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Research Article

The m-bipartite Ramsey number of the $K_{2,2}$ versus $K_{6,6}$

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Abstract

For the given bipartite graphs $G_1, \dots, G_n$, the bipartite Ramsey number $BR(G_1, \dots, G_n)$ is the least positive integer b such that any complete bipartite graph $K_{b,b}$ having edges coloured with $1, 2, \dots, n$, contains a copy of some G_i ($1 \le i \le n$), where all the edges of G_i have colour i. For the given bipartite graphs $G_1, \dots, G_n$ and a positive integer m, the m-bipartite Ramsey number $BR_m(G_1, \dots, G_n)$ is defined as the least positive integer b ($b \ge m$) such that any complete bipartite graph $K_{m,b}$ having edges coloured with $1, 2, \dots, n$, contains a copy of some G_i ($1 \le i \le n$), where all the edges of G_i have colour i. The values of $BR_m(G_1, G_2)$ (for each m), $BR_m(K_{3,3}, K_{3,3})$ and $BR_m(K_{2,2}, K_{5,5})$ (for particular values of m) have already been determined in several articles, where $G_1 = K_{2,2}$ and $G_2 \in \{K_{3,3}, K_{4,4}\}$. In this article, the value of $BR_m(K_{2,2}, K_{6,6})$ is computed for each $m \in \{2, 3, \dots, 8\}$.

Keywords: Ramsey numbers; bipartite Ramsey numbers; complete graphs; m-bipartite Ramsey number.

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1. Introduction

Extremal graph theory problems generally ask for the maximum or minimum order or size of a graph having certain characteristics. Such problems are often quite natural in the construction of networks or circuits. Ramsey theory explores the question of how big a structure must be to contain a certain substructure or substructures. For the graphs G and H, the Ramsey number $R(G, H)$ is the smallest order of a complete graph such that any 2-colouring of the edges results in either a copy of G in the first colour or a copy of H in the second colour. It is a well-known fact that $R(G, H) \le R(K_m, K_n)$, where G and H are two arbitrary graphs of orders m and n, respectively. Bipartite Ramsey problems deal with the same questions but the graph under investigation in this case is the complete bipartite graph instead of the complete graph. For the given bipartite graphs $G_1, \dots, G_n$, the bipartite Ramsey number $BR(G_1, \dots, G_n)$ is the least positive integer b such that any complete bipartite graph $K_{b,b}$ having edges coloured with $1, 2, \dots, n$ contains a copy of some G_i ($1 \le i \le n$), where all the edges of G_i have colour i. One can refer to [3, 4, 6–10, 15, 16] and their references for further detail on this topic.

For the given bipartite graphs $G_1, \dots, G_n$ and a positive integer m, the m-bipartite Ramsey number $BR_m(G_1, \dots, G_n)$ is defined as the least positive integer b ($b \ge m$) such that any complete bipartite graph $K_{m,b}$ having edges coloured with $1, 2, \dots, n$, contains a copy of some G_i ($1 \le i \le n$), where all the edges of G_i have colour i. The value of $BR_m(G_1, G_2)$ have already been determined in several papers for $G_1 \in \{K_{2,2}, K_{3,3}\}$ and $G_2 \in \{K_{3,3}, K_{4,4}, K_{5,5}\}$. One can refer to [1, 2, 5, 11–14] and their references for further studies on m-bipartite Ramsey numbers.

Theorem 1.1 (see [1, 14]). *If $m \ge 2$, then*

$$BR_m(K_{2,2}, K_{3,3}) = \begin{cases} \textit{does not exist}, & where \quad m = 2, 3, \\ 15 & where \quad m = 4, \\ 12 & where \quad m = 5, 6, \\ 9 & where \quad m = 7, 8, \\ m & where \quad m \ge 9. \end{cases}$$

Theorem 1.2 (see [2]). *If $m \in \{2, 3, \dots, 8\}$, then*

$$BR_m(K_{3,3}, K_{3,3}) = \begin{cases} \textit{does not exist}, & where \quad m = 2, 3, 4, \\ 41 & where \quad m = 5, 6, \\ 29 & where \quad m = 7, 8. \end{cases}$$

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Theorem 1.3 (see [13]). *If $m \in \{2,3,\dots,8\}$, then*

$$BR_m(K_{2,2},K_{5,5}) = \begin{cases} \textit{does not exist}, & where \ \ m = 2,3,4,5,\\ 40 & where \ \ m = 6,\\ 30 & where \ \ m = 7,8. \end{cases}$$

In this paper, the exact value of $BR_m(K_{2,2},K_{6,6})$ is computed for some $m \geq 2$, as given in the following theorem.

Theorem 1.4. *If $m \in \{2,3,\dots,8\}$, then*

$$BR_m(K_{2,2},K_{6,6}) = \begin{cases} \textit{does not exist}, & where \ \ m = 2,3,4,5,6,\\ 57 & where \ \ m = 7,\\ 45 & where \ \ m = 8. \end{cases}$$

Suppose that $G[X,Y]$ is a bipartite graph with the partite sets X and Y. Let $E(G[X',Y'])$ be the edge set of $G[X',Y']$, where $X' \subseteq X$ and $Y' \subseteq Y$. We use $\Delta(G_X)$ and $\Delta(G_Y)$ to denote the maximum degree of vertices in the parts X and Y of G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $\deg_G(v)$. For each $v \in V(G)$, $N_G(v) = \{u \in V(G),\ \ vu \in E(G)\}$. For the given graphs G, H, and F, we say G is 2-colourable to (H,F) if there is a subgraph of G, say G', such that $H \nsubseteq G'$ and $F \nsubseteq \overline{G'}$. We use $G \to (H,F)$ to indicate that G is 2-colourable to (H,F). Two graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ are isomorphic if there is a bijection ϕ from V_1 to V_2 such that $vw \in E_1$ if and only if $\phi(v)\phi(w) \in E_2$. We write $G_1 \cong G_2$ when G_1 is isomorphic to G_2. For simplification, we use $[n] = \{1,2,\dots,n\}$.

2. Proof of Theorem 1.4

We start with the following lemma which is helpful for proving Theorem 1.4.

Lemma 2.1. *For $m \geq 7$ and $n \geq 12$, let $X = \{x_1,\dots,x_m\}$ and $Y = \{y_1,\dots,y_n\}$ be the partite sets of $K = K_{m,n}$. Let G be a subgraph of $K_{m,n}$. If $\Delta(G_X) \geq 12$, then either $K_{2,2} \subseteq G$ or $K_{6,6} \subseteq \overline{G}$.*

Proof. Without loss of generality, let $\Delta(G_X) = 12$ and $N_G(x) = Y'$, where $|Y'| = 12$ and $K_{2,2} \nsubseteq G$. Therefore,

$$|N_G(x') \cap Y'| \leq 1$$

for each $x' \in X \setminus \{x\}$. Since $|X| \geq 7$ and $|Y'| = 12$, one has $K_{6,6} \subseteq \overline{G}[X \setminus \{x\}, Y']$. □

To establish Theorem 1.4, we need the following result.

Theorem 2.1. *For each $m \in \{2,3,4,5,6\}$, the number $BR_m(K_{2,2},K_{6,6})$ does not exist.*

Proof. Suppose that $m \in \{2,3,4,5,6\}$. For an arbitrary integer $t \geq 6$, set $K = K_{m,t}$, and let G be a subgraph of K such that $G = K_{1,t}$. Therefore, we have $\overline{G} = K_{m-1,t}$. Hence, one concludes that neither $K_{2,2} \subseteq G$ nor $K_{6,6} \subseteq \overline{G}$. Therefore, for each $m \in \{2,3,4,5,6\}$, the number $BR_m(K_{2,2},K_{6,6})$ does not exist. □

In the next result, we determine the value of $BR_m(K_{2,2},K_{6,6})$ for $m = 7$.

Theorem 2.2. $BR_7(K_{2,2},K_{6,6}) = 57$.

Proof. Suppose that $X = \{x_1,\dots,x_7\}$ and $Y = \{y_1,y_2,\dots,y_{56}\}$ are the partite sets of $K = K_{7,56}$. Suppose that $G \subseteq K$ such that $N_G(x_i) = Y_i$ satisfying the following properties:

(A1): $Y_1 = \{y_1,y_2,\dots,y_{11}\}$,

(A2): $Y_2 = \{y_1,y_{12},y_{13},\dots,y_{21}\}$,

(A3): $Y_3 = \{y_2,y_{12},y_{22},y_{23},\dots,y_{30}\}$,

(A4): $Y_4 = \{y_3,y_{13},y_{22},y_{31},\dots,y_{38}\}$,

(A5): $Y_5 = \{y_4,y_{14},y_{23},y_{31},y_{39},y_{40},\dots,y_{45}\}$,

(A6): $Y_6 = \{y_5,y_{15},y_{24},y_{32},y_{39},y_{46},y_{47},\dots,y_{51}\}$,

(A7): $Y_7 = \{y_6,y_{16},y_{25},y_{33},y_{40},y_{46},y_{52},\dots,y_{56}\}$.

For every pair $i, j \in [7]$, by using **(Ai)** and **(Aj)**, one has $|N_G(x_i) \cap N_G(x_j)| = 1$ and $| \cup_{j=1, j\neq i}^{j=7} N_G(x_j)| = 51$. Therefore, $K_{2,2} \nsubseteq G$ and $K_{6,6} \nsubseteq \overline{G}[X \setminus \{x_i\}, Y]$ for each $i \in [7]$. Hence, $BR_7(K_{2,2}, K_{6,6}) \geq 57$.

Now, suppose that $X = \{x_1, \ldots, x_7\}$ and $Y = \{y_1, \ldots, y_{57}\}$ are the partite sets of $K = K_{7,57}$. Suppose that G is a subgraph of K such that $K_{2,2} \nsubseteq G$. Consider $\Delta = \Delta(G_X)$. One can suppose that $\Delta \in \{9, 10, 11\}$. Otherwise, if $\Delta \geq 12$, then the theorem holds by Lemma 2.1. Also, for the case when $\Delta \leq 8$, it is clear that $K_{6,6} \subseteq \overline{G}$. Now, we have the following claims.

Claim 2.1. *If $\Delta = 9$, then $K_{6,6} \subseteq \overline{G}$.*

Proof of Claim 2.1. Without loss of generality, let $N_G(x_1) = Y_1 = \{y_1, \ldots, y_9\}$. As $K_{2,2} \nsubseteq G$, we have $|N_G(x_i) \cap N_G(x_j)| \leq 1$ for every pair $i, j \in [7]$. Also, it can be checked that $|N_G(x) \cap Y_1| = 1$ for at least four members of $X \setminus \{x_1\}$, otherwise $K_{6,6} \subseteq \overline{G}[X, Y_1]$. Without loss of generality, let $|N_G(x) \cap Y_1| = 1$ for each $x_i \in \{x_2, x_3, x_4, x_5\}$. Therefore, as $\Delta = 9$ and $|N_G(x) \cap Y_1| = 1$ for each $x_i \in \{x_2, x_3, x_4, x_5\}$, one gets

$$\left| \bigcup_{j=1}^{j=5} N_G(x_j) \right| \leq 41.$$

Hence, as $N_G(x_6) \leq \Delta = 9$, one has $| \cup_{j=1}^{j=6} N_G(x_j)| \leq 50$. Therefore, as $|Y| = 57$, one has $K_{6,6} \subseteq \overline{G}[X \setminus \{x_7\}, Y]$. □

Claim 2.2. *If $\Delta = 10$, then $K_{6,6} \subseteq \overline{G}$.*

Proof of Claim 2.2. Without loss of generality, let $N_G(x_1) = Y_1 = \{y_1, \ldots, y_{10}\}$. As $K_{2,2} \nsubseteq G$, we have $|N_G(x_i) \cap N_G(x_j)| \leq 1$ for each $i, j \in [7]$. Also, it can be checked that $|N_G(x) \cap Y_1| = 1$ for at least five members of $X \setminus \{x_1\}$, otherwise $K_{6,6} \subseteq \overline{G}[X, Y_1]$. Without loss of generality, let $|N_G(x) \cap Y_1| = 1$ for each $x \in X' = \{x_2, x_3, x_4, x_5, x_6\}$. Therefore, as $\Delta = 10$ and $|N_G(x) \cap Y_1| = 1$ for each $x \in X'$, then one can check that $| \cup_{j=1}^{j=6} N_G(x_j)| \leq 55$. If $|N_G(x) \cap (Y \setminus Y_1)| \leq 8$ for at least four members of X', then one can check that $| \cup_{j=1}^{j=6} N_G(x_j)| \leq 51$. Therefore, as $|Y| = 57$, we have $K_{6,6} \subseteq \overline{G}[X \setminus \{x_7\}, Y]$, that is the claim is true. Now, suppose that $|N_G(x) \cap (Y \setminus Y_1)| = 9$ for at least two members of X'. Without loss of generality, let $|N_G(x) \cap (Y \setminus Y_1)| = 9$ for each $x \in \{x_2, x_3\}$. For $i = 2, 3$, we may suppose that $N_G(x_i) \cap (Y \setminus Y_1) = Y_i$. As $|Y_i| = 9$, if $|Y_2 \cap Y_3| = 0$, then it is easy to check that $|N_G(x) \cap Y_i| = 1$ for each $x \in \{x_4, x_5, x_6\}$. Otherwise, one can say that $K_{6,6} \subseteq \overline{G}[X \setminus \{x_i\}, Y_i]$ for some $i \in \{2, 3\}$. So, we have $| \cup_{j=1}^{j=6} N_G(x_j)| \leq 10 + 9 + 9 + 7 + 7 + 7 = 49$, hence the proof is the same. So, let $|Y_2 \cap Y_3| = 1$. Therefore, we have $| \cup_{j=1}^{j=3} N_G(x_j)| = 27$. If $|N_G(x) \cap Y_i| = 1$ for each $x \in \{x_4, x_5, x_6\}$, then the proof is the same. Now, for each $j = 4, 5, 6$, let $|N_G(x_j) \cap Y_i| = 1$ for at least one $i \in \{2, 3\}$. Therefore, we have $|N_G(x) \cap (Y_1 \cup Y_2 \cup Y_3)| \geq 2$, that is $|N_G(x) \cap (Y \setminus Y_1 \cup Y_2 \cup Y_3)| \leq 8$ for each $x \in \{x_4, x_5, x_6\}$. So, we have $| \cup_{j=1}^{j=6} N_G(x_j)| \leq 10 + 9 + 8 + 8 + 8 + 8 = 51$. Therefore, we have $K_{6,6} \subseteq \overline{G}[X \setminus \{x_7\}, Y]$. Now, let there is a member of $\{x_4, x_5, x_6\}$ say x, such that $|N_G(x) \cap Y_i| = 0$ for each $i = 2, 3$. Without loss of generality, let $x = x_4$. As $|Y_3| = 9$, $|Y_2 \cap Y_3| = 1$, and $|N_G(x_4) \cap Y_i| = 0$, it is easy to check that $|N_G(x) \cap Y_i| = 1$ for each $x \in \{x_5, x_6\}$ and each $i \in \{2, 3\}$. Otherwise, one can say that $K_{6,6} \subseteq \overline{G}[X \setminus \{x_i\}, Y_i]$ for some $i = 2, 3$. So, we have

$$\left| \bigcup_{j=1, j\neq 4}^{j=6} N_G(x_j) \right| \leq 10 + 9 + 8 + 7 + 7 = 41.$$

Therefore, as $\Delta = 10$, $|N_G(x_4) \cap Y_1| = 1$, we have $| \cup_{j=1}^{j=6} N_G(x_j)| \leq 50$, that is $K_{6,6} \subseteq \overline{G}[X \setminus \{x_7\}, Y]$. □

By Claims 2.1 and 2.2, we can assume that $\Delta = 11$. Without loss of generality, let $|N_G(x_1)| = 11$ and $Y_1 = N_G(x_1) = \{y_1, \ldots, y_{11}\}$. Next, we have the following claim.

Claim 2.3. *If either $|N_G(x_i) \cap Y_1| = 0$ or $N_G(x_i) \cap Y_1 = N_G(x_j) \cap Y_1$ for some $i, j \in \{2, \ldots, 7\}$, then $K_{6,6} \subseteq \overline{G}$.*

Proof of Claim 2.3. As $K_{2,2} \nsubseteq G$, so $|N_G(x_i) \cap Y_1| \leq 1$ for each i. Now, let $|N_G(x_2) \cap Y_1| = 0$. Therefore, it is clear that $K_{6,6} \subseteq \overline{G}[X \setminus \{x_1\}, Y_1]$. Also, without loss of generality, let $N_G(x_2) \cap N_G(x_3) \cap Y_1 = \{y\}$, then as $|X| = 7$ and $|Y_1| = 11$, we have $K_{6,6} \subseteq \overline{G}[X \setminus \{x_1\}, Y_1 \setminus \{y\}]$. □

By Claim 2.3, it is clear that $K_{6,5} \subseteq \overline{G}[X \setminus \{x_1\}, Y_1]$. If $|N_G(x)| = 11$ for each $x \in X$, then by Claim 2.3 we have $| \cup_{j=1}^{j=7} N_G(x_j)| = 56$, that is there exists a member of Y say y_{57}, such that $|N_G(y_{57})| = 0$. Therefore, we have $K_{6,6} \subseteq \overline{G}[X \setminus \{x_1\}, Y_1 \cup \{y_{57}\}]$ and the proof is complete. For the case that $|N_G(x)| = \Delta = 11$ for at least three members of X, the proof is the same. For example, assume that $|N_G(x)| = 11$ for three members of X, and without loss of generality, suppose that $|N_G(x_i)| = 11$ for $i = 1, 2, 3$. Now, by Claim 2.3 we have $| \cup_{j=1}^{j=3} N_G(x_j)| = 30$. Also, by Claim 2.3 we have $| \cup_{j=1}^{j=6} N_G(x_j)| \leq 51$, so, the proof is the same. Hence, we may assume that $|N_G(x)| = 11$ for at most two members of X. Assume that $|N_G(x)| = 11$ for two members of X (for other case the proof is the same). Without loss of generality, suppose that $|N_G(x_i)| = 11$ for $i = 1, 2$. Now, by Claim 2.3 we have $|\cup_{j=1}^{j=2} N_G(x_j)| = 21$. One can say that there exist at least four members of $X \setminus \{x_1, x_2\}$ say $\{x_3, x_4, x_5, x_6\}$ such that for each $i = 3, 4, 5, 6$ we have $|N_G(x_i)| = 10$, otherwise $|\cup_{j=1}^{j=6} N_G(x_j)| \leq 51$, and the proof is the same. Hence, for some $i \in \{3, 4, 5, 6\}$ say $i = 3$, it is clear that there exist at least two members of $\{x_4, x_5, x_6\}$ say x_4, x_5 such that $|N_G(x_3) \cap N_G(x_j)| = 1$, for $j = 4, 5$, otherwise we have $K_{6,6} \subseteq \overline{G}[X \setminus \{x_3\}, Y_3]$ and the proof is complete. So, by Claim 2.3 one can say that $|\cup_{j=1}^{j=6} N_G(x_j)| \leq 51$, hence the proof is the same. So, the theorem holds. □

In the next result, we determine the value of $BR_8(K_{2,2}, K_{6,6})$.

Theorem 2.3. $BR_8(K_{2,2}, K_{6,6}) = 45$.

Proof. Let $X = \{x_1, \ldots, x_8\}$ and $Y = \{y_1, \ldots, y_{44}\}$ be the partite sets of $K = K_{8,44}$. Suppose that G be a subgraph of K such that for each $i \in [8]$, $N_G(x_i) = Y_i$ with the following properties:

(B1): $Y_1 = \{y_1, \ldots, y_9\}$,

(B2): $Y_2 = \{y_1, y_{10}, y_{11}, \ldots, y_{17}\}$,

(B3): $Y_3 = \{y_2, y_{10}, y_{18}, y_{19}, \ldots, y_{24}\}$,

(B4): $Y_4 = \{y_3, y_{11}, y_{18}, y_{25}, y_{26}, y_{27}, y_{28}, y_{29}, y_{30}\}$,

(B5): $Y_5 = \{y_4, y_{12}, y_{19}, y_{25}, y_{31}, y_{32}, y_{33}, y_{34}, y_{35}\}$,

(B6): $Y_6 = \{y_5, y_{13}, y_{20}, y_{26}, y_{31}, y_{36}, y_{37}, y_{38}, y_{39}\}$,

(B7): $Y_7 = \{y_6, y_{14}, y_{21}, y_{27}, y_{32}, y_{36}, y_{40}, y_{41}, y_{42}\}$,

(B8): $Y_8 = \{y_7, y_{15}, y_{22}, y_{28}, y_{33}, y_{37}, y_{40}, y_{43}, y_{44}\}$.

By considering **(Bi)** and **(Bj)**, it can be said that:

(C1): $|N_G(x_i) \cap N_G(x_j)| = 1$, for each $i, j \in [8]$,

(C2): $|\cup_{i=1}^{i=6} N_G(x_{j_i})| = 39$, for each $j_1, \ldots, j_6 \in [8]$.

Therefore, by **(C1)**, we have $K_{2,2} \nsubseteq G$. Also, by **(C2)**, one can check that $K_{6,6} \nsubseteq \overline{G}$, which means that $K_{8,44} \to (K_{2,2}, K_{6,6})$. Therefore, the lower bound holds.

Now, we prove the upper bound. Suppose that $X = \{x_1, \ldots, x_8\}$ and $Y = \{y_1, \ldots, y_{45}\}$ are the partition sets of $K = K_{8,45}$. Let G be a subgraph of K such that $K_{2,2} \nsubseteq G$. We show that $K_{6,6} \subseteq \overline{G}$. Consider $\Delta = \Delta(G_X)$. As $K_{2,2} \nsubseteq G$, by Lemma 2.1, one can assume that $\Delta \leq 11$. We have the following claim.

Claim 2.4. *If* $\Delta = 11$*, then* $K_{6,6} \subseteq \overline{G}$.

Proof of Claim 2.4. Without loss of generality, let $|N_G(x_1) = Y_1| = 11$. Since $K_{2,2} \nsubseteq G$, $|X| = 8$, and $|Y_1| = 11$, then for each pair $i, j \in \{2, \ldots, 8\}$ one can suppose that $|N_G(x_i) \cap Y_1| = 1$, and that x_i and x_j have a different neighborhood in Y_1. Otherwise, in any case, it is clear that $K_{6,6} \subseteq \overline{G}[X, Y_1]$. Therefore, for each $x \neq x_1$, we have $K_{6,5} \subseteq \overline{G}[X \setminus \{x_1, x\}, Y_1]$. If there is a member of $Y \setminus Y_1$ say y, such that $|N_{\overline{G}}(y) \cap (X \setminus \{x_1\})| \geq 6$, then $K_{6,6} \subseteq \overline{G}[X \setminus \{x_1\}, Y_1 \cup \{y\}]$. Hence, let $|N_G(y) \cap (X \setminus \{x_1\})| \geq 2$ for each $y \in Y \setminus Y_1$. Therefore, $|E(G[X \setminus \{x_1\}, Y \setminus Y_1])| \geq 34 \times 2 = 68$. Hence, by pigeon-hole principle, there is at least one member of $X \setminus \{x_1\}$ say x_2, such that $|N_G(x_2) \cap (Y \setminus Y_1)| \geq 10$. Set $N_G(x_2) \cap (Y \setminus Y_1) = Y_2$. Now, as $K_{2,2} \nsubseteq G$, then $|N_G(x_i) \cap Y_2| \leq 1$ for each $i \in \{3, \ldots, 8\}$. Therefore, since $|Y_2| \geq 10$, one can check that $K_{6,1} \subseteq \overline{G}[X \setminus \{x_1, x_2\}, Y_2]$. Hence, as $K_{6,5} \subseteq \overline{G}[X \setminus \{x_1, x_2\}, Y_1]$, we have $K_{6,6} \subseteq \overline{G}[X \setminus \{x_1, x_2\}, Y_1 \cup Y_2]$. So, the claim holds. □

Therefore, by Claim 2.4, one can assume that $\Delta \leq 10$. Now, we have the following claim.

Claim 2.5. *If there exist* $V \subseteq X$*, such that* $|V| = 5$ *and* $|\cup_{x \in V} N_G(x)| \leq 35$*, then we have* $K_{6,6} \subseteq \overline{G}$.

Proof of Claim 2.5. Without loss of generality, assume that $V = \{x_1, x_2, x_3, x_4, x_5\}$ and let $Y' = \cup_{x \in V} N_G(x)$ where $|Y'| \leq 35$. If $|Y'| \leq 29$, then as $\Delta \leq 10$, it is easy to say that $|Y' \cup N_G(x)| \leq 39$ for each $x \in X \setminus V$, that is $K_{6,6} \subseteq \overline{G}[V \cup \{x\}, Y]$, hence the claim holds. So, suppose that $|Y'| \in \{30, \ldots, 34, 35\}$. Assume that $|Y'| = 35$. Hence, for each $x \in \{x_6, x_7, x_8\}$, one can assume that $|N_G(x) \cap (Y \setminus Y')| \geq 5$, otherwise we have $|Y' \cup N_G(x)| \leq 39$ for some $x \in X \setminus V$, that is $K_{6,6} \subseteq \overline{G}$. Therefore, as $|Y \setminus \cup_{i=1}^{i=5} N_G(x_i)| = 10$ and $|\{x_6, x_7, x_8\}| = 3$, it is easy to say that $K_{2,2} \subseteq G[\{x_6, x_7, x_8\}, Y \setminus \cup_{i=1}^{i=5} N_G(x_i)]$, a contradiction. Now, let $|Y'| = 34$. Hence, for each $x \in \{x_6, x_7, x_8\}$, one can assume that $|N_G(x) \cap (Y \setminus Y')| \geq 6$, otherwise $|Y' \cup N_G(x)| \leq 39$

for some $x \in X \setminus V$, that is we have $K_{6,6} \subseteq \overline{G}$. Therefore, as $|Y \setminus \cup_{i=1}^{i=5} N_G(x_i)| = 11$ and $|\{x_6, x_7, x_8\}| = 3$, it is easy to say that $K_{2,2} \subseteq G[\{x_6, x_7, x_8\}, Y \setminus \cup_{i=1}^{i=5} N_G(x_i)]$, a contradiction again. For the case that $|Y'| \in \{30, 31, 32, 33\}$, the proof is the same. Hence, the claim holds. □

If $\Delta \le 6$, then it is clear that $K_{6,6} \subseteq \overline{G}$. Next, we suppose that $\Delta = 7$, and without loss of generality, let $|N_G(x_1) = Y_1| = 7$. One can assume that $|N_G(x) \cap Y_1| = 1$ for at least three members of $X \setminus \{x_1\}$. Otherwise, as $|Y_1| = |X \setminus \{x_1\}| = 7$ and $|N_G(x) \cap Y_1| \le 1$ for each member of $X \setminus \{x_1\}$. Then it is easy to say that $K_{6,6} \subseteq \overline{G}[X, Y_1]$. Hence, without loss of generality, assume that $|N_G(x) \cap Y_1| = 1$ for each members of $\{x_2, x_3, x_4\}$. Hence as $\Delta = 7$, one can check that $| \cup_{i=1}^{i=4} N_G(x_i)| = 25$. Therefore, we have $| \cup_{i=1}^{i=5} N_G(x_i)| \le 32$, and by Claim 2.5, we have $K_{6,6} \subseteq \overline{G}$. So, we may suppose that $\Delta \in \{8, 9, 10\}$. Now, we consider the following cases.

Case 1. $\Delta = 8$. Without loss of generality, suppose that $\Delta = |N_G(x_1) = Y_1|$. As $K_{2,2} \nsubseteq G$, one can suppose that there exist at least four members of $X \setminus \{x_1\}$ say $X' = \{x_2, \dots, x_5\}$, such that $|N_G(x_i) \cap Y_1| = 1$ and $N_G(x_i) \cap Y_1 \neq N_G(x_j) \cap Y_1$ for each $i, j \in \{2, \dots, 5\}$. Otherwise, one can check that $K_{7,5} \subseteq \overline{G}[X \setminus \{x_1\}, Y_1]$. Therefore, for each $y \in Y \setminus Y_1$, one can suppose that $|N_G(y) \cap (X \setminus \{x_1\})| \ge 2$. Otherwise $K_{6,6} \subseteq \overline{G}$. So, as $|Y \setminus Y_1| = 37$, we have $|E(G[X \setminus \{x_1\}, Y \setminus Y_1])| \ge 74$. Therefore, by pigeon-hole principle there is at least one member of $X \setminus \{x_1\}$ say x, such that $|N_G(x)| \ge 10$, a contradiction. Now, without loss of generality, suppose that $Y_1 = \{y_1, \dots, y_8\}$ and $x_i y_{i-1} \in E(G)$ for each $i = 2, 3, 4, 5$. Set $Y' = \cup_{i=1}^{i=5} N_G(x_i)$. Hence, it is easy to say that $|Y'| \le 36$. If $|Y'| \le 35$, then the proof is complete by Claim 2.5. So, let $|Y'| = 36$. That is $\Delta = 8 = |N_G(x_i)|$ for each $i \in [5]$ and $|N_G(x_i) \cap N_G(x_j)| = 0$ for each $i, j \in \{2, 3, 4, 5\}$. Now consider $i = 2, 3$, as $K_{2,2} \nsubseteq G$, we have $|N_G(x_j) \cap (N_G(x_i) \setminus \{y_{i-1}\})| \le 1$ for each $j \in \{6, 7, 8\}$. Hence one can say that $K_{6,6} \subseteq \overline{G}[X \setminus \{x_2, x_3\}, N_G(x_2) \cup N_G(x_3) \setminus \{y_1, y_2\}]$.

Case 2. $\Delta = 9$. Without loss of generality, suppose that $N_G(x_1) = Y_1 = \{y_1, y_2, \dots, y_9\}$. Now, set A as follow:

$$A = \{x \in X, \ \ |N_G(x)| = \Delta = 9\}.$$

As $K_{2,2} \nsubseteq G$, we have $|N_G(x_i) \cap Y_1| \le 1$. Hence, one can say that $K_{6,3} \subseteq \overline{G}[X \setminus \{x_1, x\}, Y_1]$ for each $x \in \{x_2, \dots, x_8\}$. Therefore, by considering the members of A, one can check that the following claim is true.

Claim 2.6. *If* $|N_G(x) \cap N_G(x')| = 0$ *for some* $x, x' \in A$*, then* $K_{6,6} \subseteq \overline{G}$.

Next, by using Claim 2.6, we prove the following claim.

Claim 2.7. *If* $|N_G(x) \cap N_G(x') \cap N_G(x'')| = 1$ *for some* $x, x', x'' \in A$*, then* $K_{6,6} \subseteq \overline{G}$.

Proof of Claim 2.7. Without loss of generality, assume that $x_1, x_2, x_3 \in A$, $\{y_1\} = Y_1 \cap Y_2 \cap Y_3$, where $Y_i = N_G(x_i)$ for $i = 1, 2, 3$. Since $K_{2,2} \nsubseteq G$, for each $i \in [3]$ and each $x \in X \setminus \{x_1, x_2, x_3\}$ we have $|N_G(x) \cap Y_i| \le 1$. Therefore, as $|Y_i| = 9$, and $|N_G(x) \cap Y_i| \le 1$ for each $x \in X \setminus \{x_1, x_2, x_3\}$, it is easy to say that $K_{5,3} \subseteq \overline{G}[X \setminus \{x_1, x_2, x_3\}, Y_i \setminus \{y_1\}]$ for each $i \in [3]$. Therefore, we have $K_{5,6} \subseteq \overline{G}[X \setminus \{x_1, x_2, x_3\}, Y_1 \cup Y_2 \setminus \{y_1\}]$. So, as $y_1 \in Y_1 \cap Y_2 \cap Y_3$ and $K_{2,2} \nsubseteq G$, then $N_G(x_3) \cap (Y_1 \cup Y_2 \setminus \{y_1\}) = \emptyset$. Therefore, $K_{6,6} \subseteq \overline{G}[X \setminus \{x_1, x_2\}, Y_1 \cup Y_2 \setminus \{y_1\}]$. Hence, the claim holds. □

Consider $|A|$. First suppose that $|A| \ge 5$, and without loss of generality, assume that $\{x_1, x_2, x_3, x_4, x_5\} \subseteq A$. Therefore, by Claims 2.6 and 2.7, it can be said that $| \cup_{i=1}^{i=5} N_G(x_i)| = 35$. Hence, by Claim 2.5 the proof is complete. So, we may assume that $|A| \le 4$. Now, we verify the following two claims.

Claim 2.8. *If* $|A| = 4$*, then* $K_{6,6} \subseteq \overline{G}$.

Proof of Claim 2.8. Without loss of generality, assume that $A = \{x_1, x_2, x_3, x_4\}$. Therefore, by Claims 2.6 and 2.7, one can check that $| \cup_{i=1}^{i=4} N_G(x_i)| = 30$. Set $Y' = \cup_{i=1}^{i=4} N_G(x_i)$. If there is a member of $X \setminus A$ say x, so that $3 \le |N_G(x) \cap Y'|$, then

$$\left| \bigcup_{i=1}^{i=4} N_G(x_i) \cup N_G(x) \right| \le 35$$

and the proof is complete by Claim 2.5.

Hence, we may suppose that $|N_G(x) \cap Y'| \le 2$ for each $x \in X \setminus A$. So as $|Y'| = 30$, one can check that $K_{4,22} \subseteq \overline{G}[X \setminus A, Y']$. Without loss of generality, let $K_{4,22} \cong \overline{G}[X \setminus A, Y'']$, where $Y'' \subseteq Y'$ and $|Y''| = 22$. Therefore, it is easy to check that there is at least two members of A say $x_{i_1} x_{i_2}$, such that $|(N_G(x_{i_1}) \cup N_G(x_{i_1})) \cap Y''| \le 16$. Without loss of generality, let $i_1 = 1, i_2 = 2$. So, we have $K_{6,6} \subseteq \overline{G}[X \setminus \{x_3, x_4\}, Y'']$. Hence, the claim holds. □

Claim 2.9. *If* $|A| = 3$*, then* $K_{6,6} \subseteq \overline{G}$*.*

Proof of Claim 2.9. Without loss of generality, suppose that $A = \{x_1, x_2, x_3\}$. Therefore, by Claims 2.6 and 2.7, it can be said that $|\cup_{i=1}^{i=3} N_G(x_i)| = 24$. Set $Y' = \cup_{i=1}^{i=3} N_G(x_i)$. Suppose that there exists a vertex of $X \setminus A$ say x, such that $|N_G(x) \cap Y'| \geq 2$, then we have $|\cup_{i=1}^{i=4} N_G(x_i) \cup N_G(x)| \leq 30$. Without loss of generality, assume that $x = x_4$. If $|\cup_{i=1}^{i=4} N_G(x_i)| \leq 27$, then the proof is complete by Claim 2.5. So, suppose that $28 \leq |\cup_{i=1}^{i=4} N_G(x_i)| \leq 30$. Let $|\cup_{i=1}^{i=4} N_G(x_i)| = 30$. Set $X' = \{x_5, x_6, x_7, x_8\}$. In this case, one can suppose that $|N_G(x) \cap (Y \setminus \cup_{i=1}^{i=4} N_G(x_i))| \geq 6$ for each $x \in X'$. Otherwise, the proof is complete by Claim 2.5. Therefore, as $|X'| = 4$ and $|Y \setminus \cup_{i=1}^{i=4} N_G(x_i))| = 15$, then one can check that $K_{2,2} \subseteq G$, a contradiction. For the case that $|\cup_{i=1}^{i=4} N_G(x_i)| = 28, 29$, the proof is the same.

So, let $|N_G(x) \cap Y'| \leq 1$ for each $x \in X \setminus A$. Therefore, it is clear that $K_{5,19} \subseteq \overline{G}[X \setminus A, Y']$. Without loss of generality, let $K_{5,19} \cong \overline{G}[X \setminus A, Y'']$, where $Y'' \subseteq Y'$ and $|Y''| = 19$. Therefore, one can say that there is at least one member of A say y, so that $|N_G(y) \cap Y''| \leq 9$. Without loss of generality, let $y = y_1$. So, $K_{6,6} \subseteq \overline{G}[X \setminus \{x_2, x_3\}, Y'']$. Hence, the claim holds. □

Hence, by Claims 2.8 and 2.9, one can suppose that $|A| \leq 2$. First, assume that $|A| = 2$ and without loss of generality, suppose that $A = \{x_1, x_2\}$. By Claim 2.7, we have $|\cup_{i=1}^{i=2} N_G(x_i)| = 17$. For $i = 1, 2$, set $Y_i = N_G(x_i)$. By Claim 2.7, without loss of generality, let $y_1 \in Y_1 \cap Y_2$. Set $X' = X \setminus A$. Suppose that there is at least two vertices of X' say x_3, x_4, such that $|N_G(x_j) \cap (Y_i \setminus \{y_1\})| = 0$ for at least one $i \in [2]$ and $j = 3, 4$. Without loss of generality, let $|N_G(x_j) \cap (Y_1 \setminus \{y_1\})| = 0$, therefore as $|N_G(x_j) \cap (Y_2 \setminus \{y_1\})| \leq 1$ and $|Y_i| = 9$, one can say that $K_{6,4} \subseteq \overline{G}[X \setminus A, Y_1 \setminus \{y_1\}]$. Also, one can check that $K_{6,2} \subseteq \overline{G}[X \setminus A, Y_2 \setminus \{y_1\}]$, hence $K_{6,6} \subseteq \overline{G}[X \setminus A, Y_1 \cup Y_2 \setminus \{y_1\}]$. Therefore for any $i \in \{1, 2\}$, we may suppose that $|N_G(x) \cap (Y_i \setminus \{y_1\})| = 1$ for at least five members of X'. Hence as $|X'| = 6$, it is clear that there is at least three members of X' say $\{x_3, x_4, x_5\}$, so that for any $x \in \{x_3, x_4, x_5\}$, we have $|N_G(x) \cap (Y_1 \cup Y_2 \setminus \{y_1\})| = 2$. Therefore, as $|N_G(x)| \leq 8$ for each $i = 3, 4, 5$, one can check that $|\cup_{i=1}^{i=5} N_G(x_i)| \leq 17 + 18 = 35$. Hence, the proof is complete by Claim 2.5.

Now, let $|A| = 1$ and without loss of generality, let $A = \{x_1\}$. In this case, one can say that there exist at least five vertices of $X \setminus \{x_1\}$ say $X'' = \{x_2, x_3, x_4, x_5, x_6\}$, such that $|N_G(x) \cap Y_1| = 1$ for each $x \in X''$. Otherwise, as $|Y_1| = 9$ and $|N_G(x) \cap Y_1| \leq 1$ for each $x \in X \setminus \{x_1\}$, then one can say that $K_{6,6} \cong \overline{G}[X \setminus \{x_1\}, Y_1]$. Therefore, there is at least one vertex of X'' say x_2, so that $|N_G(x)| = 8$. Otherwise, we have $|\cup_{i=2}^{i=6} N_G(x_i)| \leq 35$ and the proof is complete by Claim 2.5. Without loss of generality, assume that $Y_2 = N_G(x_2) \cap (Y \setminus Y_1)$ and $|Y_2| = 7$. Therefore, one can say that there is at least two vertices of $X'' \setminus \{x_2\}$ say $\{x_3, x_4\}$, so that for each $x \in \{x_3, x_4\}$, we have $|N_G(x) \cap Y_2| = 1$, otherwise as $|Y_2| = 7$, then one can say that $K_{6,6} \cong \overline{G}[X \setminus \{x_2\}, Y_2]$. Now, one can check that $|\cup_{i=1}^{i=5} N_G(x_i)| \leq 9 + 7 + 6 + 6 + 7 = 35$, and the proof is complete by Claim 2.5.

Case 3. $\Delta = 10$. Without loss of generality, let $N_G(x_1) = Y_1 = \{y_1, \ldots, y_{10}\}$. Therefore by $K_{2,2} \nsubseteq G$ it is clear to say that $K_{6,4} \subseteq \overline{G}[X \setminus \{x_1, x\}, Y_1]$ for each $x \in X \setminus \{x_1\}$. Let there is a member of $X \setminus \{x_1\}$ say x_2, so that $|N_G(x_2) \cap (Y \setminus Y_1) = Y_2| = 8$. Therefore, as $K_{2,2} \nsubseteq G$, we have $|N_G(x_i) \cap Y_2| \leq 1$. Hence, since $|Y_2| = 8$ and $|X \setminus \{x_1, x_2\}| = 6$, one can say that $K_{6,2} \subseteq \overline{G}[X \setminus \{x_1, x_2\}, Y_2]$. So, $K_{6,6} \subseteq \overline{G}[X \setminus \{x_1, x_2\}, Y_1 \cup Y_2]$. Now, one can suppose that $|N_G(x) \cap (Y \setminus Y_1)| \leq 7$ for any member of $X \setminus \{x_1\}$. Hence we have the following claim:

Claim 2.10. *Suppose that* $|Y' = N_G(x) \cap (Y \setminus Y_1)| = 7$*. If either* $|N_G(x') \cap Y'| = 0$ *for one* $x' \in X \setminus \{x_1, x\}$*, or* $|N_G(x') \cap N_G(x'') \cap Y'| = 1$ *for some* $x', x'' \in X \setminus \{x_1, x\}$*, then* $K_{6,6} \subseteq \overline{G}$*.*

Proof of Claim 2.10. Without loss of generality, let $|Y' = N_G(x_2) \cap (Y \setminus Y_1)| = 7$. Also, without loss of generality, let $|N_G(x_3) \cap Y'| = 0$. Therefore, Since $K_{2,2} \nsubseteq G$, so $|N_G(x_i) \cap (Y_1 \cup Y')| \leq 2$ for each $i \in \{3, 4, 5, 6, 7, 8\}$. As $|Y_1 \cup Y'| = 17$, $|N_G(x_3) \cap Y'| = 0$, and $|N_G(x_i) \cap (Y_1 \cup Y')| \leq 2$, one can say that $|\cup_{i=3}^{i=8} (N_G(x_i) \cap (Y_1 \cup Y'))| \leq 11$, which means that $K_{6,6} \subseteq \overline{G}[X \setminus \{x_1, x_2\}, Y_1 \cup Y']$. For the case that $|N_G(x') \cap N_G(x'') \cap Y'| = 1$ for some $x', x'' \in X \setminus \{x_1, x\}$, the proof is the same. Hence, the claim holds. □

Set M as follow:

$$M = \{x \in X \setminus \{x_1\}, \ \ |N_G(x) \cap (Y \setminus Y_1)| = 7\}.$$

By considering M, we have:

Claim 2.11. *If* $|M| \neq 0$*, then we have* $K_{6,6} \subseteq \overline{G}$*.*

Proof of Claim 2.11. Without loss of generality, let $x_2 \in M$, and $N_G(x_2) \cap (Y \setminus Y_1) = Y_2 = \{y_{11}, y_{12} \ldots, y_{17}\}$. If $|M| \geq 5$, then by Claim 2.10, it can be said that $|\cup_{x_j \in M'} N_G(x_j)| \leq 35$, where $M' \subseteq M$ and $|M'| = 5$. Hence, the proof is complete by Claim 2.5. Now, assume that $|M| = i$, and without loss of generality, suppose that $M = \{x_2, x_3, \ldots, x_{i+1}\}$, where $i \in \{1, 2, 3, 4\}$. If $|M| \leq 2$, then it can be said that $|\cup_{x_j \in M''} N_G(x_j)| \leq 35$, where $M'' \subseteq X \setminus M$ and $|M''| = 5$. Hence, the proof is complete by Claim 2.5. Now assume that $|M| = i$, where $i \in \{3, 4\}$.

By Claim 2.10, for the case that $i = 4$, we have $| \cup_{j=1}^{j=5} N_G(x_j)| = 10 + 7 + 6 + 5 + 4 = 32$. Hence, the proof is complete by Claim 2.5. Also for the case that $i = 3$, by Claim 2.10, we have $| \cup_{j=1}^{j=4} N_G(x_j)| = 10 + 7 + 6 + 5 = 28$. Therefore, we have $| \cup_{j=1}^{j=5} N_G(x_j)| \leq 34$. Hence, the proof is complete by Claim 2.5.

□

Now, by Claim 2.11, let $|M| = 0$, that is $|N_G(x) \cap (Y \setminus Y_1)| \leq 6$ for each $x \in X \setminus \{x_1\}$. In this case, the proof is complete by Claim 2.5.

Hence, by Cases 1, 2, and 3, the upper bound holds. Consequently, it holds that $BR_8(K_{2,2}, K_{6,6}) = 45$, which completes the proof of Theorem 2.3. □

Proof of Theorem 1.4. By combining Theorems 2.1, 2.2, and 2.3, one gets Theorem 1.4. □

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