/2}$, $\nu_2(y,z)=\big(y^2+z^2\big)^{-1/3}$ and $\nu_3(y,z)=\big(y^2+z^2\big)^{-1}$ (note that $\nu_1,\nu_2,\nu_3\in A_4$, $p=4$, $q=3$ and $s=2$), and we define $b:\Omega\times\mathbb{R}\times\mathbb{R}^2\longrightarrow\mathbb{R}^2$, $a:\Omega\times\mathbb{R}^2\longrightarrow\mathbb{R}^2$ and $g:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ by

$$a\Big((y,z),\delta\Big)=h_1(y,z)|\delta|^3 sgn(\delta),$$

$$b\Big((y,z),\mu,\delta\Big)=|\delta|^2 sgn(\delta),$$

$$g\Big((y,z),\mu\Big)=h_4(y,z)|\mu|sgn(\mu),$$

with $h_1(y,z)=2e^{(y^2+z^2)}$ and $h_4(y,z)=2-cos^2(yz)$. Let us look at the problem

$$\begin{cases}\mathcal{A}\varphi(y,z)=cos(y+z) & \text{in } \Omega,\\ \varphi(y,z)=0 & \text{on } \partial\Omega,\end{cases} \tag{8}$$

where

$$\mathcal{A}\varphi(y,z)=-\mathrm{div}\Big[\nu_1 a\Big((y,z),\nabla\varphi(y,z)\Big)+\nu_2 b\Big((y,z),\varphi(y,z),\nabla\varphi(y,z)\Big)\Big]+\nu_3 g\Big((y,z),\varphi(y,z)\Big).$$

From Theorem 4.1, it follows that the problem (8) admits the unique weak solution in $W_0^{1,4}(\Omega,\nu_1)$.

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References

[1] A. Abbassi, C. Allalou, A. Kassidi, Topological degree methods for a Neumann problem governed by nonlinear elliptic equation, *Moroccan J. Pure Appl. Anal.* **6** (2020) 231–242.
[2] A. Abbassi, C. Allalou, A. Kassidi, Existence results for some nonlinear elliptic equations via topological degree methods, *J. Elliptic Parabol. Equ.* **7** (2021) 121–136.
[3] Y. Akdim, C. Allalou, Existence and uniqueness of renormalized solution of nonlinear degenerated elliptic problems, *Anal. Theory Appl.* **30** (2014) 318–343.
[4] A. Bensoussan, L. Boccardo, F. Murat, On a non linear partial differential equation having natural growth terms and unbounded solution, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **5** (1988) 347–364.
[5] A. C. Cavalheiro, Existence results for degenerate quasilinear elliptic equations in weighted Sobolev spaces, *Bull. Belg. Math. Soc. Simon Stevin* **17** (2010) 141–153.
[6] A. C. Cavalheiro, Existence results for a class of nonlinear degenerate elliptic equations, *Moroccan J. Pure Appl. Anal.* **5** (2019) 164–178.
[7] F. Chiarenza, Regularity for solutions of quasilinear elliptic equations under minimal assumptions, In: M. Biroli (Ed.), *Potential Theory and Degenerate Partial Differential Operators*, Springer, Dordrecht, 1995, pp. 325–334.
[8] P. Drabek, A. Kufner, V. Mustonen, Pseudo-monotonicity and degenerated or singular elliptic operators, *Bull. Aust. Math. Soc.* **56** (1998) 213–221.
[9] S. Gala, M. A. Ragusa, Y. Sawano, H. Tanaka, Uniqueness criterion of weak solutions for the dissipative quasi-geostrophic equations in Orlicz-Morrey spaces, *Appl. Anal.* **93** (2014) 356–368.
[10] J. Garcia-Cuerva, J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, Elsevier, Amsterdam, 2011.
[11] A. Kufner, O. John, S. Fučík, *Function Spaces*, Noordhoff International Publishing, Leyden, 1977.
[12] A. Kufner, B. Opic, How to define reasonably weighted Sobolev spaces, *Comment. Math. Univ. Carolin.* **25** (1984) 537–554.
[13] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.* (1972) 207–226.
[14] M. E. Ouaarabi, A. Abbassi, C. Allalou, Existence result for a Dirichlet problem governed by nonlinear degenerate elliptic equation in weighted Sobolev spaces, *J. Elliptic Parabol. Equ.* **7** (2021) 221–242.
[15] M. E. Ouaarabi, A. Abbassi, C. Allalou, Existence result for a general nonlinear degenerate elliptic problems with measure datum in weighted Sobolev spaces, *Int. J. Optim. Appl.* **1** (2021) 1–9.
[16] M. E. Ouaarabi, A. Abbassi, C. Allalou, Existence and uniqueness of weak solution in weighted Sobolev spaces for a class of nonlinear degenerate elliptic problems with measure data, *Int. J. Nonlinear Anal. Appl.* **13** (2022) 2635–2653.
[17] M. E. Ouaarabi, C. Allalou, A. Abbassi, Strongly nonlinear degenerated elliptic equations with weight and L^1 data, *International Conference on Research in Applied Mathematics and Computer Science*, Casablanca, 2021, #174.
[18] M. E. Ouaarabi, C. Allalou, A. Abbassi, On the Dirichlet problem for some nonlinear degenerated elliptic equations with weight, *7th International Conference on Optimization and Applications*, Wolfenbüttel, 2021, pp. 1–6.
[19] M. A. Ragusa, On weak solutions of ultraparabolic equations, *Nonlinear Anal.* **47** (2001) 503–511.
[20] M. A. Ragusa, Local Hölder regularity for solutions of elliptic systems, *Duke Math. J.* **113** (2002) 385–397.
[21] M. A. Ragusa, Parabolic Herz spaces and their applications, *Appl. Math. Lett.* **25** (2012) 1270–1273.
[22] M. A. Ragusa, A. Razani, Weak solutions for a system of quasilinear elliptic equations, *Contrib. Math.* **1** (2020) 11–26.
[23] M. A. Ragusa, A. Tachikawa, Regularity for minimizers for functionals of double phase with variable exponents, *Adv. Nonlinear Anal.* **9** (2020) 710–728.
[24] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Academic Press, Orlando, 1986.
[25] B. O. Turesson, *Nonlinear Potential Theory and Weighted Sobolev Spaces*, Springer-Verlag, Berlin, 2000.
[26] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, Vol. II/B, Springer-Verlag, Berlin, 1990.