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Research Article

Averaging principle for fuzzy stochastic differential equations

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Abstract

This study offers the averaging principle for fuzzy stochastic differential equations (FSDEs). The solutions to FSDEs can be approximated in the sense of mean square solutions of averaged fuzzy stochastic system under certain assumptions.

Keywords: fuzzy stochastic differential equations; averaging principle; Gronwall inequality.

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1. Introduction

The averaging method is a powerful tool for investigating the qualitative property of dynamical system in physics as well as in variety of other fields. This method shows a connection between the solutions of averaged systems and the solutions of a standard form [15, 16]. Nevertheless, up to now, the averaging principle for fuzzy stochastic differential equations (FSDEs) has not yet been studied in literature. In the present paper, we make the first attempt to study this method for FSDEs.

For crisp stochastic differential equations (SDEs), seminal results on the averaging principle can be found in [5, 6, 12]. Tan et al. [13] established the averaging method for stochastic differential delay equations (SDDEs) under non-Lipschitz conditions. In [11, 14], the authors investigated the averaging principle for SDDEs with jumps and with fractional Brownian motion. Recently, Guo et al. [3] established the averaging method for a class of SDEs with nonlinear terms satisfying the monotone condition, and Luo et al. [7, 8] investigated the averaging principle for a class of stochastic fractional differential equations (SFDEs) with time-delays. Ahmed et al. [1] established the averaging principle for Hilfer fractional stochastic delay differential equations with Poisson jumps. On the other hand, FSDEs are utilised in real-world systems where the phenomena is connected to randomness and fuzziness as two types of uncertainty. In [2, 4], the authors presented a definition of the fuzzy stochastic Itô integral using a method that allows embedding of a crisp Itô stochastic integral into fuzzy space for building a fuzzy random variable. The present paper aims at extending the averaging principle to FSDEs.

The rest of this paper is organised as follows. Section 2 provides the fundamental tools that are required in upcoming sections. In Section 3, the averaging method for FSDEs under some conditions is investigated. An example is given in Section 4 to illustrate the main result of this paper. Finally, the conclusion is given in Section 5.

2. Preliminaries

In this section, we introduce some notations, definitions, and preliminary facts which are used in the rest of this paper. Let $\mathcal{K}(\mathbb{R}^n)$ be the family of nonempty convex and compact subsets of $\mathbb{R}^n$. In $\mathcal{K}(\mathbb{R}^n)$, the distance d_H is defined by

$$d_H(M,N) := \max\left(\sup_{m\in M}\inf_{n\in N}\|m-n\|, \sup_{n\in N}\inf_{m\in M}\|m-n\|\right), \quad M,N\in\mathcal{K}(\mathbb{R}^n).$$

It is know that $\mathcal{K}(\mathbb{R}^n)$ is a complete and separable metric space with respect to d_H. Let $\mathbf{E}^n$ be the fuzzy set space of $\mathbb{R}^n$, i.e. the set of functions $v:\mathbb{R}^n \longrightarrow [0,1]$ such that $[v]^\alpha \in \mathcal{K}(\mathbb{R}^n)$, $\forall\, \alpha\in[0,1]$, where

$$[v]^\alpha := \{a\in\mathbb{R}^n : v(a)\geq\alpha\}, \quad \text{for } \alpha\in[0,1],$$

and

$$[v]^0 := cl\{a\in\mathbb{R}^n : v(a) > 0\}.$$

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Let

$$d_\infty(x,y) := \sup_{\alpha\in[0,1]} d_H\big([x]^\alpha,[y]^\alpha\big)$$

be the metric satisfying the following properties

$$d_\infty(x+z,y+z)=d_\infty(x,y),\quad d_\infty(x+y,z+w)\le d_\infty(x,z)+d_\infty(y,w),\quad \text{and}\quad d_\infty(\lambda x,\lambda y)=|\lambda| d_\infty(x,y),\quad \lambda\in\mathbb{R}.$$

Let $\langle . \rangle : \mathbb{R}^n \longrightarrow \mathbf{E}^n$ be an embedding of $\mathbb{R}^n$ into $\mathbf{E}^n$, i.e. for $r\in\mathbb{R}^n$, one has

$$\langle r\rangle(a)=\begin{cases} 1\,, & if \quad a=r,\\ 0\,, & if \quad a\neq r.\end{cases}$$

Remark 2.1. *Let $v:[0,T]\times\Omega\longrightarrow\mathbb{R}^n$ be an $\mathbb{R}^n$-valued stochastic process, then $\langle v\rangle:[0,T]\times\Omega\longrightarrow\mathbf{E}^n$ is a fuzzy stochastic process.*

Let $\{B(t),\ t\in I:=[0,T]\}$ be an one-dimensional Brownian motion defined on a complete probability space $(\Omega,\mathcal{A},\mathbb{P})$ with a filtration $\{\mathcal{A}_t\}_{t\in[0,T]}$ satisfying usual hypotheses.

Definition 2.1 (see [10]). *By fuzzy stochastic Itô integral we mean the fuzzy random variable $\langle\int_0^t v(s)dB(s)\rangle$. For every $t\in I$, consider the fuzzy stochastic Itô integral $\langle\int_0^t v(s)dB(s)\rangle$, which may be interpreted as follows*

$$\left\langle \int_0^t v(s)dB(s)\right\rangle := \left\langle \int_0^T \chi_{[0,t]}(s)v(s)dB(s)\right\rangle.$$

Proposition 2.1 (see [10]). *If $u,v\in\mathcal{L}^2\big(I\times\Omega,\mathbf{N};\mathbb{R}^n\big)$, then $\forall\, t\in I$ we have*

$$d_\infty^2\left(\left\langle \int_0^t u(s)dB(s)\right\rangle,\left\langle \int_0^t v(s)dB(s)\right\rangle\right)=\int_0^t d_\infty^2\big(\langle u(s)\rangle,\langle v(s)\rangle\big)ds.$$

Proposition 2.2 (see [9]). *For $u,v\in\mathcal{L}^p(I\times\Omega,\mathbf{N};\mathbf{E}^n)$ and $p\ge 1$, we have*

$$\mathbb{E}\sup_{a\in[0,t]} d_\infty^p\left(\int_0^a u(s)ds,\int_0^a v(s)ds\right)\le t^{p-1}\int_0^t \mathbb{E}d_\infty^p(u(s),v(s))ds.$$

3. Main result

Consider the following FSDEs

$$\begin{cases} dx(s)=f(t,x(t))dt+\big\langle g(t,x(t))dB(t)\big\rangle,\\ x(0)=x_0\in\mathbf{E}^n,\end{cases}\tag{1}$$

where $f:I\times\mathbf{E}^n\longrightarrow\mathbf{E}^n$, $g:I\times\mathbf{E}^n\longrightarrow\mathbb{R}^n$ and $x_0:\Omega\longrightarrow\mathbf{E}^n$ is a fuzzy random variable. Equation (1) is equivalent to the following fuzzy stochastic integral equation

$$x(t)=x_0+\int_0^t f(s,x(s))ds+\left\langle\int_0^t g(s,x(s))dB(s)\right\rangle,\qquad t\in I.\tag{2}$$

We apply conditions on the coefficient functions to ensure that the solution to (1) exists and is unique.

($\mathcal{A}$**1**). There exists a constant $C_1>0$ such that $\forall\, t\in I$ and $\forall\, x\in\mathbf{E}^n$ we have

$$d_\infty^2\big(f(t,x),\widehat{0}\big)\le C_1^2\big(1+d_\infty^2(x,\widehat{0})\big)\ \text{ and }\ \big\|g(t,x)\big\|^2:=d_\infty^2\big(\langle g(t,x)\rangle,\widehat{0}\big)\le C_1^2\big(1+d_\infty^2(x,\widehat{0})\big).$$

($\mathcal{A}$**2**). There exists a constant $C_2>0$ such that $\forall\, t\in I$ and $\forall\, x,y\in\mathbf{E}^n$ we have

$$d_\infty^2\big(f(t,x),f(t,y)\big)\le C_2 d_\infty^2(x,y)\ \text{ and }\ \big\|g(t,x)-g(t,y)\big\|^2=d_\infty^2\big(\langle g(t,x)\rangle,\langle g(t,y)\rangle\big)\le C_2 d_\infty^2(x,y).$$

By the work of Malinowski and Michta [10], we know that under the assumptions $(\mathcal{A}1)$ and $(\mathcal{A}2)$, FSDEs (1) has a unique solution $x(t)$ with the initial data x_0.

Let us consider the standard form of Equation (2)

$$x_\epsilon(t)=x_0+\epsilon\int_0^t f(s,x_\epsilon(s))ds+\sqrt{\epsilon}\left\langle\int_0^t g(s,x_\epsilon(s))dB(s)\right\rangle,\tag{3}$$

where the initial value x_0, functions f and g have the same conditions as in Equation (2), and $\epsilon\in(0,\epsilon_0)$ is a small positive parameter with ϵ_0 a fixed number. Based on the existence and uniqueness results, Equation (3) also has a unique solution $x_\epsilon(t)$ for every fixed $\epsilon\in(0,\epsilon_0)$ and $t\in I$. We set certain assumptions on the coefficients to see if the solution $x_\epsilon(t)$ can be approximated by a simple process.

Let $\tilde{f} : \mathbf{E}^n \longrightarrow \mathbf{E}^n$ and $\tilde{g} : \mathbf{E}^n \longrightarrow \mathbb{R}^n$ be measurable functions satisfying $(\mathcal{A}1)$ and $(\mathcal{A}2)$, as well as the following additional inequalities:

($\mathcal{A}$3). For $x \in \mathbf{E}^n$ and $T' \in I$ we have

$$\frac{1}{T'}\int_0^{T'} d_\infty^2\Big(f(s,x),\tilde{f}(x)\Big)ds \le \beta_1(T')\big(1+d_\infty^2(x,\widehat{0})\big) \text{ and } \frac{1}{T'}\int_0^{T'} \big\|g(s,x)-\tilde{g}(x)\big\|^2 ds \le \beta_2(T')\big(1+d_\infty^2(x,\widehat{0})\big),$$

where $\lim_{T'\longrightarrow\infty}\beta_i(T') = 0$, $i = 1, 2$.

With the above appropriate preparations, we now show that the solution x_ϵ converges to the solution y_ϵ of the following averaged FSDEs

$$y_\epsilon(t) = x_0 + \epsilon\int_0^t \tilde{f}(y_\epsilon(s))ds + \sqrt{\epsilon}\Big\langle \int_0^t \tilde{g}(y_\epsilon(s))dB(s)\Big\rangle, \tag{4}$$

as $\epsilon \longrightarrow 0$. Clearly, under similar assumptions as of Equation (3), Equation (4) also has a unique solution y_ϵ. The main result of this paper is now presented in the form of the following theorem, in which we consider the connections between x_ϵ and y_ϵ.

Theorem 3.1. *Assume that the assumptions* $(\mathcal{A}1)-(\mathcal{A}3)$ *are satisfied. For a given arbitrarily small number* $\Delta > 0$ *and a constant* $k > 0$, $\alpha \in (0,1)$, *there exists* $\epsilon_1 \in (0,\epsilon_0]$ *such that* $\forall\, \epsilon \in (0,\epsilon_1]$, *we have*

$$\sup_{t\in[0,k\epsilon^{-\alpha}]} \mathbb{E}\, d_\infty^2\big(x_\epsilon(t), y_\epsilon(t)\big) \le \Delta.$$

Proof. For any $t \in [0,u] \subset I$, we have

$$\begin{aligned}\sup_{t\in[0,u]} \mathbb{E}d_\infty^2\big(x_\epsilon(t), y_\epsilon(t)\big) &= \sup_{t\in[0,u]} \mathbb{E}d_\infty^2\Big(x_0 + \epsilon\int_0^t f(s,x_\epsilon(s))ds + \sqrt{\epsilon}\Big\langle \int_0^t g(s,x_\epsilon(s))dB(s)\Big\rangle,\\ &\qquad x_0 + \epsilon\int_0^t \tilde{f}(y_\epsilon(s))ds + \sqrt{\epsilon}\Big\langle \int_0^t \tilde{g}(y_\epsilon(s))dB(s)\Big\rangle\Big)\\ &\le 2\epsilon^2 \sup_{t\in[0,u]} \mathbb{E}d_\infty^2\Big(\int_0^t f(s,x_\epsilon(s))ds, \int_0^t \tilde{f}(y_\epsilon(s))ds\Big) +\\ &\qquad 2\epsilon \sup_{t\in[0,u]} \mathbb{E}d_\infty^2\Big(\Big\langle \int_0^t g(s,x_\epsilon(s))dB(s)\Big\rangle, \Big\langle \int_0^t \tilde{g}(y_\epsilon(s))dB(s)\Big\rangle\Big).\end{aligned}$$

Denote by

$$J_1 = 2\epsilon^2 \sup_{t\in[0,u]} \mathbb{E}d_\infty^2\Big(\int_0^t f(s,x_\epsilon(s))ds, \int_0^t \tilde{f}(y_\epsilon(s))ds\Big)$$

and

$$J_2 = 2\epsilon \sup_{t\in[0,u]} \mathbb{E}d_\infty^2\Big(\Big\langle \int_0^t g(s,x_\epsilon(s))dB(s)\Big\rangle, \Big\langle \int_0^t \tilde{g}(y_\epsilon(s))dB(s)\Big\rangle\Big).$$

Then, by using the properties of the metric d_∞, we get

$$\begin{aligned}J_1 &\le 4\epsilon^2 \sup_{t\in[0,u]} \mathbb{E}d_\infty^2\Big(\int_0^t f(s,x_\epsilon(s))ds, \int_0^t f(s,y_\epsilon(s))ds,\Big) + 4\epsilon^2 \sup_{t\in[0,u]} \mathbb{E}d_\infty^2\Big(\int_0^t f(s,y_\epsilon(s))ds, \int_0^t \tilde{f}(y_\epsilon(s))ds\Big),\\ &:= J_{11} + J_{12}.\end{aligned}$$

By using Proposition 2.2 and the assumption $(\mathcal{A}2)$, we have

$$\begin{aligned}J_{11} &\le 4\epsilon^2 \sup_{t\in[0,u]}\Big(t\int_0^t \mathbb{E}d_\infty^2\big(f(s,x_\epsilon(s)), f(s,y_\epsilon(s))\big)ds\Big)\\ &\le 4\epsilon^2 C_2 u \int_0^u \mathbb{E}d_\infty^2\big(x_\epsilon(s), y_\epsilon(s)\big)ds.\end{aligned}$$

For J_{12}, we use Proposition 2.2 and the assumption $(\mathcal{A}3)$, and hence we get

$$\begin{aligned} J_{12} &\leq 4\epsilon^2 \sup_{t\in[0,u]} \left(t \int_0^t \mathbb{E}d^2_\infty\big(f(s,y_\epsilon(s)), \tilde{f}(y_\epsilon(s))\big)ds \right) \\ &\leq 4\epsilon^2 \sup_{t\in[0,u]} \left(t^2 \frac{1}{t} \int_0^t \mathbb{E}d^2_\infty\big(f(s,y_\epsilon(s)), \tilde{f}(y_\epsilon(s))\big)ds \right) \\ &\leq 4\epsilon^2 u^2 \beta_1(u)\big[1 + \sup_{t\in[0,u]} \mathbb{E}d^2_\infty\big(y_\epsilon(t), \widehat{0}\big)\big] \\ &:= 4\epsilon^2 u^2 \lambda_1. \end{aligned}$$

Therefore,

$$J_1 \leq 4\epsilon^2 C_2 u \int_0^u \mathbb{E}d^2_\infty\big(x_\epsilon(s), y_\epsilon(s)\big)ds + 4\epsilon^2 u^2 \lambda_1. \tag{5}$$

For the second term J_2, by using Proposition 2.1, we have

$$\begin{aligned} J_2 &\leq 2\epsilon \sup_{t\in[0,u]} \int_0^t \mathbb{E}\big\| g(s,x_\epsilon(s)) - \tilde{g}(y_\epsilon(s))\big\|^2 ds \\ &\leq 4\epsilon \sup_{t\in[0,u]} \int_0^t \mathbb{E}\big\| g(s,x_\epsilon(s)) - g(s,y_\epsilon(s))\big\|^2 ds + 4\epsilon \sup_{t\in[0,u]} \int_0^t \mathbb{E}\big\| g(s,y_\epsilon(s)) - \tilde{g}(y_\epsilon(s))\big\|^2 ds \\ &:= J_{21} + J_{22}. \end{aligned}$$

Using the assumption $(\mathcal{A}2)$, we get

$$J_{21} \leq 4\epsilon C_2 \int_0^u \mathbb{E}d^2_\infty\big(x_\epsilon(s), y_\epsilon(s)\big)ds.$$

Also, by using the assumption $(\mathcal{A}3)$, we have

$$\begin{aligned} J_{22} &\leq 4\epsilon \sup_{t\in[0,u]} \left(t\frac{1}{t} \int_0^t \mathbb{E}\big\| g(s,y_\epsilon(s)) - \tilde{g}(y_\epsilon(s))\big\|^2 ds \right) \\ &\leq 4\epsilon u \beta_2(u)\big[1 + \sup_{t\in[0,u]} \mathbb{E}d^2_\infty\big(y_\epsilon(t), \widehat{0}\big)\big] \\ &:= 4\epsilon u \lambda_2. \end{aligned}$$

Therefore,

$$J_2 \leq 4\epsilon C_2 \int_0^u \mathbb{E}d^2_\infty\big(x_\epsilon(s), y_\epsilon(s)\big)ds + 4\epsilon u \lambda_2. \tag{6}$$

By combining (5) and (6), we get

$$\begin{aligned} \sup_{t\in[0,u]} \mathbb{E}d^2_\infty\big(x_\epsilon(t), y_\epsilon(t)\big) &\leq 4\epsilon u\big(\lambda_2 + \epsilon u \lambda_1\big) + 4\epsilon C_2\big(1+\epsilon u\big) \int_0^u \mathbb{E}d^2_\infty\big(x_\epsilon(s), y_\epsilon(s)\big)ds \\ &\leq 4\epsilon u\big(\lambda_2 + \epsilon u \lambda_1\big) + 4\epsilon C_2\big(1+\epsilon u\big) \int_0^u \sup_{v\in[0,s]} \mathbb{E}d^2_\infty\big(x_\epsilon(v), y_\epsilon(v)\big)ds. \end{aligned}$$

Hence, by using the Gronwall inequality, we get

$$\sup_{t\in[0,u]} \mathbb{E}d^2_\infty\big(x_\epsilon(t), y_\epsilon(t)\big) \leq 4\epsilon u\big(\lambda_2 + \epsilon u \lambda_1\big)e^{4\epsilon C_2(1+\epsilon u)}.$$

Choose $\alpha \in (0,1)$ and $L > 0$ such that for every $t \in [0, L\epsilon^{-\alpha}] \subseteq I$, we have

$$\sup_{t\in[0,L\epsilon^{-\alpha}]} \mathbb{E}d^2_\infty\big(x_\epsilon(t), y_\epsilon(t)\big) \leq kL\epsilon^{1-\alpha},$$

where $k = 4\big(\lambda_2 + L\epsilon^{1-\alpha}\lambda_1\big)\exp\{4\epsilon C_2\big(1 + L\epsilon^{1-\alpha}\big)\}$ is a constant. Therefore, for any given number Δ, $\exists\ \epsilon_1 \in (0, \epsilon_0]$ such that for each $\epsilon \in (0, \epsilon_1]$ and $t \in [0, L\epsilon^{-\alpha}]$, we have

$$\sup_{t\in[0,L\epsilon^{-\alpha}]} \mathbb{E}d^2_\infty\big(x_\epsilon(t), y_\epsilon(t)\big) \leq \Delta.$$

□

4. An example

In this section, we give an example to illustrate our main result of this paper. Consider the following FSDEs

$$\begin{cases} dX(t) = 4\cos^2(t)X(t)dt + \big\langle X(t)dB(t)\big\rangle, \\ X(0) = 0. \end{cases} \tag{7}$$

The standard form of the above FSDEs is given as

$$dX^\epsilon = 4\epsilon\cos^2(t)X^\epsilon dt + \sqrt{\epsilon}\big\langle X^\epsilon dB(t)\big\rangle.$$

Note that $f(t, X^\epsilon) = 4\cos^2(t)X^\epsilon$ and $g(t, X^\epsilon) = X^\epsilon$. Hence,

$$\tilde{f}(X^\epsilon) = \frac{1}{\pi}\int_0^\pi 4\cos^2(t)X^\epsilon dt = 2X^\epsilon \quad \text{and} \quad \tilde{g}(X^\epsilon) = \frac{1}{\pi}\int_0^\pi g(t, X^\epsilon)dt = X^\epsilon.$$

Therefore, the averaging form of (7) is

$$dY^\epsilon = 2\epsilon Y^\epsilon dt + \sqrt{\epsilon}\big\langle Y^\epsilon dB(t)\big\rangle. \tag{8}$$

The coefficients $f(t, X^\epsilon)$ and $g(t, X^\epsilon)$ satisfy the assumptions $(\mathcal{A}1) - (\mathcal{A}2)$, and hence FSDEs (7) has a unique fuzzy solution. Also, it is observed that the coefficients $\tilde{f}(X^\epsilon)$ and $\tilde{g}(X^\epsilon)$ satisfy the assumption $(\mathcal{A}3)$. Therefore, by Theorem 3.1, as $\epsilon \longrightarrow 0$, the solutions X^ϵ and Y^ϵ to Equations (7) and (8) are equivalent in the sense of mean square.

5. Conclusion

In this work, we have established the averaging principle for FSDEs. We have proved that the solution to the averaged FSDEs converges to that of the standard FSDEs in the sense of mean square. For future researches, we plan to study the averaging principle for fuzzy fractional stochastic differential equations with/without Hilfer fractional derivative.

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