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Research Article

On the Hermite-Hadamard type inequalities involving generalized integrals

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Abstract

In this paper, using a generalized integral operator, new Hermite-Hadamard type inequalities are obtained for differentiable modified (h, m)-convex functions of the second type.

Keywords: generalized integral operators; Hermite-Hadamard integral inequality; modified (h, m)-convex functions.

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1. Introduction

In recent years, integral inequalities became one of the most attractive areas in mathematics. Consequently, in this area, there has been a significant growth in the number of researchers and the findings gained in recent years. In this field of research, there is a classic inequality: the Hermite-Hadamard inequality (1) for convex functions, which is more than 130 years old and continues to attract mathematicians all over the world (for example, see [10, 19, 21]).

Let $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}$ be the sets of real numbers, positive real numbers, and positive integers, respectively. Also, we take $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. If $I \subset \mathbb{R}$ is an interval and $\phi : I \to \mathbb{R}$ is a convex function, then for $a, b \in I$ with $a < b$, the inequality

$$\phi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \phi(x)dx \leq \frac{\phi(a)+\phi(b)}{2} \tag{1}$$

holds. Convexity is a fundamental concept in geometry, but it is also utilized frequently in other areas of mathematics; for example, theory of optimization, theory of inequalities, functional analysis, mathematical programming, game theory, number theory, and variational calculus. The relationship between convexity and these branches becomes deeper and more beneficial day by day [14–16, 41]. We recommend the paper [35] to readers who want a more comprehensive understanding of the many expansions and generalizations of the classical notion of convexity.

The inequality (1) has become an object of research for many mathematicians, not only with the refinement of the classical concept of convexity, but also with the use of new integral operators, such as Riemann integral, fractional integrals of Riemann-Liouville type, and generalized integrals; for example, see [1, 3, 5, 8, 11–13, 20, 23, 25, 30, 33, 34, 36, 37, 44, 46] and the references cited therein.

The concept of m-convexity was introduced by Toader in [45]. The definition of an m-convex function is given as follows.

Definition 1.1. *The function $\phi : [0, \xi_2] \to \mathbb{R}$, $\xi_2 > 0$, is said to be m-convex, where $m \in [0, 1]$, if the inequality*

$$\phi(tx + m(1-t)y) \leq t\phi(x) + m(1-t)\phi(y) \tag{2}$$

holds for all $x, y \in [0, \xi_2]$ and $t \in [0, 1]$.

In Definition 1.1, the function ϕ is said to be m-concave if the reverse case in (2) is fulfilled. The following definitions are successive extensions of the concept of convex functions.

Definition 1.2 (see [6, 22])**.** *Let $s \in (0, 1]$ be a real number. A function $\phi : [0, \xi_2] \to [0, +\infty)$, with $\xi_2 > 0$, is said to be s-convex in the first sense if the inequality*

$$\phi(tx + (1-t)y) \leq t^s\phi(x) + (1-t^s)\phi(y), \tag{3}$$

holds for all $x, y \in [0, \xi_2]$ and $t \in (0, 1)$.

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Definition 1.3 (see [6,22]). *Let $s \in (0,1]$ be a real number. A function $\phi : [0,\xi_2] \to [0,+\infty)$, with $\xi_2 > 0$, is said to be s-convex in the second sense if the inequality*

$$\phi(tx+(1-t)y) \le t^s\phi(x) + (1-t)^s\phi(y) \tag{4}$$

holds for all $x, y \in [0,\xi_2]$ and $t \in (0,1)$.

Definition 1.4 (see [48]). *Let $s \in [-1,1]$ be a real number. A function $\phi : [0,\xi_2] \to [0,+\infty)$, with $\xi_2 > 0$, is said to be extended s-convex if the inequality*

$$\phi(tx+(1-t)y) \le t^s\phi(x) + (1-t)^s\phi(y), \tag{5}$$

holds for all $x, y \in [0,\xi_2]$ and $t \in (0,1)$.

Definition 1.5 (see [29]). *The function $\phi : [0,\xi_2] \to [0,+\infty)$, with $\xi_2 > 0$, is said to be (a,m)-convex, where $(\alpha,m) \in [0,1]^2$, if for every $x, y \in [0,\xi_2]$ and $t \in [0,1]$ the following inequality holds:*

$$\phi(tx+m(1-t)y) \le t^a\phi(x) + m(1-t^a)\phi(y). \tag{6}$$

Definition 1.6 (see [31]). *Let $h : [0,1] \to \mathbb{R}$ be a non-negative function. The non-negative function $\phi : [0,\xi_2] \to [0,+\infty)$, with $\xi_2 > 0$, is said to be (h,m)-convex on $[0,\xi_2]$ if the inequality*

$$\phi\left(tx+(1-t)y\right) \le h(t)\phi(x) + mh(1-t)\phi(y) \tag{7}$$

is fulfilled for $m \in [0,1]$ and for all $x, y \in I$, and $t \in [0,1]$.

If (7) is reversed, then ϕ is said to be (h,m)-concave. In Definition 1.6, note that if $h(t) = t$ then this definition coincides with the definition of an m-convex function; if in addition, we put $m = 1$ then we obtain the definition of a convex function. In [32], the authors presented the class of s-(a,m)-convex functions as follows ("redefined" in [47]).

Definition 1.7. *A function $\phi : [0,+\infty) \to [0,+\infty)$ is said to be s-(a,m)-convex in the second sense if for all $\xi_1, \xi_2 \in [0,+\infty)$ and $t \in [0,1]$, the inequality*

$$\phi\big(t\xi_1 + m(1-t)\xi_2\big) \le \big(t^a\big)^s\phi(\xi_1) + m\big(1-t^a\big)^s\phi(\xi_2), \tag{8}$$

holds, where $(a,m) \in [0,1]^2$ and $s \in (0,1]$.

On the basis of the definitions listed before, we now present the classes of functions that are crucial for our main results (see [2]).

Definition 1.8. *Let $h : [0,1] \to \mathbb{R}$ be a non-negative function. The non-negative function $\phi : [0,+\infty) \to [0,+\infty)$ is said to be modified (h,m)-convex of the second type on $[0,+\infty)$ if the inequality*

$$\phi\left(tx+m(1-t)y\right) \le h^s(t)\phi(x) + m(1-h(t))^s\phi(y) \tag{9}$$

holds for $m \in [0,1]$, $s \in [-1,1]$, for all $x, y \in I$, and $t \in [0,1]$.

Remark 1.1. *Here, we list some special cases of Definition 1.8.*

(1). *If we take $h(t) = t$ and $m = s = 1$, then ϕ is a convex function on $[0,+\infty)$* (see [7]).

(2). *If $h(t) = t$, m, $s = 1$, then ϕ is an m-convex function on $[0,+\infty)$* (see [45]).

(3). *If $h(t) = t$, $m = 1$, and $s \in (0,1]$, then ϕ is a s-convex function on $[0,+\infty)$* (see [6,22]).

(4). *If we take $h(t) = t$, $m = 1$, and $s \in [-1,1]$, then ϕ is an extended s-convex function on $[0,+\infty)$* (see [48]).

(5). *If $h(t) = t$, m, and $s \in (0,1]$, then ϕ is an (s,m)-convex function on $[0,+\infty)$* (see [39]).

(6). *If $h(t) = t^\alpha$, $m = 1$, s, with $\alpha \in (0,1]$, then ϕ is an (α,s)-convex function on $[0,+\infty)$* (see [4]).

(7). *If $h(t) = t^\alpha$, m, $s = 1$, with $\alpha \in (0,1]$, then ϕ is an (α,m)-convex function on $[0,+\infty)$* (see [29]).

(8). *If $h(t) = t^\alpha$, m, s, with $\alpha \in (0,1]$, then ϕ is an s-(α,m)-convex function on $[0,+\infty)$* (see [47]).

(9). *If we take $h(t)$, m, $s = 1$, then we have a variant of the (h,m)-convex function on $[0,+\infty)$* (see [38]).

In the rest of this paper, we utilize the functions Γ (see [40, 41, 49, 50]) and Γ_k (see [10]) as defined below:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, \mathrm{d}t, \quad \Re(z) > 0, \tag{10}$$

$$\Gamma_k(z) = \int_0^{\infty} t^{z-1} e^{-t^k/k}\, \mathrm{d}t, \quad k > 0. \tag{11}$$

It is noted here that

$$\Gamma_k(z) \to \Gamma(z) \quad \text{whenever} \quad k \to 1, \quad \Gamma_k(z) = (k)^{\frac{z}{k}-1} \Gamma\left(\frac{z}{k}\right), \quad \text{and} \quad \Gamma_k(z+k) = z\Gamma_k(z).$$

Next, we provide some of the most well-known fractional operators (with the assumption that $0 \le \xi_1 < t < \xi_2 \le \infty$) to make the main results of this paper easier to read. The well-known Riemann-Liouville fractional integrals are the first of these operators.

Definition 1.9. *If $\phi \in L_1[\xi_1, \xi_2]$, then Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{C}$, with $\Re(\alpha) > 0$, are defined by (right and left, respectively):*

$${}^{\alpha}I_{\xi_1+}\phi(x) = \frac{1}{\Gamma(\alpha)} \int_{\xi_1}^{x} (x-t)^{\alpha-1} \phi(t)\, dt, \quad x > \xi_1 \tag{12}$$

$${}^{\alpha}I_{\xi_2-}\phi(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\xi_2} (t-x)^{\alpha-1} \phi(t)\, dt, \quad x < \xi_2. \tag{13}$$

The generalized integral operators that we use in this paper are defined in the next definition (see [18]).

Definition 1.10. *The generalized fractional Riemann-Liouville integrals (of order $\alpha \in \mathbb{R}$) of an integrable function $f(x)$ on $[0, \infty)$ are given as follows:*

$$\left({}^{\beta}J^{\alpha}_{F,\xi_1+}\phi\right)(x) = \frac{1}{\Gamma(\beta)} \int_{\xi_1}^{x} \frac{\phi(t)dt}{F(\mathbb{F}(x,t),\beta)F\,(t,\alpha)}, \tag{14}$$

$$\left({}^{\beta}J^{\alpha}_{F,\xi_2-}\phi\right)(x) = \frac{1}{\Gamma(\beta)} \int_{x}^{\xi_2} \frac{\phi(t)dt}{F(\mathbb{F}(x,t),\beta)F\,(t,\alpha)}, \tag{15}$$

where

$$\mathbb{F}(x,t) = \int_t^x \frac{ds}{F(t,s)},$$

and F is an absolutely continuous positive function.

Definition 1.11. *The left and right fractional generalized integrals of order $\beta \in \mathbb{C}$, with $Re(\beta) > 0$, are defined by*

$$\left({}^{\beta}J^{\alpha}_{F,\xi_1+}\phi\right)(x) = \frac{1}{\Gamma(\beta)} \int_{\xi_1}^{x} \frac{\phi(t)dt}{F(\mathbb{F}_{+}(x,t),\beta)F\,(t-a,\alpha)} \tag{16}$$

$$\left({}^{\beta}J^{\alpha}_{F,\xi_2-}\phi\right)(x) = \frac{1}{\Gamma(\beta)} \int_{x}^{\xi_2} \frac{\phi(t)dt}{F(\mathbb{F}_{-}(x,t),\beta)F\,(b-t,\alpha)} \tag{17}$$

where

$$\mathbb{F}_{+}(x,t) = \int_t^x \frac{ds}{F(x-s,\alpha)} = J^{\alpha}_{F,x+}(t),$$

$$\mathbb{F}_{-}(x,t) = \int_x^t \frac{ds}{F(s-x,\alpha)} = J^{\alpha}_{F,x-}(t),$$

and $F(\mathbb{F}_{+}(t,x),1) = F(\mathbb{F}_{-}(t,x),1) = 1$.

Remark 1.2. *If we take $F(z,r) = z^{(1-r)}$ in the previous definitions, then the integral operators used in [42, 43] and defined in [24] are obtained. A generalization of the classical Riemann-Liouville fractional integral is obtained from the operators of the kernel pointed at the beginning when $\beta = 1$. Obviously, we get the classical Riemann integral under the case of the previous kernel if $\alpha = 1$. Other well-known fractional integrals, such as Hadamard's [19, 27] and Katugampola's [9, 26], can also be easily obtained from our definition.*

In this paper, we present some variants of the inequality (1) for modified (h,m)-convex functions, within the framework of the generalized integral operators given in Definition 1.11.

2. Main results

Theorem 2.1. *Let $\psi : [0,+\infty) \to \mathbb{R}$ be a modified (h,m)-convex function of the second type such that $m \in (0,1]$. If $0 \le \xi_1 < m\xi_2 < +\infty$, $\psi \in L^1[\xi_1, m\xi_2]$ and $h \in L^1[0,1]$, then the following inequality holds:*

$$\begin{aligned}\mathbb{F}\psi\left(\frac{\xi_1+\xi_2}{2}\right) &\le \frac{1}{\xi_2-\xi_1}\left(h^s\left(\frac{1}{2}\right) J^{\alpha}_{F,\xi_1+}(\psi)(\xi_2) + \left(1-h\left(\frac{1}{2}\right)\right)^s J^{\alpha}_{F,\xi_2-}(\psi)(\xi_1)\right) \\ &\le \left(h^s\left(\frac{1}{2}\right)\psi(\xi_1) + \left(1-h\left(\frac{1}{2}\right)\right)^s \psi(\xi_2)\right)\int_0^1 \frac{h^s(t)dt}{F(t,\alpha)} \\ &\quad + m\left(h^s\left(\frac{1}{2}\right)\psi\left(\frac{\xi_1}{m}\right) + \left(1-h\left(\frac{1}{2}\right)\right)^s \psi\left(\frac{\xi_2}{m}\right)\right)\int_0^1 \frac{(1-h(t))^s dt}{F(t,\alpha)},\end{aligned} \tag{18}$$

where

$$\mathbb{F} = \int_0^1 \frac{dt}{F(t,\alpha)}.$$

Proof. For $x, y \in [0,+\infty)$, $t = \frac{1}{2}$, and $m = 1$, we have

$$\psi\left(\frac{x+y}{2}\right) \le h^s\left(\frac{1}{2}\right)\psi(x) + \left(1-h\left(\frac{1}{2}\right)\right)^s \psi(y),$$

If we choose $x = t\xi_1 + (1-t)\xi_2$ and $y = t\xi_2 + (1-t)\xi_1$, with $t \in [0,1]$, we get

$$\psi\left(\frac{\xi_1+\xi_2}{2}\right) \le h^s\left(\frac{1}{2}\right)\psi(t\xi_1 + (1-t)\xi_2) + \left(1-h\left(\frac{1}{2}\right)\right)^s \psi(t\xi_2 + (1-t)\xi_1). \tag{19}$$

By integrating (19) with respect to t, on $[0,1]$, and then by changing variables, brings us to the first inequality of (18). Rewriting the right member of (19), we have

$$\begin{aligned}&h^s\left(\frac{1}{2}\right)\psi(t\xi_1 + (1-t)\xi_2) + \left(1-h\left(\frac{1}{2}\right)\right)^s \psi(t\xi_2 + (1-t)\xi_1) \\ = \;& h^s\left(\frac{1}{2}\right)\psi\left(t\xi_1 + m(1-t)\frac{\xi_2}{m}\right) + \left(1-h\left(\frac{1}{2}\right)\right)^s \psi\left(t\xi_2 + m(1-t)\frac{\xi_1}{m}\right) \\ \le \;& h^s\left(\frac{1}{2}\right)\left(h^s(t)\psi(\xi_1) + m(1-h(t))^s\psi\left(\frac{\xi_2}{m}\right) + h^s(t)\psi(\xi_2) + m(1-h(t))^s\psi\left(\frac{\xi_1}{m}\right)\right).\end{aligned}$$

After integrating this inequality with respect to t, between 0 and 1, we obtain

$$\begin{aligned}&\frac{1}{\xi_2-\xi_1}\left(h^s\left(\frac{1}{2}\right) J^{\alpha}_{F,\xi_1+}(\psi)(\xi_2) + \left(1-h\left(\frac{1}{2}\right)\right)^s J^{\alpha}_{F,\xi_2-}(\psi)(\xi_1)\right) \\ \le \;& \left(h^s\left(\frac{1}{2}\right)\psi(\xi_1) + \left(1-h\left(\frac{1}{2}\right)\right)^s \psi(\xi_2)\right)\int_0^1 \frac{h^s(t)dt}{F(t,\alpha)} \\ & + m\left(h^s\left(\frac{1}{2}\right)\psi\left(\frac{\xi_1}{m}\right) + \left(1-h\left(\frac{1}{2}\right)\right)^s \psi\left(\frac{\xi_2}{m}\right)\right)\int_0^1 \frac{(1-h(t))^s dt}{F(t,\alpha)}.\end{aligned}$$

Thus, we obtain the second inequality. □

Remark 2.1. *In Theorem 2.1, if we consider the Riemann integral (or equivalently, if we take $F \equiv 1$ and ψ as a convex function ($h(t) = t$, $s = 1$, $a = 1$ and $m = 1$)), then from (19) we obtain the classical Hermite-Hadamard inequality (1). Also, this result is a variant of Theorem 9 of [38].*

As we will see, the following result "complements" Theorem 2.1.

Theorem 2.2. *Let $\psi : [0,+\infty) \to \mathbb{R}$ be a modified (h,m)-convex function of the second type such that $m \in (0,1]$. If $0 < m\xi_1 \le \xi_1 < m\xi_2 < +\infty$, $\psi \in L^1[m\xi_1, m\xi_2]$ and $h \in L^1[0,1]$, then the following inequality holds:*

$$\frac{1}{\xi_2-\xi_1}\left\{J^{\alpha}_{F,\xi_1+}(\psi)(\xi_2) + J^{\alpha}_{F,\xi_2-}(\psi)(\xi_1)\right\} \le (\psi(\xi_1)+\psi(\xi_2))\int_0^1 \frac{h^s(t)dt}{F(t,\alpha)} + m\left(\psi\left(\frac{\xi_1}{m}\right)+\psi\left(\frac{\xi_2}{m}\right)\right)\int_0^1 \frac{(1-h(t))^s dt}{F(t,\alpha)}. \tag{20}$$

Proof. By using the modified (h,m)-convexity of the second type of ψ, we have

$$\psi(t\xi_1+(1-t)\xi_2)\le h^s(t)\psi(\xi_1)+m(1-h(t))^s\psi\left(\frac{\xi_2}{m}\right),$$

$$\psi(t\xi_2+(1-t)\xi_1)\le h^s(t)\psi(\xi_2)+m(1-h(t))^s\psi\left(\frac{\xi_1}{m}\right).$$

Integrating these two inequalities, with respect to t between 0 and 1, and then adding member to member, we obtain the required result by changing variables in the first integrals. □

Remark 2.2. *If $F\equiv 1$ and ψ is a convex function, then from (20) we get the right member of the classical Hermite-Hadamard inequality (1). Similarly, working with the classical Riemann integral, that is, by taking $F\equiv 1$ and simultaneously taking $\psi(t\xi_1+(1-t)\xi_2)$ with $\psi(t\xi_2+(1-t)\xi_1)$, and $s=1$, we obtain Theorem 10 of [38]. Analogously, in Theorem 2.2, if we put $F\equiv 1$ and $m=s=1$, then we obtain Theorem 2.1 of [31] (Remark 2.1 of the aforementioned work remains valid). If, on the contrary, we consider the kernel $F(t,\alpha)=\Gamma(\alpha)t^{1-\alpha}$, then the following inequality (not reported in the literature) is obtained, which is valid for Riemann-Liouville fractional integrals:*

$$\frac{1}{(\xi_2-\xi_1)^\alpha}\left[{}^{RL}J^\alpha_{\xi_1+}\psi(\xi_2)+{}^{RL}J^\alpha_{\xi_2-}\psi(\xi_1)\right]\le(\psi(\xi_1)+\psi(\xi_2))\left[{}^{RL}J^\alpha_{0+}(h^s(t))(1)+m\,{}^{RL}J^\alpha_{0+}((1-h(t))^s)(1)\right].$$

Of course, if we consider different kernels, then we get new variants of (20).

The next result is a more general variation of the previous two results, in which two modified (h,m)-convex functions of second type are involved.

Theorem 2.3. *Let ψ_1 be a modified (h_1,m)-convex of the second type and ψ_2 be a modified (h_2,m)-convex function of the second type such that $\psi_1\psi_2\in L^1[m\xi_1,m\xi_2]$ and $h_1h_2\in L^1[\xi_1,\xi_2]$. The following inequality*

$$\begin{aligned}
&\frac{1}{\xi_2-\xi_1}\left\{J^\alpha_{F,\xi_1+}(\psi_1\psi_2)(m\xi_2)+J^\alpha_{F,\xi_2-}(\psi_1\psi_2)(m\xi_1)\right\}\\
\le\;&(\psi_1(\xi_1)\psi_2(\xi_1)+\psi_1(\xi_2)\psi_2(\xi_2))\,J^\alpha_{F,0}(h_1^sh_2^s)(1)\\
&+m\psi_1(\xi_1)\psi_2(\xi_2)\int_0^1 h_1^s(t)(1-h_2(t))^s d_Ft+m\psi_1(\xi_2)\psi_2(\xi_1)\int_0^1(1-h_1(t))^sh_2^s(t)d_Ft\\
&+m\psi_1(\xi_2)\psi_2(\xi_1)\int_0^1 h_1^s(t)(1-h_2(t))^s d_Ft+m\psi_1(\xi_1)\psi_2(\xi_2)\int_0^1(1-h_1(t))^sh_2^s(t)d_Ft\\
&+m^2(\psi_1(\xi_1)\psi_2(\xi_1)+\psi_1(\xi_2)\psi_2(\xi_2))\int_0^1(1-h_1(t))^s(1-h_2(t))^s d_Ft
\end{aligned}\tag{21}$$

holds, where

$$d_Ft=\frac{dt}{F(t,\alpha)}.$$

Proof. By using the definitions of the functions ψ_1 and ψ_2, we have

$$\begin{aligned}
&\psi_1(t\xi_1+m(1-t)\xi_2)\psi_2(t\xi_1+m(1-t)\xi_2)\\
\le\;&\left(h_1^s(t)\psi_1(\xi_1)+m(1-h_1(t))^s\psi_1(\xi_2)\right)\left(h_2^s(t)\psi_2(\xi_1)+m(1-h_2(t))^s\psi_2(\xi_2)\right),
\end{aligned}\tag{22}$$

$$\begin{aligned}
&\psi_1(t\xi_2+m(1-t)\xi_1)\psi_2(t\xi_2+m(1-t)\xi_1)\\
\le\;&\left(h_1^s(t)\psi_1(\xi_2)+m(1-h_1(t))^s\psi_1(\xi_1)\right)\left(h_2^s(t)\psi_2(\xi_2)+m(1-h_2(t))^s\psi_2(\xi_1)\right).
\end{aligned}\tag{23}$$

After multiplying and ordering, we get from (22) and (23)

$$\begin{aligned}
&\psi_1(t\xi_1+m(1-t)\xi_2)\psi_2(t\xi_1+m(1-t)\xi_2)+\psi_1(t\xi_2+m(1-t)\xi_1)\psi_2(t\xi_2+m(1-t)\xi_1)\\
\le\;&h_1^s(t)h_2^s(t)\left(\psi_1(\xi_1)\psi_2(\xi_1)+\psi_1(\xi_2)\psi_2(\xi_2)\right)\\
&+mh_1^s(t)(1-h_2(t))^s\psi_1(\xi_1)\psi_2(\xi_2)+m(1-h_1(t))^sh_2^s(t)\psi_1(\xi_2)\psi_2(\xi_1)\\
&+mh_1^s(t)(1-h_2(t))^s\psi_1(\xi_2)\psi_2(\xi_1)+m(1-h_1(t))^sh_2^s(t)\psi_1(\xi_1)\psi_2(\xi_2)\\
&+m^2(1-h_1(t))^s(1-h_2(t))^s\left(\psi_1(\xi_1)\psi_2(\xi_1)+\psi_1(\xi_2)\psi_2(\xi_2)\right).
\end{aligned}$$

The desired inequality is obtained after integrating this last inequality, with respect to t between 0 and 1 and change of variables in the integrals of the left member. □

Remark 2.3. *In Theorem 2.3, if we put $F \equiv 1$, $s = 1$, and if we consider only (22), then we obtain a complement to Theorem 2.2 of [31] for (h, m)-convex. If we consider the kernel $F(t,\alpha) = t^{1-\alpha}$, then we obtain new inequalities under Riemann-Liouville fractional integrals. If we use another kernel F, then we obtain inequalities not reported in the literature.*

The next result gives a more general conclusion than Theorem 2.2.

Theorem 2.4. *Under the conditions on ψ_1 and ψ_2 specified in Theorem 2.3, the following inequality is satisfied:*

$$\frac{1}{m\xi_2 - \xi_1}\left[J^{\alpha}_{F,\xi_1+}\psi_1\psi_2(\xi_2) + J^{\alpha}_{F,\xi_2-}\psi_1\psi_2(\xi_1)\right] \leq Min\,\mathbb{A} \tag{24}$$

where

$$\begin{aligned}\mathbb{A} \;=\;& \left(\psi_1(\xi_1)\psi_2(\xi_1) + \psi_1(\xi_2)\psi_2(\xi_2)\right) J^{\alpha}_{F,0}(h_1^s h_2^s)(1) + m_1 m_2\left[\psi_1\left(\frac{\xi_2}{m_1}\right)\psi_2\left(\frac{\xi_2}{m_2}\right)\right] J^{\alpha}_{F,0}(h_1^s h_2^s)(1)\\ &+\left[\psi_1\left(\frac{\xi_2}{m_1}\right)\psi_2(\xi_1) + m_1\psi_1\left(\frac{\xi_1}{m_1}\right)\psi_2(\xi_2)\right]\int_0^1 \left((1-h_1(t))^s h_2^s(t)\right) d_F t\\ &+m_2\left[\psi_1(\xi_1)\psi_2\left(\frac{\xi_2}{m_2}\right) + \psi_1(\xi_2)\psi_2\left(\frac{\xi_1}{m_2}\right)\right]\int_0^1 \left(h_1^s(t)(1-h_2(t))^s\right) d_F t.\end{aligned}$$

Proof. We have

$$\begin{aligned}\psi_1(t\xi_1 + (1-t)\xi_2)\psi_2(t\xi_1 + (1-t)\xi_2) &= \psi_1\left(t\xi_1 + m_1(1-t)\frac{\xi_2}{m_1}\right)\psi_2\left(t\xi_1 + m_2(1-t)\frac{\xi_2}{m_2}\right)\\ &\leq \left(h_1^s(t)\psi_1(\xi_1) + m_1(1-h_1(t))^s\psi_1\left(\frac{\xi_2}{m_1}\right)\right)\left(h_2^s(t)\psi_2(\xi_1) + m_2(1-h_2(t))^s\psi_2\left(\frac{\xi_2}{m_1}\right)\right),\end{aligned} \tag{25}$$

$$\begin{aligned}\psi_1(t\xi_2 + (1-t)\xi_1)\psi_2(t\xi_2 + (1-t)\xi_1) &= \psi_1\left(t\xi_2 + m_1(1-t)\frac{\xi_1}{m_1}\right)\psi_2\left(t\xi_2 + m_2(1-t)\frac{\xi_1}{m_2}\right)\\ &\leq \left(h_1^s(t)\psi_1(\xi_2) + m_1(1-h_1(t))^s\psi_1\left(\frac{\xi_1}{m_1}\right)\right)\left(h_2^s(t)\psi_2(\xi_2) + m_2(1-h_2(t))^s\psi_2\left(\frac{\xi_1}{m_2}\right)\right).\end{aligned} \tag{26}$$

By adding member to member of (25) and (26), we obtain

$$\begin{aligned}&\psi_1(t\xi_1 + (1-t)\xi_2)\psi_2(t\xi_1 + (1-t)\xi_2) + \psi_1(t\xi_2 + (1-t)\xi_1)\psi_2(t\xi_2 + (1-t)\xi_1)\\ \leq\;& \left(h_1^s(t)\psi_1(\xi_1) + m_1(1-h_1(t))^s\psi_1\left(\frac{\xi_2}{m_1}\right)\right)\left(h_2^s(t)\psi_2(\xi_1) + m_2(1-h_2(t))^s\psi_2\left(\frac{\xi_2}{m_1}\right)\right)\\ &+\left(h_1^s(t)\psi_1(\xi_2) + m_1(1-h_1(t))^s\psi_1\left(\frac{\xi_1}{m_1}\right)\right)\left(h_2^s(t)\psi_2(\xi_2) + m_2(1-h_2(t))^s\psi_2\left(\frac{\xi_1}{m_2}\right)\right).\end{aligned}$$

By proceeding in the same way as in Theorem 2.3, we obtain the required inequality. □

Remark 2.4. *In Theorem 2.3, if we take $F \equiv 1$, $m = s = 1$, and use only (25), then we get Theorem 2.3 of [31]. By using different kernels, we obtain new integral inequalities.*

3. Conclusions

In this work, we have obtained several extensions and generalizations of the classical Hermite-Hadamard inequality, in the context of generalized integral operators. We have shown that several previously published results are particular cases of the ones that we have obtained. As a future work, it seems to be interesting to study other inequalities (for example, see [17]) by using generalized operators.

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