Contributions to Mathematics
www.shahindp.com/locate/cm

Contrib. Math. **5** (2022) 32–35
DOI: 10.47443/cm.2022.015

Research Article

Notes on several integral inequalities of Hermite–Hadamard type for s-geometrically convex functions

Chun-Ying He[1], Feng Qi[2,3,*]

[1]*School of Mathematics and Statistics, Hulunbuir University, Hulunbuir 021008, Inner Mongolia, China*
[2]*Institute of Mathematics, Henan Polytechnic University, Jiaozuo 454010, Henan, China*
[3]*School of Mathematical Sciences, Tiangong University, Tianjin 300387, China*

(Received: 6 April 2022. Received in revised form: 27 April 2022. Accepted: 28 April 2022. Published online: 2 May 2022.)

Abstract

In this paper, several erroneous results appeared in the papers [T.-Y. Zhang, A.-P. Ji, F. Qi, *Abstr. Appl. Anal.* **2012** (2012) #560586] and [T.-Y. Zhang, M. Tunç, A.-P. Ji, B.-Y. Xi, *Abstr. Appl. Anal.* **2014** (2014) #294739] are corrected.[†]

Keywords: Hermite–Hadamard type inequalities; integral inequality; s-geometrically convex function; Hölder's integral inequality.

2020 Mathematics Subject Classification: 26D15, 26A51.

1. Preliminaries

In [2, Definition 1.9], the concept of s-geometrically convex functions was introduced as follows.

Definition 1.1 (see [2, Definition 1.9])**.** *For some $s \in (0,1]$, a function $f : I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}_+$ is said to be an s-geometrically convex function if the inequality*

$$f\left(x^{\lambda} y^{1-\lambda}\right) \le [f(x)]^{\lambda^s} [f(y)]^{(1-\lambda)^s}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

The following two integral identities were established in the papers [2,3].

Lemma 1.1 (see [2, Lemma 2.1] and [3, Lemma 2.1])**.** *Let $f : I \subset \mathbb{R} = (-\infty,\infty) \to \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $f' \in L([a,b])$, then*

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x = \frac{b-a}{4}\int_0^1 \left[tf'\left((1-t)a + t\frac{a+b}{2}\right) + (t-1)f'\left((1-t)\frac{a+b}{2} + tb\right)\right]\mathrm{d}\,t$$

and

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x = \frac{b-a}{4}\int_0^1 \left[(t-1)f'\left((1-t)a + t\frac{a+b}{2}\right) + tf'\left((1-t)\frac{a+b}{2} + tb\right)\right]\mathrm{d}\,t.$$

In this paper, we need also the following lemmas.

Lemma 1.2. *Let $s \in (0,1)$ be a constant. If $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ is an s-geometrically convex function, then $f(x) \ge 1$ for all $x \in I$.*

Proof. For $x \in I$ and $\lambda \in (0,1)$, using the geometric convexity of f on I, we obtain

$$f(x) = f\left(x^{\lambda} x^{1-\lambda}\right) \le [f(x)]^{\lambda^s} [f(x)]^{(1-\lambda)^s} = [f(x)]^{\lambda^s + (1-\lambda)^s}.$$

Therefore, the inequality $f(x) \ge 1$ holds for all $x \in I$.

□

Lemma 1.3 (see [1, p. 4])**.** *Ig $0 < \mu \le 1 \le \eta$ and $0 < s, t \le 1$, then*

$$\mu^{t^s} \le \mu^{st} \quad \text{and} \quad \eta^{t^s} \le \eta^{st+1-s}.$$

The aim of this paper is to correct several errors appeared in the papers [2,3].

*Corresponding author (qifeng618@gmail.com).

[†]It was customary and appropriate to publish this article in the same journal (that is, Abstract and Applied Analysis) that published the flawed results. Unfortunately, at the time of the submission of this paper, the journal "Abstract and Applied Analysis" had a fee of 1025 US dollars for every accepted manuscript. The current authors were unwilling to bear this publishing cost.

2. Corrections

In this section, we state and prove corrected versions of some erroneous results appeared in [2,3].

Theorem 2.1 (Corrected version of [2, Theorem 3.1] and [3, pp. 1–3]). *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L([a,b])$ for $0 < a < b < \infty$. If $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$ for $q \ge 1$ and $s \in (0,1]$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}\left[|f'(a)||f'(b)|^{1-s}[g_1(\alpha)]^{1/q}+|f'(a)f'(b)|^{1-s/2}[g_2(\alpha)]^{1/q}\right] \tag{1}$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}\left[|f'(a)||f'(b)|^{1-s}[g_2(\alpha)]^{1/q}+|f'(a)f'(b)|^{1-s/2}[g_1(\alpha)]^{1/q}\right], \tag{2}$$

where

$$\alpha = \left|\frac{f'(b)}{f'(a)}\right|^{sq/2}, \quad g_1(\alpha) = \begin{cases} \dfrac{1}{2}, & \alpha = 1; \\ \dfrac{\alpha\ln\alpha-\alpha+1}{\ln^2\alpha}, & \alpha \ne 1, \end{cases}$$

and

$$g_2(\alpha) = \begin{cases} \dfrac{1}{2}, & \alpha = 1; \\ \dfrac{\alpha-\ln\alpha-1}{\ln^2\alpha}, & \alpha \ne 1. \end{cases}$$

Proof. Since $|f'|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$, using Lemma 1.1 and Hölder's inequality, we have

$$\begin{aligned}
\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| &\le \frac{b-a}{4}\int_0^1\left[t\left|f'\left((1-t)a+t\frac{a+b}{2}\right)\right|+(1-t)\left|f'\left((1-t)\frac{a+b}{2}+tb\right)\right|\right]\mathrm{d}\,t \\
&\le \frac{b-a}{4}\left\{\left(\int_0^1 t\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1 t|f'(a)|^{q((2-t)/2)^s}|f'(b)|^{q(t/2)^s}\,\mathrm{d}\,t\right]^{1/q}\right. \\
&\quad \left.+\left[\int_0^1(1-t)\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1(1-t)|f'(a)|^{q((1-t)/2)^s}|f'(b)|^{q((1+t)/2)^s}\,\mathrm{d}\,t\right]^{1/q}\right\} \\
&\le \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{\left[\int_0^1 t|f'(a)|^{q((2-t)/2)^s}|f'(b)|^{q(t/2)^s}\,\mathrm{d}\,t\right]^{1/q}\right. \\
&\quad \left.+\left[\int_0^1(1-t)|f'(a)|^{q((1-t)/2)^s}|f'(b)|^{q((1+t)/2)^s}\,\mathrm{d}\,t\right]^{1/q}\right\}.
\end{aligned} \tag{3}$$

By Lemmas 1.2 and 1.3, we obtain

$$\int_0^1 t|f'(a)|^{q((2-t)/2)^s}|f'(b)|^{q(t/2)^s}\,\mathrm{d}\,t \le \int_0^1 t|f'(a)|^{q[s(2-t)/2+1-s]}|f'(b)|^{q[st/2+1-s]}\,\mathrm{d}\,t = |f'(a)|^q|f'(b)|^{(1-s)q}g_1(\alpha) \tag{4}$$

and

$$\begin{aligned}
\int_0^1(1-t)|f'(a)|^{q((1-t)/2)^s}|f'(b)|^{q((1+t)/2)^s}\,\mathrm{d}\,t &\le \int_0^1(1-t)|f'(a)|^{q[s(1-t)/2+1-s]}|f'(b)|^{q[s(1+t)/2+1-s]}\,\mathrm{d}\,t \\
&= |f'(a)f'(b)|^{(1-s/2)q}g_2(\alpha).
\end{aligned} \tag{5}$$

Combining the inequalities (3), (4), and (5) leads to the inequality (1).

Since $|f'|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$, by Lemma 1.1 and Hölder's inequality, we acquire

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| &\le \frac{b-a}{4}\int_0^1\left[(1-t)\left|f'\left((1-t)a+t\frac{a+b}{2}\right)\right|+t\left|f'\left((1-t)\frac{a+b}{2}+tb\right)\right|\right]\mathrm{d}\,t \\
&\le \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{\left[\int_0^1(1-t)|f'(a)|^{q((2-t)/2)^s}|f'(b)|^{q(t/2)^s}\,\mathrm{d}\,t\right]^{1/q}+\left[\int_0^1 t|f'(a)|^{q((1-t)/2)^s}|f'(b)|^{q((1+t)/2)^s}\,\mathrm{d}\,t\right]^{1/q}\right\}.
\end{aligned} \tag{6}$$

Using Lemmas 1.2 and 1.3, we arrive at

$$\int_0^1 (1-t)|f'(a)|^{q((2-t)/2)^s}|f'(b)|^{q(t/2)^s}\,\mathrm{d}\,t \le |f'(a)|^q|f'(b)|^{(1-s)q}g_2(\alpha) \tag{7}$$

and

$$\int_0^1 t|f'(a)|^{q((1-t)/2)^s}|f'(b)|^{q((1+t)/2)^s}\,\mathrm{d}\,t \le |f'(a)f'(b)|^{(1-s/2)q}g_1(\alpha). \tag{8}$$

From the inequalities (6), (7), and (8), the inequality (2) follows readily. Theorem 2.1 is thus proved. □

Corollary 2.1 (Corrected version of [2, Corollary 3.2] and [3, p. 3]). *Under conditions of Theorem 2.1, we have the following conclusions:*

1. *When $q=1$, the inequalities*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\left[|f'(a)||f'(b)|^{1-s}g_1(\alpha)+|f'(a)f'(b)|^{1-s/2}g_2(\alpha)\right]$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\left[|f'(a)||f'(b)|^{1-s}g_2(\alpha)+|f'(a)f'(b)|^{1-s/2}g_1(\alpha)\right]$$

hold.

2. *When $s=1$, the inequalities*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}\left[|f'(a)|[g_1(\alpha)]^{1/q}+|f'(a)f'(b)|^{1/2}[g_2(\alpha)]^{1/q}\right]$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}\left[|f'(a)|[g_2(\alpha)]^{1/q}+|f'(a)f'(b)|^{1/2}[g_1(\alpha)]^{1/q}\right]$$

hold.

Theorem 2.2 (Corrected version of [2, Theorem 3.3] and [3, pp. 3–4]). *Let $f: I\subset\mathbb{R}_+\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L([a,b])$ for $0<a<b<\infty$. If $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$ for $q>1$ and $s\in(0,1]$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left[|f'(a)||f'(b)|^{1-s}+|f'(a)f'(b)|^{1-s/2}\right]\left[g_3(\alpha)\right]^{1/q} \tag{9}$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left[|f'(a)||f'(b)|^{1-s}+|f'(a)f'(b)|^{1-s/2}\right]\left[g_3(\alpha)\right]^{1/q}, \tag{10}$$

where α is the same as in Theorem 2.1 and

$$g_3(\alpha)=\begin{cases}1, & \alpha=1;\\ \dfrac{\alpha-1}{\ln\alpha}, & \alpha\ne1.\end{cases}$$

Proof. Since $|f'|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$, by Lemma 1.1 and Hölder's inequality, we get

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| &\le \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left\{\left[\int_0^1|f'(a)|^{q((2-t)/2)^s}|f'(b)|^{q(t/2)^s}\,\mathrm{d}\,t\right]^{1/q}\right.\\ &\quad\left.+\left[\int_0^1|f'(a)|^{q((1-t)/2)^s}|f'(b)|^{q((1+t)/2)^s}\,\mathrm{d}\,t\right]^{1/q}\right\}\end{aligned} \tag{11}$$

and

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| &\le \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left\{\left[\int_0^1|f'(a)|^{q((2-t)/2)^s}|f'(b)|^{q(t/2)^s}\,\mathrm{d}\,t\right]^{1/q}\right.\\ &\quad\left.+\left[\int_0^1|f'(a)|^{q((1-t)/2)^s}|f'(b)|^{q((1+t)/2)^s}\,\mathrm{d}\,t\right]^{1/q}\right\}.\end{aligned} \tag{12}$$

Furthermore, we have

$$\int_0^1 |f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q(t/2)^s} \,\mathrm{d}\, t \le |f'(a)|^q |f'(b)|^{(1-s)q} g_3(\alpha) \tag{13}$$

and

$$\int_0^1 |f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+t)/2)^s} \,\mathrm{d}\, t \le |f'(a)f'(b)|^{(1-s/2)q} g_3(\alpha). \tag{14}$$

By substituting (13) and (14) into (11) and (12), respectively, we obtain the inequalities (9) and (10), respectively.

□

Corollary 2.2 (Corrected version of [2, Corollary 3.4] and [3, p. 4])**.** *Under conditions of Theorem 2.2, when* $s = 1$*, we have*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x) \,\mathrm{d}\, x \right| \le \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q} \left[|f'(a)| + |f'(a)f'(b)|^{1/2}\right] \left[g_3(\alpha)\right]^{1/q}$$

and

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x) \,\mathrm{d}\, x \right| \le \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q} \left[|f'(a)| + |f'(a)f'(b)|^{1/2}\right] \left[g_3(\alpha)\right]^{1/q}.$$

Acknowledgement

This work was partially supported by the National Natural Science Foundation of China (Grant No. 12061033) and by the Teaching Program at Hulunbuir University (Grant No. JYZD2021031), China.

References

[1] R.-F. Bai, F. Qi, B.-Y. Xi, Hermite–Hadamard type inequalities for the m- and (α, m)-logarithmically convex functions, *Filomat* **27** (2013) 1–7.

[2] T.-Y. Zhang, A.-P. Ji, F. Qi, On integral inequalities of Hermite-Hadamard type for s-geometrically convex functions, *Abstr. Appl. Anal.* **2012** (2012) #560586.

[3] T.-Y. Zhang, M. Tunç, A.-P. Ji, B.-Y. Xi, Erratum to "On integral inequalities of Hermite–Hadamard type for s-geometrically convex functions", *Abstr. Appl. Anal.* **2014** (2014) #294739.