Notes on several integral inequalities of Hermite–Hadamard type for \(s\)-geometrically convex functions

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Abstract


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\section{Preliminaries}

In [2, Definition 1.9], the concept of \(s\)-geometrically convex functions was introduced as follows.

**Definition 1.1** (see [2, Definition 1.9]). For some \(s \in (0,1)\), a function \(f : I \subseteq \mathbb{R}_+ = (0, \infty) \rightarrow \mathbb{R}_+\) is said to be an \(s\)-geometrically convex function if the inequality

\[
 f(x^s y^{1-s}) \leq [f(x)]^s [f(y)]^{(1-s)s}
\]

holds for all \(x, y \in I\) and \(\lambda \in [0,1]\).

The following two integral identities were established in the papers [2, 3].

**Lemma 1.1** (see [2, Lemma 2.1] and [3, Lemma 2.1]). Let \(f : I \subseteq \mathbb{R} = (-\infty, \infty) \rightarrow \mathbb{R}\) be differentiable on \(I^0\) and \(a, b \in I\) with \(a < b\). If \(f' \in L([a, b])\), then

\[
 f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx = \frac{b-a}{4} \int_0^1 \left[ t f'\left( (1-t)a + t \frac{a+b}{2} \right) + (1-t) f'\left( (1-t)a + t \frac{a+b}{2} + tb \right) \right] \, dt
\]

and

\[
 \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx = \frac{b-a}{4} \int_0^1 \left[ (t-1) f'\left( (1-t)a + t \frac{a+b}{2} \right) + t f'\left( (1-t)a + t \frac{a+b}{2} + tb \right) \right] \, dt.
\]

In this paper, we need also the following lemmas.

**Lemma 1.2.** Let \(s \in (0,1)\) be a constant. If \(f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}_+\) is an \(s\)-geometrically convex function, then \(f(x) \geq 1\) for all \(x \in I\).

**Proof.** For \(x \in I\) and \(\lambda \in (0,1)\), using the geometric convexity of \(f\) on \(I\), we obtain

\[
 f(x) = f\left(x^\lambda x^{1-\lambda}\right) \leq [f(x)]^\lambda [f(x)]^{(1-\lambda)s} = [f(x)]^{\lambda s + (1-\lambda)s}.
\]

Therefore, the inequality \(f(x) \geq 1\) holds for all \(x \in I\).

\section*{Acknowledgments}

The aim of this paper is to correct several errors appeared in the papers [2, 3].

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\(\dagger\) It was customary and appropriate to publish this article in the same journal (that is, Abstract and Applied Analysis) that published the flawed results. Unfortunately, at the time of the submission of this paper, the journal "Abstract and Applied Analysis" had a fee of 1025 US dollars for every accepted manuscript. The current authors were unwilling to bear this publishing cost.
2. Corrections

In this section, we state and prove corrected versions of some erroneous results appeared in [2, 3].

**Theorem 2.1** (Corrected version of [2, Theorem 3.1] and [3, pp. 1–3]). Let \( f : I \subset \mathbb{R}_+ \to \mathbb{R} \) be a differentiable function on \( I^0 \) such that \( f' \in L([a, b]) \) for \( 0 < a < b < \infty \). If \( |f'(x)|^q \) is \( s \)-geometrically convex and monotonically decreasing on \([a, b]\) for \( q \geq 1 \) and \( s \in (0, 1] \), then

\[
|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx| \leq \frac{b-a}{4} \left(\frac{1}{2}\right)^{1-1/q} \left[ |f'(a)| |f'(b)|^{1-s} |g_1(\alpha)|^{1/q} + \left| f'(a)f'(b)\right|^{1-s/2} [g_2(\alpha)]^{1/q}\right]
\]

and

\[
\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left(\frac{1}{2}\right)^{1-1/q} \left[ |f'(a)| |f'(b)|^{1-s} |g_1(\alpha)|^{1/q} + \left| f'(a)f'(b)\right|^{1-s/2} [g_2(\alpha)]^{1/q}\right],
\]

where

\[
\alpha = \left| \frac{f'(b)}{f'(a)} \right|^{s/q}, \quad g_1(\alpha) = \begin{cases} \frac{1}{2}, & \alpha = 1; \\ \frac{\alpha \ln \alpha - \alpha + 1}{\ln^2 \alpha}, & \alpha \neq 1, \end{cases}
\]

and

\[
g_2(\alpha) = \begin{cases} \frac{1}{2}, & \alpha = 1; \\ \frac{\alpha - \ln \alpha - 1}{\ln^2 \alpha}, & \alpha \neq 1. \end{cases}
\]

**Proof.** Since \( |f'|^q \) is \( s \)-geometrically convex and monotonically decreasing on \([a, b]\), using Lemma 1.1 and Hölder’s inequality, we have

\[
|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx| \leq \frac{b-a}{4} \left(\frac{1}{2}\right)^{1-1/q} \left[ \int_0^1 t |f'(a)|^{q(1-t)} |f'(b)|^{q(1+t)} \, dt \right]^{1/q} + \left( \int_0^1 (1-t) \, dt \right)^{1-1/q} \left[ \int_0^1 |f'(a)|^{q(1-t)} |f'(b)|^{q(1+t)} \, dt \right]^{1/q}
\]

\[
\leq \frac{b-a}{4} \left(\frac{1}{2}\right)^{1-1/q} \left\{ \int_0^1 t |f'(a)|^{q(1-t)} |f'(b)|^{q(1+t)} \, dt \right\}^{1/q} + \left( \int_0^1 (1-t) |f'(a)|^{q(1-t)} |f'(b)|^{q(1+t)} \, dt \right\}^{1/q}.
\]

By Lemmas 1.2 and 1.3, we obtain

\[
\int_0^1 t f'(a)^{q((1-t)/2)^2} f'(b)^{q(t/2)^2} \, dt \leq \int_0^1 t f'(a)^{q(s_{(1-t)/2}+1-s)} f'(b)^{q(t/2)+1-s} \, dt = f'(a)^{q(s_{(t/2)+1-s})} f'(b)^{(1-s)q} g_{1}(\alpha)
\]

and

\[
\int_0^1 (1-t) f'(a)^{q((1-t)/2)^2} f'(b)^{q((1+t)/2)^2} \, dt \leq \int_0^1 (1-t) f'(a)^{q(s_{(1-t)/2}+1-s)} f'(b)^{q(s_{(1+t)/2}+1-s}) \, dt = f'(a)^{q(s_{(1-t)/2}+1-s)} f'(b)^{q(s_{(1+t)/2}+1-s)} g_{2}(\alpha).
\]

Combining the inequalities (3), (4), and (5) leads to the inequality (1).

Since \( |f'|^q \) is \( s \)-geometrically convex and monotonically decreasing on \([a, b]\), by Lemma 1.1 and Hölder’s inequality, we acquire

\[
\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left(\frac{1}{2}\right)^{1-1/q} \left[ \int_0^1 (1-t) |f'(a)|^{q(s_{(1-t)/2}+1-s)} f'(b)^{q(t/2)^2} \, dt \right]^{1/q} + \left( \int_0^1 (1-t) |f'(a)|^{q(s_{(1-t)/2}+1-s)} f'(b)^{q((1+t)/2)^2} \, dt \right)\right].
\]
Using Lemmas 1.2 and 1.3, we arrive at

\[ \int_0^1 (1-t)|f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q((1/2)^s} \, dt \leq |f'(a)|^q |f'(b)|^{(1-s)^{q}g_2(\alpha)} \]  

(7)
and

\[ \int_0^1 t|f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+1/2)^s} \, dt \leq |f'(a)|f'(b)^{(1-s/2)^q}g_1(\alpha). \]  

(8)

From the inequalities (6), (7), and (8), the inequality (2) follows readily. Theorem 2.1 is thus proved.

Corollary 2.1 (Corrected version of [2, Corollary 3.2] and [3, p. 3]). Under conditions of Theorem 2.1, we have the following conclusions:

1. When \( q = 1 \), the inequalities

\[ \left| f\left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left[ \frac{q-1}{2q-1} \right] \left[ \int_0^1 |f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q((1/2)^s} \, dt \right]^{1/q} \]

and

\[ \left| f(a) + f(b) \right| - \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{b-a}{4} \left( \frac{q-1}{2q-1} \right)^{1-q} \left[ \int_0^1 |f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+1/2)^s} \, dt \right]^{1/q} \]

hold.

2. When \( s = 1 \), the inequalities

\[ \left| f\left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left( \frac{q-1}{2q-1} \right)^{1-q} \left[ \int_0^1 |f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q((1/2)^s} \, dt \right]^{1/q} \]

and

\[ \left| f(a) + f(b) \right| - \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{b-a}{4} \left( \frac{q-1}{2q-1} \right)^{1-q} \left[ \int_0^1 |f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+1/2)^s} \, dt \right]^{1/q} \]

hold.

Theorem 2.2 (Corrected version of [2, Theorem 3.3] and [3, pp. 3–4]). Let \( f : I \subset \mathbb{R}_+ \rightarrow \mathbb{R} \) be a differentiable function on \( I^o \) such that \( f' \in L([a, b]) \) for \( 0 < a < b < \infty \). If \( |f'(x)|^q \) is \( s \)-geometrically convex and monotonically decreasing on \([a, b]\) for \( q > 1 \) and \( s \in (0, 1) \), then

\[ \left| f\left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left( \frac{q-1}{2q-1} \right)^{1-q} \left( \int_0^1 |f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q((1/2)^s} \, dt \right)^{1/q} \]

and

\[ \left| f(a) + f(b) \right| - \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{b-a}{4} \left( \frac{q-1}{2q-1} \right)^{1-q} \left( \int_0^1 |f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+1/2)^s} \, dt \right)^{1/q} \]

where \( \alpha \) is the same as in Theorem 2.1 and

\[ g_3(\alpha) = \begin{cases} 1, & \alpha = 1; \\ \frac{\alpha - 1}{\ln \alpha}, & \alpha \neq 1. \end{cases} \]

Proof: Since \( |f'|^q \) is \( s \)-geometrically convex and monotonically decreasing on \([a, b]\), by Lemma 1.1 and Hölder’s inequality, we get

\[ \left| f\left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left( \frac{q-1}{2q-1} \right)^{1-q} \left\{ \int_0^1 |f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q((1/2)^s} \, dt \right\}^{1/q} \]

and

\[ \left| f(a) + f(b) \right| - \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{b-a}{4} \left( \frac{q-1}{2q-1} \right)^{1-q} \left\{ \int_0^1 |f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+1/2)^s} \, dt \right\}^{1/q} \]
Furthermore, we have
\[
\int_0^1 |f'(a)|^{q((2-t)/2)} |f'(b)|^{q(t/2)} \, dt \leq |f'(a)|^q |f'(b)|^{(1-s)q} g_3(\alpha) \tag{13}
\]
and
\[
\int_0^1 |f'(a)|^{q((1-t)/2)} |f'(b)|^{q((1+t)/2)} \, dt \leq |f'(a)|^q |f'(b)|^{(1-s/2)q} g_3(\alpha). \tag{14}
\]

By substituting (13) and (14) into (11) and (12), respectively, we obtain the inequalities (9) and (10), respectively.

\begin{proof}

\textbf{Corollary 2.2} (Corrected version of [2, Corollary 3.4] and [3, p. 4]). \textit{Under conditions of Theorem 2.2, when } s = 1, \textit{we have}
\[
\left| f\left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left( \frac{q-1}{2q-1} \right)^{1-1/q} \left[ |f'(a)| + |f'(a)f'(b)|^{1/2} \right] [g_3(\alpha)]^{1/q}
\]
and
\[
\left| f(a) + f(b) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left( \frac{q-1}{2q-1} \right)^{1-1/q} \left[ |f'(a)| + |f'(a)f'(b)|^{1/2} \right] [g_3(\alpha)]^{1/q}.
\]

\end{proof}

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\textbf{References}

