Contributions to Mathematics
www.shahindp.com/locate/cm

Contrib. Math. **5** (2022) 30–31
DOI: 10.47443/cm.2022.014

Research Article

On arc reversal in digraphs with a restricted number of cycles

Zbigniew R. Bogdanowicz*

CCDC Armaments Center, Picatinny, New Jersey 07806, USA

(Received: 16 March 2022. Accepted: 1 April 2022. Published online: 4 April 2022.)

Abstract

Let C_G be a longest cycle in the directed graph $G = (V, A)$. Let q be the total number of cycles in G. It is proved that if $1 \leq q \leq |V(C_G)|$ then there exists an arc $a \in A(G)$ whose reversal decreases the total number of cycles in G.

Keywords: Ádám's conjecture; arc reversal; number of directed cycles; multidigraph; simple digraph.

2020 Mathematics Subject Classification: 05C20, 05C38.

1. Introduction

In this paper, by the graph $G = (V, A)$ we mean a directed graph (i.e., a digraph) on the vertex set $V(G)$ and arcs set $A(G)$, which is either a simple digraph or a multidigraph without loops. Ádám's conjecture introduced in 1963 [1] on arc-reversal in G is one of the most important and fundamental conjectures in graph theory. It states that if G contains a directed cycle then G also contains an arc whose reversal results in a digraph H with a fewer number of directed cycles than in G.

Unfortunately, Ádám's conjecture does not hold in general since it was independently disproved for multidigraphs by Thomassen [8] and Grinberg [3], in 1987 and 1988, respectively. On the other hand, Ádám's conjecture remains open for simple digraphs. Hence, the arc-reversal problem became a hot topic for investigating special families of digraphs, especially after 1988. In particular, most of the papers focused on special families of simple digraphs.

In 1984, Reid proved the conjecture for 2-arc-connected tournaments, which are not 3-arc-connected [7]. Note that tournaments represent a special family of simple digraphs, and the conjecture remains open for them as well. In 1987, Jirásek proved the conjecture for the special case of multidigraphs [4]. In 1992, he proved the conjecture for the special case of simple digraphs, the ones that become acyclic after the reversal of at most three arcs [5]. In 2004, Jirásek constructed counterexamples to Ádám's conjecture derived from the circulant oriented graphs that are not Hamiltonian [6]. Finally, in 2011 the present author proved that any arc reversal in a balanced digraph decreases the number of closed walks [2], which is related to Ádám's conjecture as well.

In the next section, we prove that a digraph with at least one cycle and an explicitly defined upper bound on the total number of cycles satisfies Ádám's conjecture.

2. Main result

We now consider a generic case of G with a predetermined upper bound on the total number of directed cycles, and we present the following main result of this paper.

Theorem 2.1. *Let C_G be a longest cycle and q be the total number of cycles in digraph G. If $1 \leq q \leq |C_G|$ then there exists an arc $a \in A(G)$ whose reversal decreases the number of cycles in G.*

Proof. It follows by contradiction. Suppose $1 \leq q \leq |C_G|$ and there is no arc $a \in A(G)$ whose reversal $\bar{a}$ results in digraph H with t number of cycles, where $t < q$.

Let $V(C_G)$ denote a set of vertices and $A(C_G)$ denote a set of arcs in C_G. Since $q \geq 1$ then non-empty $A(C_G)$ exists. If $a = (v, w) \in A(C_G)$ then there must also exist $a_1 = (v, v_1)$ in $A(G)$, where $a_1 \notin A(C_G)$. Otherwise, $\bar{a}$ would not belong to any cycle in H – a contradiction. Consequently, there must exist a cycle $\bar{C} = vv_1v_2 \cdots v_kwv$ in H for some $k \geq 0$ (in particular, there must exist $\bar{C} = a_1\bar{a}$ for $k = 0$).

*E-mail addresses: zbigniew.bogdanowicz.civ@mail.mil, zrb2@aol.com

If $k \geq 1$ and $v_1, v_2, \ldots, v_k \notin V(C_G)$ in $\bar{C}$ then there exists longer cycle C'_G, where $|C'_G| = |C_G|+k$, which is a contradiction. So, there must exist path $P = vv_1v_2 \cdots v_r$ for some v_r, where $v_r \in V(C_G)$ and $1 \leq r \leq k$. Without loss of generality, let P be a shortest such path (i.e., a path with minimum possible r, where $v_r \in V(C_G)$). This implies a cycles $C'' = vv_1v_2 \cdots v_rv_{r+1} \cdots v$ in G, where $v_r, v_{r+1}, \ldots, v \in V(C_G)$.

Let $|C_G| = m$ and $u_0, u_1, \ldots, u_{m-1} \in V(C_G)$. By induction, based on C'' for every u_i there is a cycle

$$C_i = u_iv_1^iv_2^i \cdots v_{r_i-1}^iu_{r_i}^iu_{r_i+1}^i \cdots u_{r_i+t_i}^iu_i$$

in G, where $(u_i, v_1^i) = a_i \notin A(C_G)$ and $(u_{r_i}^i, u_{r_i+1}^i), \ldots, (u_{r_i+t_i}^i, u_i) \in A(C_G)$. Furthermore, $a_i \in A(C_j)$ if and only if $i = j$. Hence, $C_0, C_1, \ldots, C_{m-1}$ are pairwise distinct in G. Since G also includes C_G with all arcs in $A(C_G)$, this implies that G has at least $q \geq m+1$ cycles – a contradiction, which proves the theorem. □

As a direct consequence of Theorem 2.1, for the Hamiltonian digraphs we can state the next corollary.

Corollary 2.1. *Let q be the total number of cycles in Hamiltonian digraph G. If $q \leq |V(G)|$ then there exists an arc $a \in A(G)$ whose reversal decreases the number of cycles in G.*

Proof. It is a special case of Theorem 2.1, where $|C_G| = |V(G)|$ and $1 \leq q \leq |C(G)|$ because G is Hamiltonian. □

In closing, we note that it is a challenge to come up with the higher upper bound (than the one given in this paper) on the total number of cycles in the digraph for which Ádám's conjecture is satisfied.

References

[1] A. Ádám, Problem 2, In: M. Fiedler (Ed.), *Theory of Graphs and Its Applications*, Proc. Symp. Smolenice, Czech. Acad. Sci., Prague, 1964, p. 157.
[2] Z. R. Bogdanowicz, On arc reversal in balanced digraphs, *Discrete Math.* **311** (2011) 435–436.
[3] E. Grinberg, Examples of non-Adamian multigraphs, *Latv. Mat. Ezheg.* **31** (1988) 128–138.
[4] J. Jirásek, On a certain class of multigraphs for which reversal of no arc decreases the number of their cycles, *Comment. Math. Univ. Carolina* **28** (1987) 185–189.
[5] J. Jirásek, Some remarks on Ádám's conjecture for simple directed graphs, *Discrete Math.* **108** (1992) 327–332.
[6] J. Jirásek, Arc reversal in nonhamiltonian circulant oriented graphs, *J. Graph Theory* **49** (2005) 59–68.
[7] K. Reid, Monochromatic reachability, complementary cycles, and single arc reversals in tournaments, In: K. M. Koh, H. P. Yap (Eds.), *Graph Theory Singapore 1983*, Lecture Notes in Mathematics: Vol. 1073, Springer, Berlin, 1984, pp. 11–21.
[8] C. Thomassen, Counterexamples to conjecture on arc reversals in directed graphs, *J. Combin. Theory Ser. B* **42** (1987) 128–130.