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Research Article

A short derivation of an elegant sum involving central binomial coefficients due to László via a hypergeometric series approach

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Abstract

The aim of this short note is to establish an elegant sum involving central binomial coefficients, due to László [*Amer. Math. Monthly* **108** (2001) 851–855], via a hypergeometric series approach.

Keywords: central binomial coefficients; hypergeometric series; Dixon summation theorem.

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1. Introduction

In 1999, László [1] presented a method of determining the exact norm of the Hilbert matrix of infinite size using the following sequence of functions:

$$\frac{\tanh^k x}{\cosh x} \quad (k = 0, 1, 2 \dots),\ x \in \mathbb{R}. \tag{1}$$

Later, in 2001, he [2] introduced another method through two examples based on (1) for finding the exact sums of some convergent series. In his first example, for proving the identity

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6},$$

he showed that

$$A = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8} \tag{2}$$

because $\zeta(2) = \frac{4}{3}A$. In the second example, he proved the following elegant sum involving central binomial coefficients that generalizes (2) for $n \in \mathbb{N} \cup \{0\}$:

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+2n+1)(2k+4n+1)\binom{4n+2k}{2n+k}} = \frac{\pi^2}{2^{8n+3}} \binom{2n}{n}^2$$
$$= \frac{\pi^2}{2^{4n+3}} \frac{\left(\frac{1}{2}\right)_n^2}{(1)_n^2} \tag{3}$$

where $(a)_n$ denotes the Pochhammer symbol (or rising or shifted factorials, since $(1)_n = n!$), which is defined as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)\dots(a+n-1) & \text{if } n \in \mathbb{N}, \\ 1 & \text{if } n = 0, \end{cases}$$

where $\Gamma(x)$ is the well-known Gamma function.

We recall that the generalized hypergeometric function with p numerator and q denominator parameters, in terms of the Pochhammer symbol, is

$${}_pF_q\left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; x\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{x^n}{n!}.$$

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For details about the convergence conditions of the generalized hypergeometric function, we refer the readers to the standard book of Slater [3].

It is interesting to mention here that whenever a generalized hypergeometric function reduces to the gamma function, the results are very important from the application's point of view. Thus, the classical summation theorems such as those of Gauss, Kummer, and Bailey for the series ${}_2F_1$; Watson, Dixon, Whipple, and Saalschiútz for the series ${}_3F_2$, and others play a key role. Applications of the above-mentioned classical summation theorems are well known now. However, in our present work, we shall mention here the following classical Dixon's summation theorem recorded in Slater [3, III. 8, Page 243]:

$$ {}_3F_2\left[\begin{matrix} a,\ b,\ c \\ 1+a-b,\ 1+a-c \end{matrix};1\right] = \frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(1+\frac{1}{2}a-b-c\right)}{\Gamma(1+a)\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma\left(1+\frac{1}{2}a-c\right)\Gamma(1+a-b-c)}, \tag{4} $$

provided that $\Re(a-2b-2c) > -2$.

Our aim in this note is to demonstrate that the sum (3) can be established in an elementary way by using a hypergeometric series approach. To achieve this goal, we exploit classical Dixon's summation theorem for the series ${}_3F_2$ with unit argument.

2. Derivation of the sum (3)

In order to establish (3), we denote the left-hand side of (3) by $S(n)$ and make use of the following identities

$$ \binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}, \qquad \binom{2k}{k} = \frac{2^{2k}\left(\frac{1}{2}\right)_k}{(1)_k}, $$

$$ \Gamma(2k+1) = 2^{2k}\left(\frac{1}{2}\right)_k (1)_k, \quad \Gamma(2z) = \frac{2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)}{\sqrt{\pi}}, \quad \Gamma(a-n) = (-1)^n\,\frac{\Gamma(a)}{(1-a)_n}, $$

and after little simplification, we get

$$ S(n) = \frac{\Gamma(2n+1)\,\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2^{4n+2}\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(2n+\frac{3}{2}\right)} \sum_{k=0}^{\infty} \frac{(2n+1)_k\left(\frac{1}{2}\right)_k\ \left(n+\frac{1}{2}\right)_k}{\left(n+\frac{3}{2}\right)_k\ \left(2n+\frac{3}{2}\right)_k\ k!}. $$

By summing up the series, we have

$$ S(n) = \frac{\Gamma(2n+1)\,\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2^{4n+2}\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(2n+\frac{3}{2}\right)}\, {}_3F_2\left[\begin{matrix} 2n+1,\ \frac{1}{2},\ n+\frac{1}{2} \\ 2n+\frac{3}{2},\ n+\frac{3}{2} \end{matrix};1\right]. $$

We now evaluate the ${}_3F_2$-series with the help of the classical Dixon's summation theorem (4) by letting $a = 2n+1, b = \frac{1}{2}$ and $c = n+\frac{1}{2}$, and then after some calculation, we arrive at the right-hand side of (3). This completes the proof of (3).

3. Remarks

1. In (3), by taking $n = 0$, we recover the result (2).

2. In (3), if we take $n = 1$, we get the following identity:

$$ \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{(2k+1)(2k+3)^2(2k+5)} = \frac{\pi^2}{128}. $$

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